Lecture 8 - Stochastic Processes

A. Random Processes

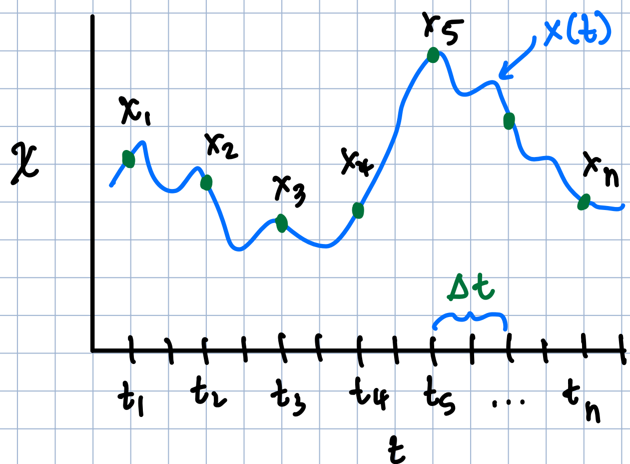
A random process is a family of an infinite number of random variables indexed by time.

$\underline{X} = \{X(t_n), n=1,2,\dots\}$ discrete e.g., Random walk

$\{X(t), t \geq 0\}$ continuous e.g., Brownian Motion aka Wiener process

Aside: X could also be discrete or continuous.

[See slides]



Discrete → continuous as $n \to \infty$ & $\Delta t \to 0$

B. Distributions

Just like all the other cases we have seen, we describe stochastic processes with distributions. We will focus on PDFs and continuous time in this lecture. However, one can use PMFs, CDFs, and discrete time when appropriate.

For a stochastic process with R.V. x_n, we have a PDF that is a function of time.

Discrete-time joint PDF

$$P(\underline{x} \in V; \underline{t}) = \int_V f(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)\, d\underline{x}$$

This is the same as the random vector case:

$$X_1 = X(t_1),\ X_2 = X(t_2),\ \dots,\ X_n = X(t_n)$$

The joint PDF tells us how all of the RVs at different times are distributed and interrelated.

One way to do continuous processes involves functionals,

$$P(\chi \in V) = \int_V Dx\ f[x(t)] \quad \leftarrow \text{continuous-time joint PDF}$$

$x(t)$ is a continuous function

We usually want to avoid this math, so we work with the discrete PDF and take the limit $\Delta t \to 0$ later on (e.g., the moments).

Recall, for random vectors, we can get marginal PDFs (e.g. for one value of $x(t_n)$)

$$f(x_n; t_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_{n-1}; t_1, t_2, \dots, t_{n-1})\ dx_1 dx_2 \dots dx_{n-1}$$

Finally, we can also define a conditional probability,

$$f(x_n; t_n \mid x_1, \dots, x_{n-1}; t_1, \dots, t_{n-1}) = \frac{f(x_1, \dots, x_n; t_1, \dots, t_n)}{f(x_1, \dots x_{n-1}; t_1, \dots, t_{n-1})}$$

joint/marginal

C. Moments

Recall for random vectors the mean is given by

$$\underline{m} = E[\underline{x}] \qquad \underline{x} = \{X_n,\ n = 0, 1, \dots 2\}$$

In the limit that $\Delta t \to 0$ the mean function is given by

$$m(t) = E[X_t] \qquad \text{Now a function of time!}$$

Similarly for second moments, the correlation matrix and cross correlation matrix are given by

$$\underline{\underline{R}} = E[\underline{X}\,\underline{X}] \quad \text{correlation matrix}$$

$$\underline{\underline{R}}_{xy} = E[\underline{X}\,\underline{Y}] \quad \text{cross-correlation matrix}$$

In the continuous limit these become kernels

like a matrix where index at t_1, t_2.

$$\lim_{\Delta t \to 0} \underline{\underline{R}} = R(t_1, t_2) = E[X(t_1)\,X(t_2)] \quad \text{correlation kernel}$$

R is symmetric: $R(t_1, t_2) = R(t_2, t_1)$

$$\lim_{\Delta t \to 0} \underline{\underline{R}}_{xy} = R_{xy}(t_1, t_2) = E[X(t_1)\,Y(t_2)] \quad \text{cross-correlation kernel}$$

Finally, the characteristic equation, MGF and CGF are given by

$$\phi(\underline{\omega}) = E[e^{i\underline{\omega}\cdot\underline{x}}] \quad \text{characteristic equation}$$

$$M(\underline{s}) = E[e^{\underline{s}\cdot\underline{x}}] \quad \text{moment generating function (MGF)}$$

$$K(\underline{s}) = \ln E[e^{\underline{s}\cdot\underline{x}}] \quad \text{cumulant generating function (CGF)}$$

D. Gaussian Random Processes

Super high-dimensional object

The full joint PDF is really hard to deal with! There are two ways we can try to simplify the joint PDF to something more manageable. The first involves a cumulant (or moment) expansion.

