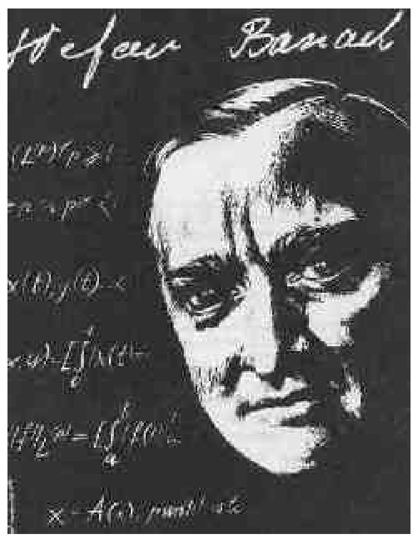
4.5.0 BANACH AND HILBERT SPACES



1. Normed Space

Norm is a map $\|\cdot\| : V \to \mathbb{R}$ such that for all $u, v \in V$ and $c \in \mathbb{R}$

1. $\|u\| \geq 0$
   $\|u\| = 0$ if and only if $u = 0$
2. $\|cu\| = |c|\|u\|$
3. $\|u+v\| \leq \|u\| + \|v\|$ (triangle inequality)

Example: in space $C[a,b]$ of all continuous functions in $[a,b]$ norm can be defined as
$\|f\|_C = \max_{x\in[a,b]} |f(x)|$

2. Metric Space

Vector space V is a **metric** space if there exists a function $\rho : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$

1. $\rho(u,u) = 0$
   $\rho(u,v) > 0$ for $u \neq v$
2. $\rho(u,v) = \rho(v,u)$ (symmetry)
3. for all $\rho(u,v) \leq \rho(u,w) + \rho(w,v)$ (triangle inequality)

$\rho(u,v)$ is called the distance between $u, v \in V$.

Vector space with introduced metric is called a metric space.

In the normed vector space the metric can be introduced as
$$\rho(u,v) = \|u - v\|$$

3. Inner Product

Inner product is a map $(,) : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$

1. $(u,v) = (v,u)$ (*for complex* $(u,v) = \overline{(v,u)}$)
2. $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$ $\alpha, \beta \in \mathbb{R}$
3. $(u,u) \geq 0$
   $(u,u) = 0$ if and only if $u = 0$

Vector space with introduced inner product is called an **inner product space**. In inner product space the norm can be defined as

$\|u\| = \sqrt{(u,u)}$ for all $u \in V$

4. Convergence

Let V be normed (metric space) and let $f_k, f \in V$, $k = 1, 2, \ldots$
The sequence $f_1, f_2, \ldots$ converges to f if
$\rho(f_k, f) \to 0$ as $k \to \infty$
$\lim_{k\to\infty} \|f_f - f\| = 0$

The sequence $f_k \in V$ is called the **Cauchy sequence** (convergent in itself) if
$\rho(f_k, f_m) \to 0$ as $k \to \infty$ and $m \to \infty$

The vector space V is called ***complete*** if all its Cauchy sequences are convergent in V.

A complete normed space is called the **Banach space**. For example, $\mathbb{R}^n$ is a Banach space with $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots x_n^2}$.

A complete inner product space is called a **Hilbert space**.

5. Orthogonality

In the inner product space $u, v \in V$ are called ***orthogonal*** if $(u,v) = 0$.

If set $\{u_k\} \in V$ consists of mutually orthogonal vectors, $(u_k, u_m) = 0$ when $k \neq m$, then this set is called an ***orthogonal set***.

If in addition, $\|u_k\| = 1$, then set $\{u_k\} \in V$ is called ***orthonormal***.

Orthogonal set is linearly independent set.

If set $\{u_k\} \in V$ is linearly independent then it can be converted to the orthonormal set $\{v_k\} \in V$ with the help of the so called Gram-Schmidt orthogonalization process:

Gram-Schmidt process

$$\begin{aligned}
v_1 &= \frac{u_1}{\|u_1\|} \\
v_2 &= \frac{u_2 - (u_2, v_1) v_1}{\|u_2 - (u_2, v_1) v_1\|} \\
&\vdots \\
v_k &= \frac{u_k - (u_k, v_1) v_1 - (u_k, v_2) v_2 - \ldots - (u_k, v_{k-1}) v_{k-1}}{\|u_k - (u_k, v_1) v_1 - (u_k, v_2) v_2 - \ldots - (u_k, v_{k-1}) v_{k-1}\|}
\end{aligned}$$

This algorithm can be formalized with the help of Gram's determinant:

$$G_k = \begin{vmatrix} (u_1,u_1) & (u_2,u_1) & \cdots & (u_k,u_1) \\ (u_1,u_2) & (u_2,u_2) & \cdots & (u_k,u_2) \\ \vdots & \vdots & \ddots & \vdots \\ (u_1,u_k) & (u_2,u_k) & \cdots & (u_k,u_k) \end{vmatrix}, \quad G_0 = 1$$

Orthonormal vectors are determined by the formula

$$v_k = \frac{1}{\sqrt{G_k G_{k-1}}} \begin{vmatrix} (u_1,u_1) & (u_2,u_1) & \cdots & (u_k,u_1) \\ (u_1,u_2) & (u_2,u_2) & \cdots & (u_k,u_2) \\ \vdots & \vdots & \ddots & \vdots \\ (u_1,u_k) & (u_2,u_{k-1}) & \cdots & (u_2,u_{k-1}) \\ u_1 & u_2 & & u_k \end{vmatrix} \qquad k = 1, 2, \ldots$$

The orthonormal set $\{u_k\} \in V$ is said to be **complete** if there does not exist a vector $v \neq \mathbf{0}$, $v \in V$ such that it is orthogonal to all vectors from $\{u_k\}$.

