Eng T 502 · Advanced Applied Engineering Mathematics

# Homework Set 6 · Hints

## Topic: Determinants, Eigenvalues, Eigenvectors

17 Sep 2003

3. Show with induction on $n$ that the *Vandermonde-Determinant* can be computed using the identity:

$$\det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le i<j\le n}(x_j - x_i).$$

Remember, to verify this assertion using induction on $n$ you need to show that it is true for $n = 1$ and under the assumption that is true for fixed $n$ also for $n+1$. The crucial part is to relate the case $n+1$ with $n$. The following hint shows how this can be done in this instance:

(a) The case $n = 4$:

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix} \underset{(D7)}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_2x_1 & x_2^3 - x_2^2x_1 \\ 1 & x_3 - x_1 & x_3^2 - x_3x_1 & x_3^3 - x_3^2x_1 \\ 1 & x_4 - x_1 & x_4^2 - x_4x_1 & x_4^3 - x_4^2x_1 \end{vmatrix}$$

$$\underset{(D9,12)}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) \\ 1 & x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) \\ 1 & x_4 - x_1 & x_4(x_4 - x_1) & x_4^2(x_4 - x_1) \end{vmatrix}$$

$$= \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) & x_2^2(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & x_3^2(x_3 - x_1) \\ x_4 - x_1 & x_4(x_4 - x_1) & x_4^2(x_4 - x_1) \end{vmatrix}$$

$$\underset{(D1b)}{=} (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \cdot \begin{vmatrix} 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{vmatrix}$$

(b) The general case:

$$V_{n+1} := \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix}$$

multiplying each column by $x_1$ and subtracting it from its right neigbor yields

$$\begin{aligned}
&\underset{(D7)}{=} \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_2 x_1 & \cdots & x_2^n - x_2^{n-1} x_1 \\ 1 & x_3 - x_1 & x_3^2 - x_3 x_1 & \cdots & x_3^n - x_3^{n-1} x_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+1} - x_1 & x_{n+1}^2 - x_{n+1} x_1 & \cdots & x_{n+1}^n - x_{n+1}^{n-1} x_1 \end{pmatrix} \\
&\underset{(D9,12)}{=} \det \begin{pmatrix} x_2 - x_1 & x_2^2 - x_2 x_1 & \cdots & x_2^n - x_2^{n-1} x_1 \\ x_3 - x_1 & x_3^2 - x_3 x_1 & \cdots & x_3^n - x_3^{n-1} x_1 \\ \vdots & \vdots & & \vdots \\ x_{n+1} - x_1 & x_{n+1}^2 - x_{n+1} x_1 & \cdots & x_{n+1}^n - x_{n+1}^{n-1} x_1 \end{pmatrix} \\
&= \det \begin{pmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{n-1}(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & \cdots & x_3^{n-1}(x_3 - x_1) \\ \vdots & \vdots & & \vdots \\ x_{n+1} - x_1 & x_{n+1}(x_{n+1} - x_1) & \cdots & x_{n+1}^{n-1}(x_{n+1} - x_1) \end{pmatrix} \\
&\underset{(D1b)}{=} \prod_{j=2}^{n+1} (x_j - x_1) \cdot \det \begin{pmatrix} 1 & x_2 & \cdots & x_2^{n-1} \\ 1 & x_3 & \cdots & x_3^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1} & \cdots & x_{n+1}^{n-1} \end{pmatrix} \\
&= \left( \prod_{j=2}^{n+1} (x_j - x_1) \right) \cdot V_n
\end{aligned}$$