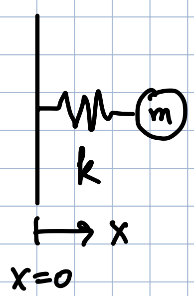
Lecture 10 – Stochastic Differential Equations

There is one last important topic to touch on in stochastic processes. We saw with the Fokker-Planck equation that we can follow the propagation of a conditional PDF to understand the process statistics. But what about following the dynamics of a single random variable in time? It seems like we might want to write an equation for such a variable.

A. Langevin equations

Consider the example of a harmonic oscillator with Brownian forces



$$m\frac{dv}{dt} = \sum_i F_i = F_s + F_D + F_B$$

F_s : spring force

F_B : Brownian force

F_D : Drag force

$\langle\,\rangle$: ensemble expectation

$$F_s = -k_s x \qquad \langle F_B \rangle = 0$$

→ separate out other forces

$$F_D = -k_D v \qquad \langle F_B(t)\, F_B(t') \rangle = \alpha_B\, \delta(t-t')$$

(Gaussian) white noise, independent

If $F_B = 0$ we have a damped harmonic oscillator with $x=0$ and $v=0$ as the equilibrium solution. See appendices for details.

In the overdamped limit (no inertia), we can write this equation as

$$0 = -k_D\frac{dx}{dt} - k_s x + F_B(t)$$

$$\frac{dx}{dt} = -\frac{k_s}{k_D}x + \frac{1}{k_D}F_B(t)$$

$F_B(t)$: a single instance of the Brownian force (a random process)

This has the form

$$\frac{dx}{dt} = a(x,t) + b(x,t)\,\xi(t) \qquad \langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = \delta(t-t')$$

We call this kind of equation a Langevin equation.

Note about units:

$$\frac{dx}{dt} [=] a(x,t) [=] b(x,t)\,\xi(t) [=] \frac{x\text{-units}}{\text{time}}$$

$$\delta(t-t') [=] \text{time}^{-1} \Rightarrow \xi(t) [=] \text{time}^{-1/2}$$

If $a=0$ and $b=1$, we get this Langevin equation,

$$\frac{dx}{dt} = \xi(t) \quad \Rightarrow \quad x(t) - \cancel{x(0)}^{\,0} = \int_0^t \underbrace{\xi(t)\,dt}_{dW}$$

This is the Wiener process! We give this integrand a new symbol, dW, a sort of differential Wiener process.

$$x(t) = W(t) = \int_0^t dW \qquad \text{white noise is } \frac{dW}{dt}^{*}$$

This stochastic integral can be rigorously defined.

*Technically, $W(t)$ is continuous but not differentiable, so we only use it under an integral.

This new Wiener increment (Itô differential) has the properties

$$\langle dW \rangle = 0 \qquad \langle dW^2 \rangle = dt \qquad dW \sim N(0, dt)$$

Furthermore, dW is Gaussian & a Markov process! I'm not going to prove it to you, but it follows from the fact that $W(t)$ is continuous.

Using this new differential, we can write our Langevin equation as a proper Stochastic Differential Equation (SDE).

$$dx(t) = a(x,t)\,dt + b(x,t)\,dW$$

There is much more we could explore here. For example there is an entire branch of math dedicated to stochastic calculus, SDEs, and stochastic numerical methods.

B. Connecting FP and SDE formalisms

We have seen two different ways to look at Markov processes, by distribution with the Fokker-Planck equation and by instance with a stochastic differential equation. We want to connect these two approaches.

Suppose I have a FP equation with a deterministic initial condition,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left[A(x) f\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(B(x) f\right) \qquad f = f(x;t\,|\,x_0;t_0)$$

$$f(x(t=0)\,|\,x_0;t_0) = \delta(x-x_0)$$

For a very short step, $(\Delta t, \Delta x)$ A and B are constant

$$\frac{\partial f}{\partial t} = -A\frac{\partial f}{\partial x} + \frac{B}{2}\frac{\partial^2 f}{\partial x^2} \qquad f(0) = \delta(x-x_0)$$

We can solve this via Fourier Transforms

$$\frac{\partial \hat{f}}{\partial t} = ikA\hat{f} - \frac{1}{2}k^2 B\hat{f} \qquad \hat{f}(0) = e^{ikx_0}$$

$$\frac{\partial \hat{f}}{\partial t} = \left(ikA - \frac{B}{2}k^2\right)\hat{f}$$

$$\hat{f}(t) = \hat{f}(0)\exp\left(\left[ikA - \frac{B}{2}k^2\right]t\right)$$

$$= \exp\left(ikAt - Bk^2t/2 + ikx_0\right)$$

Inverting the Fourier Transform gives

$$f(x,t) = \frac{1}{\sqrt{2\pi B t}}\exp\left(-\frac{\left[x-x_0-At\right]^2}{2Bt}\right) \qquad \begin{aligned}\Delta x &= x-x_0\\ \Delta t &= t-0\end{aligned}$$

We assumed that Δx and Δt were small

$$f(\Delta x, \Delta t) = \frac{1}{\sqrt{2\pi B \Delta t}} \exp\left(-\frac{(\Delta x - a\Delta t)^2}{2B\Delta t}\right)$$

This is our short-time kernel $\pi_{\Delta x, \Delta t}$ from the F-P derivation!

