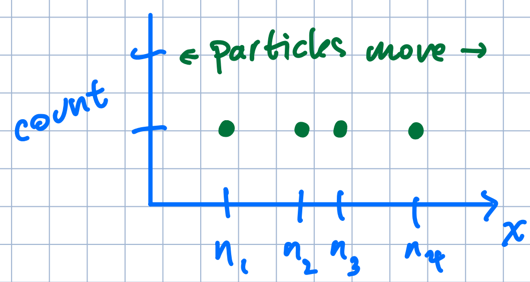
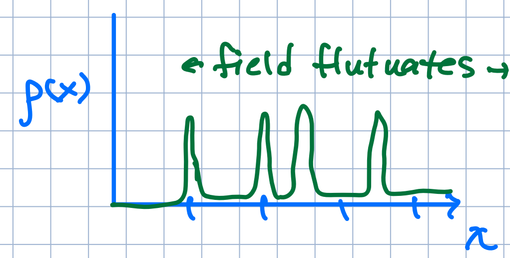
Lecture 7 – Random Fields

We will see in statistical thermo that we have many random variables, and it can make more sense to think of a random field rather than a large random vector.

6×10^{23}!

Example:

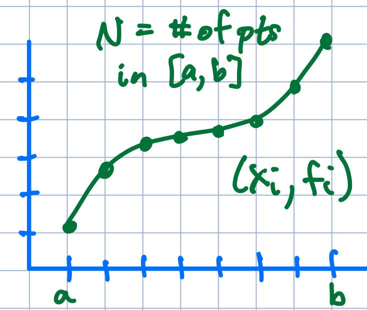




A. Vector – function analogy

An infinite-dimensional vector is a function

$$\lim_{N\to\infty} [f_1, f_2, \dots f_N]^T = f(x)$$



I don't want to go into too many formal details.

Think about programming as an example.

Similarly, an infinite-dimensional matrix is a linear operator

$$\lim_{N\to\infty} \underline{\underline{L}} \cdot \underline{f} = \mathcal{L} f(x) = g(x)$$

example: a differential operator

$$\lim_{N\to\infty} \frac{1}{\Delta x} \begin{bmatrix} 1 & 0 & 0 & \dots & -1 \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} = \frac{df}{dx} \qquad \Delta x = \frac{b-a}{N-1}$$

Functions are vectors in a Hilbert vector space, and derivatives are linear operators on that space. A linear operator maps a function to another one.

Just like we do algebra and calculus for finite dimensional vectors, we can do it for functions too.

(i) inner product

$$\underline{f} \cdot \underline{g} = \sum_i f_i g_i \quad \Rightarrow \quad (f,g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

(ii) functional

A vector function maps a vector to a number: $\underline{x} \mapsto f(\underline{x})$

A functional maps a function to a number: $f(x) \mapsto F[f]$

Example of a functional: a definite integral

$$F[f] = \int_0^1 f(x)\, dx$$

if $f(x) = x$, $F[f] = \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = 1/2$

if $f(x) = x^2$, $F[f] = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = 1/3$

(iii) functional derivative

Just like we do multivariable calculus, we can also do calculus with functionals.

$$dF = \sum_{i=1}^{N} \underbrace{\frac{\partial F}{\partial f_i}}_{\text{partial derivative}} df_i \qquad \lim_{N\to\infty} dF = \int \underbrace{\frac{\delta F}{\delta f}}_{\text{functional derivative}} \delta f$$

$\phi(x)\, dx$

How does $F(\underline{f})$ change as we change each variable f_i?

How does the number $F[f]$ change as we vary $f(x)$?

This might seem abstract, but it is actually about as straightforward as multivariable calculus. We define

$\phi(x)$ is some function we purturb $f(x)$ by.

$$\frac{\delta F}{\delta f} = \lim_{\varepsilon\to 0} \frac{F[f+\varepsilon\phi] - F[f]}{\varepsilon} = \left[\frac{d}{d\varepsilon} F[f+\varepsilon\phi]\right]_{\varepsilon=0}$$

Then we work out a bunch of examples, and we get rules that we use, instead of the definition.

