Eng T 503 · Advanced Applied Engineering Mathematics II

Homework Complex Numbers

Selected Solutions

TOPIC: ANALYTIC FUNCTIONS

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Exercise 17.3:23

If $f = u + iv$ is analytic on the region $R \subseteq \mathbb{C}$ and $u(x, y)^2 + v(x, y)^2 = C$ for some constant $C \in \mathbb{R}$ and for all $(x, y) \in \mathbb{R}^2$, show that f is constant on the region R.

PROOF: If $C = 0$, then $u(x, y)^2 = -v(x, y)^2$ for all $(x, y) \in R$, hence $u(x, y) = v(x, y) = 0$ for all $(x, y) \in R$, which implies that $f = u + iv = 0$ on R.

We can therefore assume that $C \neq 0$. We will show that $f' = u_x + iv_x$ is zero on R, which implies that f is constant. To this end, we differentiate the equation

$$u^2 + v^2 = C$$

with respect to x and y and obtain the system of linear equations

$$uu_x + vv_x = 0 \tag{1}$$
$$uu_y + vv_y = 0. \tag{2}$$

Suppose, there exists $z_0 = x_0 + iy_0 \in R$ such that $f'(z_0) \neq 0$. Then, since f is analytic on R, the coefficient matrix of the system of equations (1,2), which coincides with the Jacobian of f, is different from zero at (x_0, y_0). In fact,

$$\begin{aligned} \det(J_f)(x_0, y_0) &= \begin{vmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix} \\ &= u_x(x_0, y_0)v_y(x_0, y_0) - u_y(x_0, y_0)v_x(x_0, y_0) \end{aligned}$$

which, using the Cauchy Riemann Differential Equation ($u_x = v_y$ and $u_y = -v_x$), equals

$$= u_x(x_0, y_0)^2 + v_x(x_0, y_0)^2$$

which is different from zero, since, by assumption,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \neq 0.$$

This implies that the system of linear equations (1,2) has only the trivial solution $u(x_0, y_0) = v(x_0, y_0) = 0$ at the point (x_0, y_0). In particular,

$$u(x_0, y_0)^2 + v(x_0, y_0)^2 = 0.$$

However, by assumption, $u^2(x, y) + v^2(x, y) = C$ for all $(x, y) \in R$ and $C \neq 0$. This is a contradiction, hence, the assumption that there exists $z_0 \in R$ with $f'(z_0) \neq 0$ must be wrong. Therefore, $f'(z) = 0$ for all $z \in R$, which implies f is constant on R. $\square$