This is of course Gaussian! This kernal is one step of a Markov process. In a very short time the noise has no time for skewness or fat tails. (The central limit theorem tells us that it must be Gaussian at leading order!)

You only get a small deterministic shift of $a\Delta t$ plus a random kick of variance $B\Delta t$. But this is exactly what a Langevin equation would tell us,

$$x(t+\Delta t) = x(t) + A\Delta t + \sqrt{B\Delta t}\,\xi \qquad \langle \xi \rangle = 0, \ \langle \xi^2 \rangle = 1$$

So, we have a direct correspondence between the two approaches! The drift coefficients are equal,

$$a(x,t) = A(x,t)$$

and the noise amplitude is the square root of the diffusion term

$$b(x,t) = \sqrt{B(x,t)}\,.$$

C. Appendix: Noiseless solution to DHO

DHO equation of motion

$$m\frac{dv}{dt} = -k_s x - k_D v$$

$$\frac{dx}{dt} = v$$

initial conditions

$$x(0) = x_0$$

$$v(0) = v_0$$

Combine to 2nd order ODE

$$m\frac{d^2x}{dt^2} + k_D\frac{dx}{dt} + k_s x = 0 \qquad x(0) = x_0, \ \left.\frac{dx}{dt}\right|_{x=0} = 0$$

Dimensional analysis

vars: x, t | $m \; [=] \; kg$ $\quad x \; [=] \; m$ $\quad t \; [=] \; s$

params: m, k_D, k_s, x_0 | $k_D \; [=] \; \frac{kg}{s}$ $\quad k_s \; [=] \; kg/s^2$

6 vars/params | 3 dimensions

length scale: x_0

mass scale: m

time scales: $\underbrace{\frac{m}{k_D}}_{\text{viscous time}}, \; \overbrace{\left(\frac{m}{k_s}\right)^{1/2}}^{\text{elastic time}}, \; \underbrace{\frac{k_D}{k_s}}_{\text{relaxation time}}$ ← balances F_D & F_s

3 dimensionless groups

$$\hat{x} = x/x_0 \qquad \hat{t} = t k_s / k_D = t/\tau$$

$$\varepsilon = \frac{m k_s}{k_D^2} = \frac{m}{k_D} \cdot \frac{k_s}{k_D}$$

viscous time / relaxation time

Non-Dimensionalize

$$\frac{m \cancel{x_0}}{k_D^2/k_s^2} \frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\cancel{k_D} \cancel{x_0}}{\cancel{k_D}/k_s} \frac{d\hat{x}}{d\hat{t}} + k_s \cancel{x_0} \hat{x} = 0 \qquad \hat{x}(0) = 1, \; \hat{x}'(0) = 0$$

$$\frac{m k_s}{k_D^2} \frac{d^2\hat{x}}{d\hat{t}^2} + k_s \frac{d\hat{x}}{dt} + k_s \hat{x} = 0$$

$$\varepsilon \frac{d^2\hat{x}}{d\hat{t}^2} + \frac{d\hat{x}}{d\hat{t}} + \hat{x} = 0$$

$\varepsilon \ll 1$ when m is small and k_D is big.

Solve

$$\frac{d\hat{x}}{d\hat{t}} + \hat{x} = 0 \qquad \hat{x}(0) = 1$$

$$\hat{x} = \exp(-\hat{t})$$

$$\boxed{x = x_0 \exp(-t/\tau) \quad \tau = k_D/k_s}$$

$\hat{x} = 0$ as $\hat{t} \to \infty$

D. Appendix: Solution to SDE for Noisy DHO

⟨ ⟩ : ensemble average over many instances

Start with overdamped DHO:

$$0 = \sum F_i = F_D + F_S + F_B$$

$$0 = -k_D v - k_S x + \xi$$

$$\frac{dx}{dt} = -\frac{k_S}{k_D} x + \frac{1}{k_D}\xi$$

$$\frac{x_0}{\tau}\frac{d\hat{x}}{d\hat{t}} = -\frac{x_0}{\tau}\hat{x} + \frac{\alpha_B^{1/2}}{k_D\tau^{1/2}}\hat{\xi}$$

$$\frac{d\hat{x}}{d\hat{t}} = -\hat{x} + \underbrace{\frac{(\alpha_B\tau)^{1/2}}{x_0 k_D}}_{\beta}\hat{\xi}$$

← no dependence on x

(Dimensional Analysis)

α_B : strength of Brownian Force

FDT $\alpha_B = 2 k_B T k_D \; (=) \; \frac{kg^2 m^2}{s^3}$

$\xi(t)$: white noise (units: kgm/s^2)

$\langle \xi(t) \rangle = 0$

$\langle \xi(t)\xi(t') \rangle = \alpha_B \overbrace{\delta(t-t')}^{\text{units: } \frac{1}{s}}$

