9. Hermite's differential equation with parameter λ is

$$y'' - 2xy' + \lambda y = 0 \qquad x \in (-\infty, \infty),\ \lambda \in \mathbb{R} \qquad \text{(HE)}$$

a) Solve the HE by the power series method

b) Consider two linearly independent solutions
$y_1(x) = ...$
$y_2(x) = ...$
which include parameter λ

c) If λ is a non-negative even integer, $\lambda = 0, 2, 4, ..., 2n, ...$,
then the series terminates, and one obtains, alternating for y_1 and y_2,
polynomials of degree n, which are multiples of so called Hermitian polynomials $H_n(x)$.

d) Rewrite HE in self-adjoint form and determine the weight function $w(x)$

e) Check if the HP are orthogonal with the weight function $w(x)$ over $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w(x)\, dx = 0 \quad \text{if} \quad m \neq n$$

f) Give an example of function representation into Fourier-Hermite series

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Solution:

a) All points are ordinary points.
We will apply a power-series solution method around the ordinary point $x_0 = 0$
(the interval of convergence for this solution is $(-\infty, \infty)$).
Assume that the solution is represented by a power series

$$y = \sum_{k=0}^{\infty} a_k x^k$$

then derivatives of the solution are

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Substitute them into equation

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - 2x \sum_{k=1}^{\infty} k a_k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} (-2) k a_k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

Change of index:
$m = k - 2 \qquad m = k \qquad m = k$
$k = m + 2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=1}^{\infty} (-2) m a_m x^m + \sum_{m=0}^{\infty} \lambda a_m x^m = 0$$

$$\overbrace{2\cdot 1\cdot a_2+\lambda\cdot a_0}^{m=0}+\sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=1}^{\infty}(-2)ma_mx^m+\sum_{m=1}^{\infty}\lambda a_mx^m=0$$

$$2\cdot 1\cdot a_2+\lambda\cdot a_0+\sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2}+(\lambda-2m)a_m\right]x^m=0$$

Use identity Theorem (Theorem 2.6, Chapter 2):

$$a_2=-\frac{\lambda\cdot a_0}{2\cdot 1}$$

$$a_{m+2}=\frac{(2m-\lambda)}{(m+2)(m+1)}a_m \qquad m=1,2,\ldots$$

Write coefficients:

$$a_0=arbitrary$$
$$a_1=arbitrary$$
$$a_2=-\frac{\lambda\cdot a_0}{2\cdot 1}$$
$$a_3=\frac{(2-\lambda)}{3\cdot 2}a_1$$
$$a_4=\frac{(2\cdot 2-\lambda)}{4\cdot 3}a_2=\frac{-(2\cdot 2-\lambda)\lambda}{4\cdot 3\cdot 2\cdot 1}a_0$$
$$a_5=\frac{(2\cdot 3-\lambda)}{5\cdot 4}a_3=\frac{(2\cdot 3-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}a_1$$
$$a_6=\frac{(2\cdot 4-\lambda)}{6\cdot 5}a_4=\frac{-(2\cdot 4-\lambda)(2\cdot 2-\lambda)\lambda}{6!}a_0$$
$$a_7=\frac{(2\cdot 5-\lambda)}{7\cdot 6}a_5=\frac{(2\cdot 5-\lambda)(2\cdot 3-\lambda)(2\cdot 1-\lambda)}{7!}a_1$$
$$\vdots$$

b) Write the power series solution collecting terms with a_0 and a_1:

$$y(x)=a_0\left[1-\frac{\lambda}{2!}x^2-\frac{(2\cdot 2-\lambda)\lambda}{4!}x^4-\frac{(2\cdot 4-\lambda)(2\cdot 2-\lambda)\lambda}{6!}x^6-\ldots\right]+$$
$$+a_1\left[x+\frac{(2\cdot 1-\lambda)}{3!}x^3+\frac{(2\cdot 3-\lambda)(2\cdot 1-\lambda)}{5!}x^5+\frac{(2\cdot 5-\lambda)(2\cdot 3-\lambda)(2\cdot 1-\lambda)}{7!}x^7+\ldots\right]$$

c) Let $\lambda=2n$, where $n=0,1,2,\ldots$. Then the solution becomes

$$y_n(x)=a_0\left[1-\frac{2n}{2!}x^2-\frac{(2\cdot 2-2n)2n}{4!}x^4-\frac{(2\cdot 4-2n)(2\cdot 2-2n)2n}{6!}x^6-\ldots\right]+$$
$$+a_1\left[x+\frac{(2\cdot 1-2n)}{3!}x^3+\frac{(2\cdot 3-2n)(2\cdot 1-2n)}{5!}x^5+\frac{(2\cdot 5-2n)(2\cdot 3-2n)(2\cdot 1-2n)}{7!}x^7+\ldots\right]$$
$$=a_0y_{0,n}(x)+\quad a_1y_{1,n}(x)$$

Write the solutions which correspond to $n=0,1,2,\ldots$. Note that the infinite series terminates alternatively in one of the terms of the solution. Compare with the Hermite polynomials $H_n(x)$:

n	$y_{0,n}(x)$	$y_{1,n}(x)$	$H_n(x)$	*factor*
0	1	$y_{1,0}(x)$	1	1
1	$y_{0,1}(x)$	x	$2x$	2
2	$1-2x^2$	$y_{1,2}(x)$	$4x^2-2$	-2
3	$y_{0,3}(x)$	$x-\frac{2}{3}x^3$	$8x^3-12x$	-12
4	$1-4x^2+\frac{4}{3}x^4$	$y_{1,4}(x)$	$16x^4-48x^2+12$	12
5	$y_{0,5}(x)$	$x-\frac{4}{3}x^3+\frac{4}{15}x^5$	$32x^5-160x^3+120x$	120
6	$1-6x^2+4x^4-\frac{16}{15}x^6$	$y_{1,6}(x)$	$64x^6-480x^4+720x^2-120$	-120
7	$y_{0,7}(x)$	$x-2x^3+\frac{4}{5}x^5-\frac{16}{35}x^7$	$128x^7-1344x^5+3360x^3-1680x$	-1680
$\vdots$				

d) Rewrite the Hermite Equation in self-adjoint form:

Multiplication factor (Section 4.5.4 of Chapter 4, p.271):

$$\mu(x)=\frac{e^{\int\frac{a_1(x)}{a_0(x)}dx}}{a_0(x)}=\frac{e^{\int\frac{-2x}{1}dx}}{1}=e^{-2\int x dx}=e^{-x^2}$$

$$e^{-x^2}y''-2xe^{-x^2}y'+\lambda e^{-x^2}y=0$$

$$\left[e^{-x^2}y'\right]'+\left[0+\lambda e^{-x^2}\right]y=0$$

The weight function is

$$w(x)=e^{-x^2}$$

e) Orthogonality of the Hermite polynomials::

$$\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}dx=2^n n!\sqrt{\pi}\delta_{mn}$$

f) The Fourier-Hermite series:

$$f(x)=\sum_{n=0}^{\infty} c_n H_n(x)$$

$$c_n=\frac{\int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\,dx}{\int_{-\infty}^{\infty} H_n^2(x) e^{-x^2}\,dx}=\frac{\int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\,dx}{2^n n!\sqrt{\pi}}$$

Examples:

1. $f(x)=e^x$