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NON-NEWTONIAN BEHAVIOR OF A ROTATING-CYLINDER INDUCED FLOW

B.C. Liechty
Mechanical Engineering Department
Brigham Young University
Provo, Utah 84602

ABSTRACT

As a cylinder spins in an infinite media, viscous shear forces induce fluid motion and circular streamlines are generated. For Newtonian fluids, a simple analytical solution is possible and widely known. The hydrodynamics of non-Newtonian fluids, however, can be quite complex under such conditions. This paper investigates the behavior of a wide range of non-Newtonian fluids in the infinite cylinder problem. The effective viscosity is determined from Prazyna's visco-plasticity model and the governing equation of momentum is solved numerically using an iterative control volume approach. It is found that velocity profiles for complex fluids can differ significantly from a constant viscosity analysis.

NOMENCLATURE

k	material property [Pa s^m]
m	material property
V_θ	Fluid velocity [m/s]
r	radial coordinate [m]
r_o	cylinder radius [m]

Greek symbols

$\bar{\varepsilon}$	effective strain-rate [1/s]
$\dot{\varepsilon}_{r\theta}$	strain-rate [1/s]
η	effective viscosity [Pa s]
τ	shear stress [Pa]
ω	cylinder rotational speed [rad/s]

INTRODUCTION

The problem of an infinite cylinder spinning in a large fluid body has been extensively studied and can be found in any fluid mechanics textbook [1]. If the cylinder is spun about its axis then viscous forces will cause the surrounding fluid to move. In the simplest case, the fluid behaves as a Newtonian fluid and streamlines are purely circular, that is, $V_\theta = f(r)$ and $V_r = 0$. Note that this condition satisfies continuity identically. For such a case, an analytical solution can be obtained as $V_\theta = r_o^2\omega / r$ and the solution is independent of viscosity.

This problem has arisen in many engineering applications. In friction stir welding (FSW), for example, the welding tool consists of a blunt shoulder and a cylindrical pin. During welding, the pin is completely immersed in the weld material and an infinite cylinder/fluid model can be used as a simplified approximation to determine the effected area of the weld.

This problem, however, becomes much more complex if the fluid is non-Newtonian. In a non-Newtonian fluid, the shear stress is not proportional to the fluid strain-rate. Figure 1 illustrates several rheological behaviors. Unlike a Newtonian fluid, some fluids may be shear-thinning or shear-thickening. Bingham plastics can actually withstand a finite shear force without deformation.

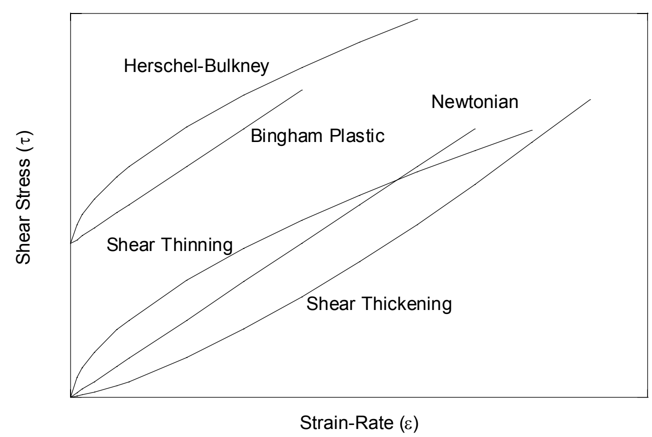

Figure 1: Rheology of various materials.

Information currently published that specifically addresses a cylinder spinning in a non-Newtonian media is limited. Craster [2] has investigated a simplified model of a longitudinal shear flow of a Herschel-Bulkley material using Legendre and hypergeometric integral transformations. Nakarmura *et al.*, have developed a method to alter the viscosity of a fluid in order to control a rotating device [3]. This paper analyzes a rotating cylinder in an infinite non-Newtonian fluid. The rheological model includes shear-thinning and shear-thickening fluids as well as simple and non-simple Bingham plastics.