6. Fourier Series

Let $\{u_k\} \in V$ be an orthonormal set.

$\sum_{k=1}^{\infty}(f,u_k)u_k$ is called the Fourier series (**generalized Fourier series**)

(f,u_k) are called the Fourier coefficients, $c_k = (f,u_k)$

Theorem The Fourier series $\sum_{k=1}^{\infty}(f,u_k)u_k$ is convergent to the function $f \in L^2(a,b)$ if and only if

$$\sum_{k=1}^{\infty}(f,u_k)^2 = \|f\|^2 \qquad \text{(Parseval's equation)}$$

Proof: Let $f(x) = \sum_{k=1}^{\infty}(f,u_k)u_k$

$$\begin{aligned}
\|f\|^2 &= (f,f) \\
&= \left(\sum_{k=1}^{\infty}(f,u_k)u_k, \sum_{k=1}^{\infty}(f,u_k)u_k\right) \\
&= \left(\sum_{k=1}^{\infty}c_k u_k, \sum_{k=1}^{\infty}c_k u_k\right) \\
&= \sum_{k=1}^{\infty}c_k^2(u_k,u_k) + 2\sum_{k\neq m}^{\infty}c_k c_m(u_k,u_m) \\
&= \sum_{k=1}^{\infty}c_k^2 \\
&= \sum_{k=1}^{\infty}(f,u_k)^2
\end{aligned} \qquad \blacksquare$$

Let $\{u_k\} \in L^2(a,b)$ be an orthonormal set .

If for any $f \in L^2(a,b)$ its Fourier series

$$\sum_{k=1}^{\infty}(f,u_k)^2$$

converges to f in $L^2(a,b)$, then $\{u_k\}$ is said **complete** in $L^2(a,b)$.

7. Vector Space L^2

Consider a particular case of Equation 3.3 from Definition 3.13 (p.205), with $p=2$ and interval $I=[a,b]$:

$$L^2(a,b) = \left\{\varphi : (a,b) \to \mathbb{R} \,\middle|\, \int_a^b \varphi^2(x)\,dx < \infty\right\}$$

Inner product in vector space $L^2(a,b)$: For $u,v \in L^2(a,b)$ define:

$(u,v) = \int_a^b u(x)v(x)\,dx$ inner product in $L^2(a,b)$

$(u,v)_p = \int_a^b u(x)v(x)p(x)\,dx$ weighted inner product in $L^2(a,b)$

with the weight function $p(x) > 0$

Inner product vector space $L^2(a,b)$ belongs to the class of Hilbert spaces.

Introduced inner product induces the norm in $L^2(a,b)$:

$$\|u\| = \int_a^b u^2(x)dx$$

$$\|u\|_p = \int_a^b u^2(x)\,p(x)dx$$

Historically, the first complete set was used by Fourier set of trigonometric functions $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx\right\},\ k = 1,2,...$ in the interval $(0,2\pi)$.

The complete orthogonal sets used in the solution of PDE will be generated by the solution of the Sturm-Liouville problems.

8. Exercizes:

The set of monoms $\{1,x,x^2,x^3,...\}$ is linearly independent in $L^2(-1,1)$.

a) Using the Gram-Schmidt orthogonalization algorithm with inner product

$$(u,v) = \int_{-1}^{1} u(x)v(x)dx$$

construct an orthonormal set in $L^2(-1,1)$
(the obtained set will be the set of the Legendre polynomials up the the scalar multiple).

b) Using the Gram-Schmidt orthogonalization algorithm with inner product

$$(u,v) = \int_{-1}^{1} \frac{u(x)v(x)}{\sqrt{1-x^2}}dx$$

construct an orthonormal set in $L^2(-1,1)$
(the obtained set will be the set of the Tchebyshev polynomials up the the scalar multiple).

c) Use the obtained orthonormal sets for generalized Fourier series expansion of the function:

$$f(x) = \begin{cases} -1 & x \in (-1,0) \\ 1 & x \in (0,1) \end{cases}$$

Compare the results for truncated series with 2,3,4 terms. Make some observations.

The Scottish Cafe in Lvov

The original **Szkocka Café** (Scottish Café) in Akamemichna Street in Lvov, Ukraine

(shown at the time when it was the Dessert Bar at Shevchenko Prospekt 27). Now it is a Bank (on the right).

The café was a meeting place for many mathematicians including Banach, Steinhaus, Ulam, Mazur, Kac, Schauder, Kaczmarz and others. Problems were written in a book kept by the landlord and often prizes were offered for their solution. A collection of these problems appeared later as the *Scottish Book*. R D Mauldin, *The Scottish Book, Mathematics from the Scottish Café* (1981) contains the problems as well as some solutions and commentaries.

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