Examples of some functional derivatives:

$$F = \int_a^b f(x)\, dx \qquad \frac{\delta F}{\delta f} = 1$$

$$F = \int_a^b f(x)^2\, dx \qquad \frac{\delta F}{\delta f} = 2f(x)$$

$$F = \int_a^b \left|\frac{df}{dx}\right|^2 dx \qquad \frac{\delta F}{\delta f} = -2\frac{d^2 f}{dx^2}$$

<u>(iv) functional integral</u> (also sometimes called a path integral)

$$\lim_{N\to\infty} \iint\cdots\int F(f_1,f_2,\dots f_N)\, C_N\, df_1\, df_2 \dots df_N = \int Df\ F[f]$$

more on this below

Functional Integral.

Add up all the numbers $F[f]$ over all the different functions $f(x)$ that could contribute.

These are harder. They don't always (usually don't) converge. Hard to find formulas for them. I really only know how to do one.

Example: Gaussian Functional Integral.

$$\int Df \; \exp\left(-\frac{1}{2}\int f^2(x)\, dx\right)$$ Gaussian with zero mean and diagonal covariance matrix equal to 1.

$$= \lim_{N\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} f_i^2 \cdot \Delta x\right) \prod_{i=1}^{N} df_i \qquad \Delta x = \frac{b-a}{N-1}$$

$$\exp\left(\sum_i\right) = \prod_i \exp(\;)$$

$$= \lim_{N\to\infty} \prod_{i=1}^{N} \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{\Delta x}{2} f_i^2\right) df_i}$$

this is a known integral: $\int_{-\infty}^{\infty} \exp\left(-\frac{az^2}{2}\right) dz = \sqrt{\frac{2\pi}{a}}$

$$= \lim_{N\to\infty} \left[\frac{2\pi}{\Delta x}\right]^{N/2} = \lim_{N\to\infty} \left[\frac{2\pi(N-1)}{b-a}\right]^{N/2}$$

This diverges as $N\to\infty$! We don't like this. So, one usually defines:

$$Df = \lim_{N\to\infty} c_N \prod_{i=1}^{N} df_i$$

where c_N is a constant. Because Gaussian integrals are so important, we set c_N so this integral is 1.

$$c_N = \left[\frac{2\pi(N-1)}{b-a}\right]^{-N/2}$$

This c_N is sort of like the Δx in the regular integral. $\sum_i f_i \Delta x$ only converges b/c $\Delta x \to 0$.

B. Distributions

Analogous to the definition for random vectors, we can define a probability density functional, $P[f]$

$$\int Df \; P[f] = 1$$

space of all functions $\nearrow$ $\qquad$ $\nwarrow$ pdf

$P[f]$ is the joint pdf.

Usually, like we've seen many times, $P[f]$ has an exponential

$$P[f] = \frac{1}{Z} e^{-S[f]}$$

S is called the "action" functional

$$Z = \int Df \; e^{-S[f]}$$

Z is the normalization constant. (keeps $\int Df \, P[f] = 1$)

It is often difficult to compute Z, but it usually cancels out, so we don't always need to do it.

Example: Gaussian Action

$$S[f] = \frac{1}{2}\int_{-\infty}^{\infty} f(x)^2 \, dx$$

zero mean, unit variance, uncorrelated.

It is hard to generalize a cdf or a pmf in this context.

Like we saw with random vectors, we can marginalize this probability density functional by integrating out some degrees of freedom.

let $f(x) = g(x) + h(x)$ split into two functions, e.g. long vs. short wavelength modes.

$$P_m[h] = \int Dg \; P[f]$$

joint

marginal

"integrate out" modes in $g(x)$

This has application in coarse-graining and in a process called "renormalization."