$$K(\underline{s}) = \underline{k}_1 \cdot \underline{s} + \frac{1}{2}\,\underline{s}\cdot\underline{\underline{k}}_2\cdot\underline{s} + \frac{1}{6}\,\underline{s}\cdot(\underline{\underline{\underline{k}}}_3\cdot\underline{s})\cdot\underline{s}$$
$$+ \tfrac{1}{24}\,\underline{s}\cdot(\underline{s}\cdot\underline{\underline{\underline{k}}}_4\cdot\underline{s})\cdot\underline{s} + \dots$$

In the continuous limit this becomes,

$$K[J] = \int k_1(t)\,J(t)\,dt + \frac{1}{2}\iint J(t_1)\,k_2(t_1,t_2)\,J(t_2)\,dt_1\,dt_2 +$$
$$\frac{1}{6}\iiint J(t_1)\,k_3(t_1,t_2,t_3)\,J(t_2)\,J(t_3)\,dt_1\,dt_2\,dt_3 + \dots$$

Truncating at second order gives a Gaussian process (GP).

$$K_1 = m(t) \quad \text{(or } \underline{m} \text{ if discrete)}$$
$$K_2 = C(t,s) \quad \text{(or } \underline{\underline{C}} \text{ if discrete)}$$

A GP has a PDF of the form of a multivariate normal distribution.

$$f(\underline{x},\underline{t}) = \frac{1}{\left[(2\pi)^n \det C\right]^{1/2}} \exp\left[-\frac{1}{2}\,\underline{\hat{x}}\cdot\underline{\underline{C}}^{-1}\cdot\underline{\hat{x}}\right]$$

$$\underline{\hat{x}} = \underline{x} - m(t) \qquad m(t) = E[X_t]$$

$$\underline{\underline{C}}(t,s) = \mathrm{Cov}(X_t, X_s) = E\left[(X_t - m(t))(X_s - m(s))\right] \quad \text{covariance kernel}$$

So, for a GP, everything is determined by $m(t)$ and $C(t,s)$, i.e., the mean function and covariance kernel.

Let's look at a few specific examples:

(i) White Noise:

$$m(t) = 0 \;, \quad C(t,s) = \sigma^2 \delta(t-s)$$

diagonal, renders all X_n independent and identically distributed (IID).

$$f(\underline{x},\underline{t}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

We only need the one-point marginal PDF to describe the process

$$f(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) = N(0,\sigma^2)$$

can get marginal PDFs by integrating out the other x_j's for $j \neq i$.

(ii) Wiener Process (i.e., Brownian motion)

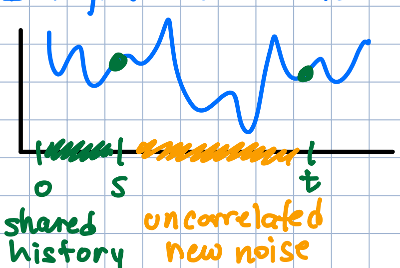
$$m(t) = 0 \;, \quad C(t,s) = \min(t,s)$$

for $0 < s < t$

$C(t,t) = t$ variance grows w/time

$C(t,s) = s$ covariance is shared history

Integral of white noise



In this case, the joint PDF doesn't diagonalize. To fully understand the process we need correlations. But we can learn a lot from the single-time marginal distribution: (It will actually be very useful)

$$f(x,t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_i, \dots, x_n; t_1, \dots, t_i, \dots, t_n)\, dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_n$$

$$f(x,t) = \frac{1}{\sqrt{2\pi\, C(t,t)}} \exp\left[-\frac{(x-m(t))^2}{2C(t,t)}\right]$$

For the wiener process: $m(t)=0$ & $C(t,t)=t$

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left[\frac{-x^2}{2t}\right] = \mathcal{N}(0,t) \quad \text{(diffusive!)}$$

no change in mean   variance grows in time

(iii) Ornstein-Uhlenbeck Process

The OU process is an exponential relaxation from a sudden change to x_0 back to zero,

$$m(t) = x_0\, e^{-\theta t} \qquad C(t,s) = \frac{\sigma^2}{2\theta}\left(e^{-\theta|t-s|} - e^{-\theta|t+s|}\right).$$

σ^2: variance of the noise θ: relaxation rate

Using the formula above, the single-time marginal PDF is given by

$$C(t,t) = \frac{\sigma^2}{2\theta}\left(e^{0} - e^{-2\theta t}\right) = \frac{\sigma^2}{2\theta}\left(1-e^{-2\theta t}\right)$$

$$f(x,t) = \left(\frac{\theta}{\pi\sigma^2} \cdot \frac{1}{1-e^{-2\theta t}}\right)^{1/2} \exp\left[-\frac{\theta}{\sigma^2}\, \frac{(x-x_0 e^{-\theta t})^2}{1-e^{-2\theta t}}\right]$$

$$\lim_{t\to\infty} f(x,t) = \sqrt{\frac{\theta}{\pi\sigma^2}} \exp\left(-\frac{\theta x^2}{\sigma^2}\right)$$