6.9 Solutions to the Exercises

HOMEWORK SET 6.1 (TOPOLOGY)

1. Let $x, y \in \mathbb{R}$. Consider the maps

   (a) $d_1(x, y) := (x - y)^2$

   (b) $d_2(x, y) := |x^2 - y^2|$

   (c) $d_3(x, y) := |x - 3y|$

   (d) $d_4(x, y) := \dfrac{|x - y|}{1 + |x - y|}$

   Determine, which of these maps are metrics on $\mathbb{R}$.

   SOLUTION: (a) d_1 is not a metric. d_1 satisfies (M1) and (M2) but not (M3): Choose for instance $x = -1$, $y = 1$ and $z = 1$. Then

   $$4 = d_1(-1, 1) \not\leq d_1(-1, 0) + d_2(0, 1) = 1 + 1$$

   (b) d_2 is not a metric. d_2 does not satisfy (M1): Choose for instance $x = 1$ and $y = -1$. Then $d_2(1, -1) = 0$, however $1 \neq -1$.

   (c) d_3 is not a metric. d_3 does not satisfy (M1): Choose for instance $x = 3$ and $y = 1$. Then $d_3(3, 1) = 0$, but $3 \neq 1$.

   (d) d_4 is a metric. (M1) and (M2) are clearly satisfied, since $d(x, y) = |x - y|$ is a metric. To verify (M3) consider

   $$\frac{|x - y|}{1 + |x - y|} = \left(\frac{1}{|x - y|} + 1\right)^{-1}$$

   which, since $|x - y| \leq |x - z| + |z - y|$,

   $$\begin{aligned}
   &\leq \left(\frac{1}{|x - z| + |z - y|} + 1\right)^{-1} \\
   &= \left(\frac{1 + |x - z| + |z - y|}{|x - z| + |z - y|}\right)^{-1} \\
   &= \frac{|x - z|}{1 + |x - z| + |z - y|} + \frac{|z - y|}{1 + |x - z| + |z - y|} \\
   &\leq \frac{|x - z|}{1 + |x - z|} + \frac{|z - y|}{1 + |z - y|}
   \end{aligned}$$

   ■

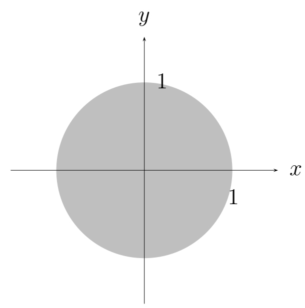
2. Let $x = (0, 0) \in \mathbb{R}^2$ and let $r = 1$. Visualize the neighborhood $N_1(x)$ with respect to

(a) the Euclidean metric on $\mathbb{R}^2$,

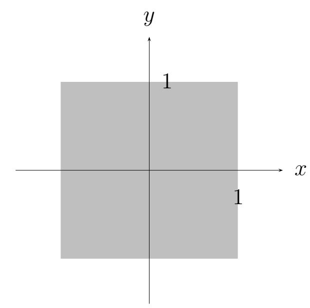
(b) the maximum metric on $\mathbb{R}^2$,

(c) the city block metric on $\mathbb{R}^2$.

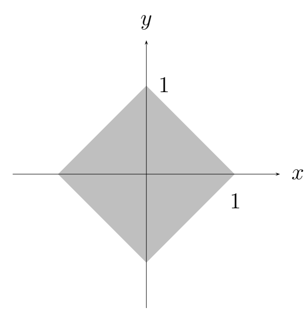
SOLUTION:



(a) The neighborhood $N_1(0)$ relative to the Euclidean metric on $\mathbb{R}^2$.



(b) The neighborhood $N_1(0)$ relative to the maximum metric on $\mathbb{R}^2$.



(c) The neighborhood $N_1(0)$ relative to the city block metric on $\mathbb{R}^2$.

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3. Construct a bounded set of real numbers with exactly three limit points.

SOLUTION: For instance, consider the set

$$E := \{\tfrac{1-n}{n} \mid n \in \mathbb{N}\} \cup \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{\tfrac{n+1}{n} \mid n \in \mathbb{N}\}.$$

■

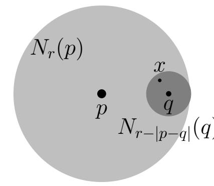
4. Let E' denote the set of all limit points of the set $E \subseteq \mathbb{R}$.

(a) Show that E' is closed.

(b) Show that E and $\overline{E}$ have the same limit points.

(c) Do E and E' always have the same limit points?

SOLUTION: (a) We have to show that every limit point of E' is a limit point of E. Suppose $p \in \mathbb{R}$ is a limit point of E'. By definition, for any $r > 0$ there exists a number $q \in N_r(p) \cap E'$ different from p. As $q \in E'$, q is a limit point of E. Hence, for all—and thus in particular—for $s := r - |p - q| > 0$ there exists a number $x \in N_s(q) \cap E$ different from q. By the choice of s, we have $N_s(q) \subseteq N_r(p)$. Thus $x \in N_r(p) \cap E$ and $x \neq p$. Hence, p is a limit point of E.



The light gray disk visualizes the neighborhood $N_r(p)$. The distance between p and q is $|p - q|$, the distance from q to the boundary of the neighborhood is $r - |p - q|$. The neighborhood $N_{r-|p-q|}(q)$ is visualized by the gray disk. The point x of $N_{r-|p-q|}(q)$ is clearly in $N_r(p)$.