ANALYSIS

Consider an infinite cylinder surrounded by an infinite non-Newtonian media. If the cylinder is spun about its axis, then viscous forces will cause the surrounding fluid to move (except, perhaps, for a Bingham plastic). The governing equation for momentum is,

$$\frac{d}{dr}\left(\eta r\frac{dV_\theta}{dr}\right)-\frac{\eta V_\theta}{r}=0 \tag{1}$$

where η (the effective viscosity) is a function of strain-rate. Since viscosity is dependent on strain-rate (and thus velocity) this equation is non-linear.

In general, the shear stress of the fluid is related to the rate of strain as $\tau = k\bar{\varepsilon}^m$ where k and m are material dependent constants and $\bar{\varepsilon}$ is the effective strain-rate. Note that for a Newtonian fluid m = 1 and k = μ (viscosity). It can be shown that the effective strain-rate, describing strain due solely from plastic motion (neglecting any elastic deformation) is,

$$\bar{\varepsilon}=\sqrt{2/3}\left|\dot{\varepsilon}_{r\theta}\right| \tag{2}$$

where the strain-rate can be determined from,

$$\dot{\varepsilon}_{r\theta}=\frac{1}{2}\left(\frac{dV_\theta}{dr}-\frac{V_\theta}{r}\right) \tag{3}$$

Finally, the effective viscosity of the fluid, η, can be determined from Prazyna's visco-plasticity model [4] as,

$$\eta = \tau/3\bar{\varepsilon} \tag{4}$$

Since the governing equation is second order, two boundary conditions must be imposed. The first bound condition, at the pin/fluid interface, can be quite complex. For many fluids (especially Bingham plastics), this interface may experience a slip or even sticking and slipping condition. Here only the simplest case of no slip is imposed:

$$r = r_o: \qquad V_\theta = r_o\omega \tag{5}$$

As the radial distance from the cylinder becomes large, the fluid velocity and the velocity gradient must both go to zero. Only one of these conditions may be applied, the latter is chosen here (the size of the domain is chosen such that the fluid velocity reaches zero before the end of the domain is reached),

$$r \to \infty: \qquad \frac{dV_\theta}{dr}=0 \tag{6}$$

Because of the non-linear nature of Eq. (1), it is solved numerically using an iterative control volume approach [5]. The radial domain is discretized and integration of the governing equation is preformed over an arbitrary cell (see Appendix A). Boundary conditions are applied by integration of Eq. (1) over the boundary cells and applying the imposed conditions. This discretization procedure yields a system of coupled linear algebraic equations for the velocity distribution at all node points. Iterations were preformed until the absolute maximum residual in V_θ (local imbalance in conservation of V_θ at any cell) was less than 10^{-5}. The solution requires specification of material properties k and m, the speed of the rotating cylinder, and the yield stress of the material (for Bingham plastics).

RESULTS

Figure 2 shows the variation of the effective viscosity in the radial domain. At m = 0.1 (highly shear-thinning), the viscosity is low at the surface of the cylinder where velocity gradients are large. Just a small distance away from the cylinder, however, the viscosity has increased several orders of magnitude. Since η increases so rapidly, once it has increased by 2-3 orders of magnitude, the fluid must come to rest because their is not sufficient momentum to move such a "stiff" fluid. In reality, the viscosity would reach a limiting value at low strain-rates, but this limit would not effect the velocity profile as long as it is a few orders of magnitude larger than that predicted at $r = r_o$. As m increases, the variation of η becomes much less dramatic. Once m becomes greater than one, the material is said to be shear-thickening, and high strain-rates near the rotating cylinder result in larger values of η compared to the far-field fluid.