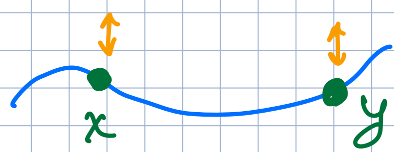
C. Expectation

The expectation operator is defined in a way very analogous to random vectors

$$E[\Theta[f]] = \int Df \; \Theta[f] \; P[f]$$

$$= \frac{1}{Z}\int Df\ \theta[f]\ e^{-S[f]}$$

- $E[f]$ is the mean value of the field. Often $\bar{f}$ or $\langle f \rangle$
- $E[f^2(x)]$ is the fluctuation of the field.
- $E[f(x)f(y)]$ is the correlation between different points in space.



- $E[f(x)g(x)]$ is the correlation between two different fields.

D. The Gaussian probability density functional

$$P[f] = \frac{1}{Z}\exp\left[-\frac{1}{2}\int dx\int dy\ \hat{f}(x)\, G^{-1}(x,y)\, \hat{f}(y)\right]$$

$$\hat{f}(x) = f(x) - E[f(x)] \qquad \hat{f}(y) = f(y) - E[f(y)]$$

$$G(x,y) = E[\hat{f}(x)\hat{f}(y)]$$, covariance function

$G^{-1}(x,y)$: inverse covariance or "propagator."

$$\int dy\ \ G(x,y)\, G^{-1}(y,z)\, dy = \delta(x-z)$$

This is usually a differential operator

Action:

$$S[f] = \frac{1}{2}\int dx\int dy\ \hat{f}(x)\, G^{-1}(x,y)\, \hat{f}(y)$$

two inner products

Normalization:

$$Z = \int Df\ \exp\left[-\frac{1}{2}\int dx\int dy\ \hat{f}(x)\, G^{-1}(x,y)\, \hat{f}(y)\right]$$

Example: $G(x,y) = \delta(x-y)$ "white noise field"

Each point is independent

In statistical mechanics $G(x,y)$ is related to the structure factor, which can be measured by scattering experiments.

E. Characteristic Functional

The characteristic functional is one of the most useful objects when working with random fields. This is because we cannot usually compute functional integrals, but we can calculate functional derivatives to get moments.

The characteristic functional is defined as

$$\phi[J] = E\left[e^{i\int dx\, J(x) f(x)} \right] \leftarrow \text{inner product.}$$

$$= \int Df \; e^{i\int dx\, J(x) f(x)} \; P[f]$$

$J(x)$ is the new function, analogous to the Fourier variable in finite dimensions.

Example: Characteristic Functional of a Gaussian field

$$\phi[J] = E\left[e^{i\int dx\, J(x) f(x)} \right] \qquad \text{As "simple" as it gets!}$$

$$= \exp\left[-\frac{1}{2} \int dx \int dy \; J(x)\, G(x,y)\, J(y) \right]$$

Functional derivatives of $\phi[J]$ give moments of the pdf.

$$E\left[f(x_1) f(x_2) \cdots f(x_n) \right] = (-i)^n \left. \frac{\delta^n \phi[J]}{\delta J(x_1)\, \delta J(x_2) \cdots \delta J(x_n)} \right|_{J=0}$$

Example:

$$G(x,y) = - \left. \frac{\delta^2 \phi[J]}{\delta J(x_1)\, \delta J(x_2)} \right|_{J=0}$$

F. Final comments

- One can define a conditional probability density

$$P[f|g] = \frac{P[f,g]}{P[g]}$$

- We have been cavalier about boundary conditions for the fields. The boundary conditions can be. ← e.g. changes bounds of integrals.
  1. $f(x) \to 0$ as $x \to \pm\infty$ (fast enough)
  2. Bounded to a finite domain: Dirichlet, Neumann or Robin condition on the boundary.
  3. Periodic boundaries
  4. Asymptotic matching (matches non-zero at $\pm\infty$)

G. Things to maybe add

- Relationship between moments and cumulants

  §2.7 in Gardiner, pp. 34–36.

- Wick's theorem:

  Eq. 2.85 in Gardiner, p. 38 (?)

  §1.2, pp. 2–6 in Zinn-Justin, "Path Integrals in Quantum Mechanics"

- Relationship between cumulants and correlation functions

  §2.7 in Gardiner, pp. 34–36.

- Feynman diagrams