(b) We have to show that $E' = \left(\overline{E}\right)'$ or, $E' \subseteq \left(\overline{E}\right)'$ and $E' \supseteq \left(\overline{E}\right)'$.

We first show $E' \subseteq \left(\overline{E}\right)'$: Let $p \in E'$ and $r > 0$. By definition, there exist a point $q \in N_r(p)$, $q \neq p$ and $q \in E$. Since $E \subseteq \overline{E}$, $q \in \overline{E}$. Hence p is a limit point of $\overline{E}$, i.e. $p \in \left(\overline{E}\right)'$.

We now show $E' \supseteq \left(\overline{E}\right)'$: Let $p \in \left(\overline{E}\right)'$ and let $r > 0$. By definition, there exists a point $q \in N_r(p)$, $q \neq p$ and $q \in \overline{E} = E \cup E'$. If $q \in E'$, we are done. If $q \in E$ then p is a limit point of E, i.e. $p \in E'$.

(c) No! Consider the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $E' = \{0\}$, while $\left(E'\right)' = \emptyset$. ■

5. Let $E \subseteq \mathbb{R}^2$ be open. Is every point of E a limit point of E?

SOLUTION: Yes! Let $x \in E$. Since E is open, there exists $r_0 > 0$ such that $N_{r_0}(x) \subseteq E$. Now let $r > 0$. Without loss of generality, we can assume that $r \leq r_0$, i.e. $N_r(x) \subseteq N_{r_0}(x) \subseteq E$. Hence there exists a point $q \in N_r(x)$, $q \neq x$ and $q \in E$. Hence x is a limit point of E, i.e. $x \in E'$. ■

6. Let $E \subseteq \mathbb{R}^2$ be closed. Is every point of E a limit point of E?

SOLUTION: No! Consider the set $E := \{0\}$. $E' = \emptyset$, thus E is closed. However 0 is not a limit point of E (0 is an isolated point of E). ■

7. Let $E \subseteq \mathbb{R}$. The set E° of all interior points of E is called the *interior* of E.

   (a) Show that E° is always open.

   (b) Show: E is open $\Leftrightarrow E = E^\circ$.

   (c) Do E and $\overline{E}$ always have the same interior?

   (d) Show that the complement of E° is the closure of the compliment of E.

SOLUTION: (a) By definition, we have to show that every point $x \in E^\circ$ is an interior point of E°, i.e. for any $x \in E^\circ$ there exists a $r > 0$ such that $N_r(x) \subseteq E^\circ$. To this end, let $x \in E^\circ$. Since x is an interior point of E, there exists $r > 0$ such that $N_r(x) \subseteq E$. To show that $N_r(x) \subseteq E^\circ$, we have to show that every point of $N_r(x)$ is an interior point of E. Let $y \in N_r(x)$. Let $s := r - |x - y| > 0$. Then, $N_s(y) \subseteq N_r(x) \subseteq E$. Hence, y is an interior point of E, i.e. $N_r(x) \subseteq E^\circ$.

(b) ($\Rightarrow$): Suppose E is open . Since, by definition, $E^\circ \subseteq E$, we only have to show that $E \subseteq E^\circ$. To this end, let $x \in E$. Since E is open, x is an interior point of E, i.e. $x \in E^\circ$.

($\Leftarrow$): Now suppose that $E = E^\circ$. We have to show that E is open. But this is clear, since every point of $E = E^\circ$ is an interior point of E.

(c) No! Consider $E = \mathbb{Q}$. Then $\mathbb{Q}^\circ = \emptyset$, while $\overline{\mathbb{Q}} = \mathbb{R}$ and thus $\left(\overline{Q}\right)^\circ = \mathbb{R}$.

(d) We have to show that $\mathbb{R} \setminus E^\circ = \overline{\mathbb{R} \setminus E}$ or equivalently: $E^\circ = \complement_{\mathbb{R}} \left(\overline{\complement_{\mathbb{R}} E}\right)$. Recall that for $X \subseteq Y$, we use the abbreviation: $\complement_Y X = Y \setminus X$.

$$
\begin{aligned}
x \in E^\circ &\Leftrightarrow \exists r > 0 : N_r(x) \subseteq E \\
&\Leftrightarrow \neg \left(\forall r > 0 : N_r(x) \not\subseteq E\right) \\
&\Leftrightarrow \neg \left(\forall r > 0 : N_r(x) \cap \complement_{\mathbb{R}} E \neq \emptyset\right) \;\Leftrightarrow x \in \overline{\complement_{\mathbb{R}} E}
\end{aligned}
$$

Here "$\neg$" denotes the negation operator, while "$\forall$" and "$\exists$" denote the "for all" and "there exists" quantifiers, respectively. Note $x \notin \complement_{\mathbb{R}} E$ since by assumption, $x \in E^\circ \subseteq E$. ■

8. Let X be an infinite set. For $x, y \in X$, we define

$$
d(x, y) := \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.
$$

Show that d is a metric on X. Which subsets of X are open. Which subsets of X are closed?

SOLUTION: (M1) and (M2) are clear. (M3) can be easily verified by considering the cases $x = y$, $x \neq y$, $z = y$, $z \neq y$. We first consider the r-neighborhoods relative to d. Let $x \in X$ and $r > 0$, then

$$
N_r(x) = \begin{cases} \{x\}, & \text{if } r < 1 \\ X, & \text{if } r \geq 1. \end{cases}
$$

Therefore, subset $E \subseteq X$ is open and closed. ■