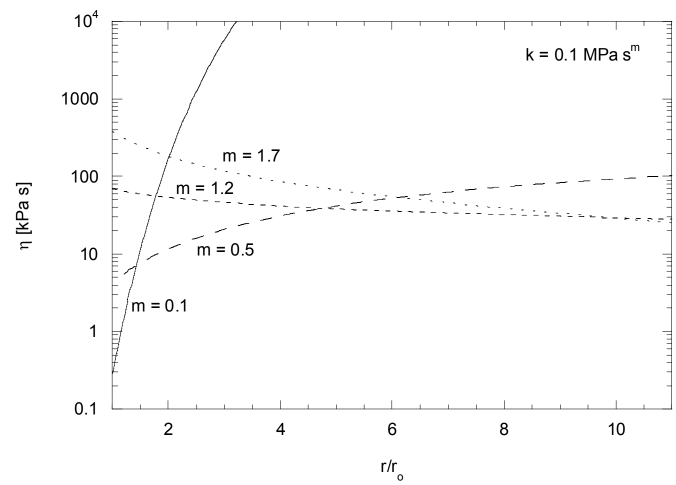

Figure 2: Radial viscosity distribution for shear-thinning (m < 1) and shear-thickening (m > 1) fluids.

Changes in viscosity throughout the fluid significantly effect fluid motion. An investigation of velocity profiles, both for shear-thinning and shear- thickening fluids are presented in Figure 3 for a constant value of k. Note that for m = 1, the

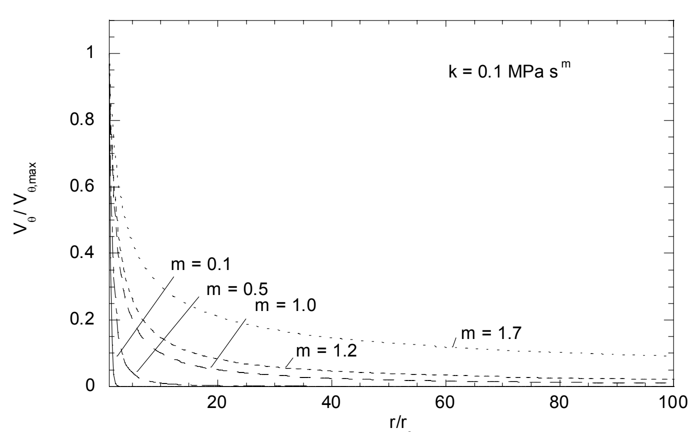

Figure 3: Radial velocity profiles for shear-thinning (m < 1) and shear-thickening (m > 1) fluids.

material behaves as a Newtonian fluid, where by definition, the shear stress is directly proportional to the strain-rate of the fluid. Although a simple analytical solution for this case is readily obtainable (V_θ = constant / r), it is shown in Figure 3 for comparison and completeness.

In a highly shear-thinning fluid (m = 0.1) very little of the fluid is affected by the spinning cylinder. This can be explained by the viscosity distribution throughout the media. Since η increases so rapidly, fluid a short distance away from the cylinder essentially behaves as a solid, where momentum in the thin region near the cylinder cannot move such a highly viscous material.

At the other extreme, in a highly shear-thickening fluid (m = 1.7), η is large near the cylinder and decreases as the fluid motion decreases. Since the momentum of the spinning shaft is imparted to such a relatively high viscous fluid (compared to the far-field fluid) momentum transfer is diffused more readily through the material. This results in a significant area of influence in the fluid, where for m = 1.7 the effected area is greater than 100 times the shaft radius.

Figure 4 shows the influence of k on the velocity profile (m being constant = 0.2). It is interesting that the velocity profile becomes independent of k at both extreme values of k. For m = 0.2, for example, values of k less than approximately 1 Pa s^m produce an identical velocity distribution and likewise for k greater than approximately 1 kPa s^m. As $m \rightarrow 1$, the range of the influence of k becomes even less, where at m = 1, k must have no influence as explained previously (constant viscosity).