$\hat{\xi} = \xi\underbrace{\left(\frac{\tau}{\alpha_B}\right)^{1/2}}_{\text{units of force}^{-1}}$ $\qquad \beta = \frac{(\alpha_B\tau)^{1/2}}{x_0 k_D}$ new dimless #

$$\beta = \frac{(\alpha_B/\tau)^{1/2}}{k_D}\frac{\tau}{x_0}\cdot x_0$$

$$\frac{kgm/s^2}{kg/s}\cdot\frac{s}{m}$$

Solution: Inhomogeneous Linear ODE

$$\frac{dx}{dt} + ax = h \qquad a = 1,\; h = \beta\xi$$

$$x = \frac{c_1}{p(t)} + \frac{1}{p(t)}\int_0^t p(s)h(s)\,ds, \quad p(t) = \exp\left(\int_0^t a(s)\,ds\right)$$

$$p(t) = \exp\left(\int_0^t 1\,ds\right) = e^t$$

$$\int_0^t p(s)h(s)\,ds = \int_0^t e^s \beta\xi(s)\,ds$$

Final solution:

$$\hat{x} = \exp(-\hat{t}) + \beta\int_0^{\hat{t}} e^{-(\hat{t}-\hat{s})}\hat{\xi}(\hat{s})\,d\hat{s} \qquad \beta = \frac{(\alpha_B\tau)^{1/2}}{x_0 k_D} \quad \tau = k_D/k_S$$

$$\frac{x}{x_0} = \exp\left(-\frac{t}{\tau}\right) + \beta\int_0^t e^{-(t-s)/\tau}\left(\frac{\tau}{\alpha_B}\right)^{1/2}\xi\,\frac{1}{\tau}\,ds \qquad d\hat{s} = d(s/\tau) = \frac{1}{\tau}ds$$

$$\frac{x}{x_0} = e^{-t/\tau} + \frac{(\cancel{\alpha_B}\cancel{\tau})^{1/2}}{x_0 k_D}\frac{\cancel{\tau}^{1/2}}{\cancel{\alpha_B}^{1/2}}\frac{1}{\cancel{\tau}}\int_0^t e^{-(t-s)/\tau}\xi(s)\,ds$$

$$\boxed{x = x_0 \exp(-t/\tau) + \frac{1}{k_0}\int_0^t e^{-(t-s)/\tau} \xi(s)\, ds}$$

← remember the integral has units $\int$ Force · time $= \frac{kg\,m}{s}$

Statistics of this process:

$$\langle x \rangle = \left\langle x_0 e^{-t/\tau} + \frac{1}{k_D}\int_0^t e^{-(t-s)/\tau} \xi(s)\, ds \right\rangle$$

$$= \langle x_0 e^{-t/\tau} \rangle + \frac{1}{k_D}\left\langle \int_0^t e^{-(t-s)/\tau} \xi(s)\, ds \right\rangle$$

$$= x_0 e^{-t/\tau} + \frac{1}{k_D}\int_0^t e^{-(t-s)/\tau} \underbrace{\langle \xi(s) \rangle}_{0}\, ds$$

$$\boxed{\langle x \rangle = x_0 e^{-t/\tau}}$$

$$\mathrm{var}[x(t)] = \langle (x(t) - \langle x \rangle)^2 \rangle$$

$$= \left\langle \left(\frac{1}{k_D}\int_0^t e^{-(t-s)/\tau} \xi(s)\, ds\right)\left(\frac{1}{k_D}\int_0^t e^{-(t-s')/\tau} \xi(s')\, ds'\right) \right\rangle$$

$$= \frac{1}{k_D^2}\int_0^t\int_0^t e^{-(t-s)/\tau} e^{-(t-s')/\tau} \underbrace{\langle \xi(s)\xi(s') \rangle}_{\alpha_\beta\, \delta(s-s')}\, ds'\, ds$$

$$= \frac{\alpha_\beta}{k_D^2}\int_0^t\int_0^t \exp\left(\frac{1}{\tau}[-t+s-t+s']\right) \delta(s-s')\, ds'\, ds$$

$$= \frac{\alpha_\beta}{k_D^2}\int_0^t \exp\left(\frac{1}{\tau}[-2t+2s]\right) ds = \frac{\alpha_\beta}{k_D^2} e^{-2t/\tau}\int_0^t e^{2s/\tau}\, ds$$

$$= \frac{\alpha_\beta}{k_D^2} e^{-2t/\tau}\left(\frac{1}{2} e^{2s/\tau}\right)\Big|_0^t = \frac{\alpha_\beta}{k_D^2} e^{-2t/\tau}\left(\frac{1}{2}e^{2t/\tau} - \frac{1}{2}e^0\right)$$

$$\boxed{\mathrm{var}[x(t)] = \frac{\alpha_\beta}{2k_D^2}\left(1 - e^{-2t/\tau}\right)}$$

Same as ornstein-uhlenbeck!