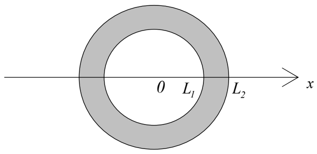


### *1 Dirichlet-Dirichlet boundary conditions*

Consider BE in the annular domain

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right)y = 0\,, \quad x \in \left(L_1, L_2\right)$$

Boundary conditions:

$$y\big|_{x=L_1} = 0 \qquad \text{(Dirichlet)}$$

$$y\big|_{x=L_2} = 0 \qquad \text{(Dirichlet)}$$

Apply boundary conditions to the general solution of BE:

$$c_1 J_\nu\left(\lambda L_1\right) + c_2 Y_\nu\left(\lambda L_1\right) = 0$$

$$c_1 J_\nu\left(\lambda L_2\right) + c_2 Y_\nu\left(\lambda L_2\right) = 0$$

This is a homogeneous system of two linear algebraic equations for $c_1$ and $c_2$. Rewrite it in the matrix form

$$\begin{bmatrix} J_\nu\left(\lambda L_1\right) & Y_\nu\left(\lambda L_1\right) \\ J_\nu\left(\lambda L_2\right) & Y_\nu\left(\lambda L_2\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We are looking for non-trivial solution of BVP, i.e. both coefficients in general solution cannot be zero

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A homogeneous linear system has a non-trivial solution only if the determinant of the system matrix is equal to zero:

**Equation for eigenvalues $\lambda_n$ :**

$$\boxed{det \begin{bmatrix} J_\nu\left(\lambda L_1\right) & Y_\nu\left(\lambda L_1\right) \\ J_\nu\left(\lambda L_2\right) & Y_\nu\left(\lambda L_2\right) \end{bmatrix} = J_\nu\left(\lambda L_1\right) Y_\nu\left(\lambda L_2\right) - J_\nu\left(\lambda L_2\right) Y_\nu\left(\lambda L_1\right) = 0}$$

The roots of this equation yield the eigenvalues $\lambda_n$ for which BVP has non-trivial solutions $y_n(x)$ (eigenfunctions). Oscillatory property of Bessel functions provides an infinite set of eigenvalues $\lambda_n$ and corresponding eigenfunctions are

$$y_n(x) = c_{1,n} J_\nu\left(\lambda_n x\right) + c_{2,n} Y_\nu\left(\lambda_n x\right)$$

Determine now the coefficients $c_{1,n}$ and $c_{2,n}$ from a system where eigenvalues are substituted

$$\begin{bmatrix} J_\nu\left(\lambda_n L_1\right) & Y_\nu\left(\lambda_n L_1\right) \\ J_\nu\left(\lambda_n L_2\right) & Y_\nu\left(\lambda_n L_2\right) \end{bmatrix} \begin{bmatrix} c_{1,n} \\ c_{2,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because a linear system has a singular matrix, solutions for $c_{1,n}$ and $c_{2,n}$ are linearly dependent and can be determined just from one equation, let it be the second one

$$c_{1,n} J_\nu\left(\lambda_n L_2\right) + c_{2,n} Y_\nu\left(\lambda_n L_2\right) = 0$$

one of the unknowns in this equation is a free parameter, choose

$c_{1,n} = \dfrac{1}{J_\nu\left(\lambda_n L_2\right)}$, then $c_{2,n} = \dfrac{-1}{Y_\nu\left(\lambda_n L_2\right)}$

Then eigenfunctions have the form:

**Eigenfunctions**

$$\boxed{y_n(x) = \frac{J_\nu\left(\lambda_n x\right)}{J_\nu\left(\lambda_n L_2\right)} - \frac{Y_\nu\left(\lambda_n x\right)}{Y_\nu\left(\lambda_n L_2\right)}}$$

The norm of eigenfunctions is given by:

$$
\begin{aligned}
N_{\nu,n}^2 &= \int_{L_1}^{L_2} x y_n^2(x)\,dx \\
&= \int_{L_1}^{L_2} x\left[\frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_2)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_2)}\right]^2 dx \\
&= \frac{1}{J_\nu^2(\lambda_n L_2)}\int_{L_1}^{L_2} x J_\nu^2(\lambda_n x)\,dx + \frac{1}{Y_\nu^2(\lambda_n L_2)}\int_{L_1}^{L_2} x Y_\nu^2(\lambda_n x)\,dx - \\
&\qquad - \frac{1}{J_\nu(\lambda_n L_2) Y_\nu(\lambda_n L_2)}\int_{L_1}^{L_2} x J_\nu(\lambda_n x) Y_\nu(\lambda_n x)\,dx
\end{aligned}
$$

$= \ldots$ express in terms of $J_{\nu+1}, \ldots$

***Summary:*** For an annular domain with boundary conditions:

$$y\big|_{x=L_1} = 0$$

$$y\big|_{x=L_2} = 0$$

Eigenvalues $\lambda_n$ are positive roots of the characteristic equation

$$J_\nu(\lambda L_1) Y_\nu(\lambda L_2) - J_\nu(\lambda L_2) Y_\nu(\lambda L_1) = 0$$

The eigenfunctions are

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_2)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_2)}$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

where $a_n = \dfrac{\int_0^L x y_n(x) f(x)\,dx}{\int_0^L x y_n^2(x)\,dx} = \dfrac{\int_0^L x J_\nu(\lambda_n x) f(x)\,dx}{N_{\nu,n}^2}$

Maple examples: $\nu = 0$ *SF-AD-1-0.mws*
$\nu = 1$ *SF-AD-1-1.mws*

$L_1 = 2$ , $L_2 = 5$ $\quad f(x) = 1 - H(x-3)$