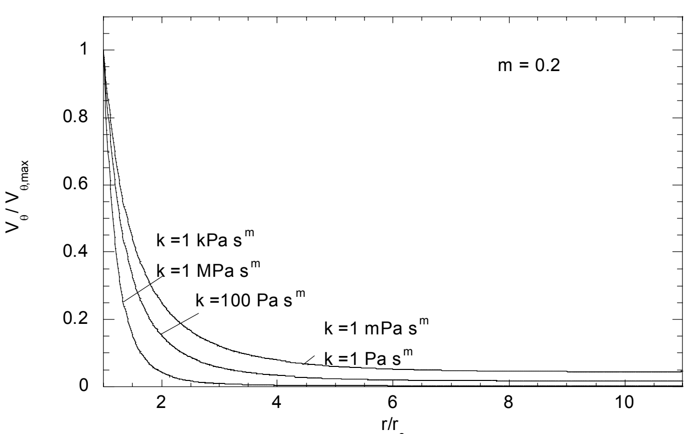

Figure 4: Radial velocity profiles of shear-thinning (m < 1) fluids at several stiffness, k, values.

Another type of fluid behavior that is commonly encountered is that of Bingham plastics. The major difference in these types of materials is that they can withstand a finite shear stress without deformation. In reality, a rotating cylinder may not induce motion in such a material unless the friction generated is larger than the yield stress. Here, for simplification, the no-slip condition is still imposed. The yield condition is imposed in the numerical model by comparing the computed shear stress with the yield stress of the material. If the shear stress computed at a particular node is not sufficient to cause yielding to occur, the effective viscosity is set several orders of magnitude larger than computed. This ensures that the material at that node is stationary.

Figure 5 illustrates velocity shapes for simple and non-simple Bingham plastics. The effect of the yield condition is apparent. Once the velocity gradient of the plastic becomes sufficiently small, such that the yield condition is not met, the material suddenly stops. The result is that penetration is retarded.

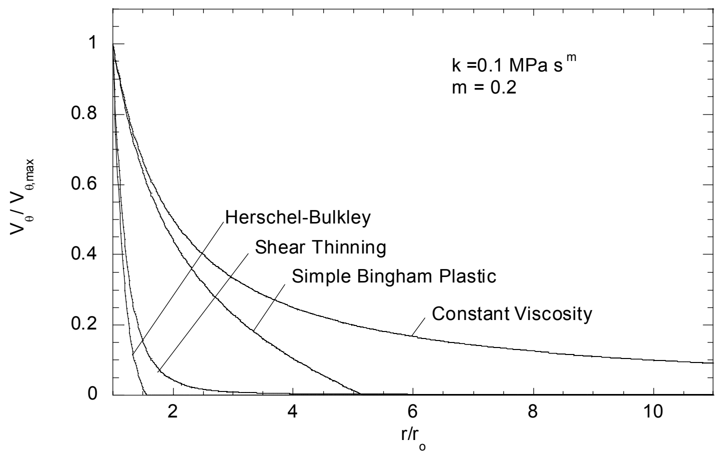

Figure 5: Velocity profiles for various rheological behaviors.

CONCLUSION

Fully-developed flow induced by a rotating cylinder has been investigated for several non-Newtonian fluids. The governing differential equation for fluid momentum has been solved using an iterative numerical approach. Results show that the velocity profile and area of influence for non-Newtonian fluids differ from a simple Newtonian fluid. A shear-thinning fluid results in reduced velocities and a smaller effected area, while the opposite is true for a shear-thickening material. The penetration distance for Bingham Plastics is reduced due to the ability of the plastic to resist a shear load.

REFERENCES

[1] F. M. White, Fluid Mechanics, New York, New York, 1999.
[2] R. V. Craster, Solutions for Herschel-Bulkley flows, J. Mech. Appl. Math 48 (1995) 343-353.
[3] T. Nakarmura, Viscous control of a rotating device using smart fluid, J. Control, Automation, Robotics and Vision 1 (2002) 447-452.
[4] O. C. Zienkiewicz, Visco-plasticity - plasticity and creep in elastic solids - a unified numerical solution approach, Int. J. Numer. Methods in Eng. 8 (1974) 821 – 845.
[5] S.V. Patanker, Numerical Heat Transfer and Fluid Flow, Hemisphere, New York, 1980.

APPENDIX A

Derivation of Finite Difference Equations for Numerical Solution

Consider viscous flow of a non-Newtonian fluid created by rotating an infinite cylinder in an infinite media. The governing differential equation can be expressed as,

$$\frac{d}{dr}\left(\eta r\frac{dV_\theta}{dr}\right)-\frac{\eta V_\theta}{r}=0 \tag{A1}$$

Assuming that the velocity profile is linear between nodes and treating the second term on the left-hand side of Eq. (A1) as a source term that is constant over the cell, integration over an arbitrary cell (P) gives,

$$\int_w^e\frac{d}{dr}\left(\eta r\frac{dV_\theta}{dr}\right)dr-\int_w^e\frac{\eta V_\theta}{r}dr=0 \tag{A2}$$

$$\eta_e r_e\frac{V_{\theta,E}-V_{\theta,P}}{\delta r}-\eta_w r_w\frac{V_{\theta,P}-V_{\theta,W}}{\delta r}-\eta_P V_{\theta,P}\ln\frac{r_e}{r_w}=0 \tag{A3}$$

where δr is the distance between nodes (also the cell width for a uniform grid), r_e and r_W are the locations of the cells right and left control surfaces respectively. (The definitions of all subscripts used here are shown graphically in Figure A1 unless specifically stated otherwise.)

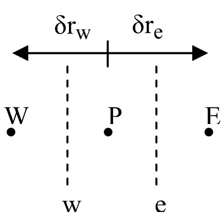

Figure A1: Notation for an arbitrary discretized cell.

Adopting a standard form for the finite difference equations as $a_P V_P = a_E V_E + a_W V_W + b$, we now recognize from Eq. (A3) that

$$a_E=\frac{\eta_e r_e}{\delta r_e}\,,\qquad a_W=\frac{\eta_w r_w}{\delta r_w}\,, \tag{A4a, b}$$

$$a_P=a_E+a_W+\eta_P\ln\frac{r_e}{r_w}\,,\qquad b=0 \tag{A4c, d}$$

Since the boundary condition at the cylinder/fluid interface is of Dirichlet type, we need only set $a_E = a_W = 0$, $a_P = 1$, and $b = r_o\omega$ (where r_o and ω are the radius and angular speed respectively, of the cylinder) and no integration is required. Integration over the boundary cell at $r \rightarrow \infty$ (denoted as node "B") yields,

$$\left.\eta r\frac{dV_\theta}{dr}\right|_w^B-\eta_B V_{\theta,B}\ln\frac{r_B}{r_w}=0 \tag{A5}$$

By applying a Neumann type boundary condition ($dV_\theta/dr = 0$) at node "B" we obtain,

$$\eta_w r_w\frac{V_{\theta,B}-V_{\theta,W}}{\delta r}+\eta_B V_{\theta,B}\ln\frac{r_B}{r_w}=0 \tag{A6}$$

The coefficients for the finite difference equation are now recognized as,

$$a_E=0\,,\qquad a_W=\frac{\eta_w r_w}{\delta r_w}\,, \tag{A7a,b}$$

$$a_P=a_E+a_W+\eta_B\ln\frac{r_B}{r_w}\,,\qquad b=0 \tag{A7c,d}$$

An important note is that the effective viscosity is evaluated at the control surfaces. Since velocities are only determined at node points, η is only determined at node locations. In order to calculate η at control surfaces, the harmonic mean is used. For example, the viscosity at the right control surface of a cell would be,

$$\eta_e=\frac{2\eta_E\eta_W}{\eta_E+\eta_W} \tag{A8}$$

