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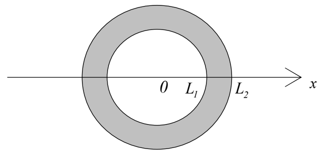
3 *Dirichlet-Neumann boundary conditions*



Consider BE in the annular domain

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right)y = 0, \quad x \in \left(L_1, L_2\right)$$

Boundary conditions:

$$\left[y\right]_{x=L_1} = 0 \qquad \text{(Dirichlet)}$$

$$\left[\frac{dy}{dx}\right]_{x=L_2} = 0 \qquad \text{(Neumann)}$$

The derivative of the general solution (use chain rule and differential identities)

$$\frac{d}{dx} y(x) = c_1\lambda\left[-J_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} J_\nu(\lambda x)\right] + c_2\lambda\left[-Y_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} Y_\nu(\lambda x)\right]$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

Apply boundary conditions to the general solution of BE:

$$c_1 J_\nu(\lambda L_1) + c_2 Y_\nu(\lambda L_1) = 0$$

$$c_1\lambda\left[-J_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} J_\nu(\lambda L_2)\right] + c_2\lambda\left[-Y_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} Y_\nu(\lambda L_2)\right] = 0$$

Denote:

$$a_{11} = \left[J_\nu(\lambda L_1)\right]$$

$$a_{12} = \left[Y_\nu(\lambda L_1)\right]$$

$$a_{21} = \left[-\lambda J_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2} J_\nu(\lambda L_2)\right]$$

$$a_{22} = \left[-\lambda Y_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2} Y_\nu(\lambda L_2)\right]$$

Then a system for coefficients has the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A necessary condition for a system to have a non-trivial solution is

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

it yields a characteristic equation for values of the parameter λ for which the BVP has a non-trivial solution:

Equation for eigenvalues λ_n :

$$\left[J_\nu(\lambda L_1)\right]\left[-\lambda Y_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2} Y_\nu(\lambda L_2)\right]$$

$$-\left[Y_\nu(\lambda L_1)\right]\left[-\lambda J_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2} J_\nu(\lambda L_2)\right] = 0$$

The positive roots of this equation provide an infinite set of eigenvalues λ_n. Then for the determined eigenvalues λ_n, coefficients $c_{1,n}$ and $c_{2,n}$ can be found from one of the equations of the system (choose the first one):

$$a_{11}c_1 + a_{12}c_2 = 0$$

One of the coefficients can be taken as a free parameter, choose

$c_1 = \dfrac{1}{a_{11}}$, then $c_2 = \dfrac{-1}{a_{12}}$

With determined coefficients, solutions of the BVP $y_n(x)$ (eigenfunctions) have the form:

Eigenfunctions:

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{a_{11,n}} - \frac{Y_\nu(\lambda_n x)}{a_{12,n}}$$

$$= \frac{J_\nu(\lambda_n x)}{[J_\nu(\lambda L_1)]} - \frac{Y_\nu(\lambda_n x)}{[Y_\nu(\lambda L_1)]}$$

The norm of eigenfunctions is given by:

$$N_{\nu,n}^2 = \int_{L_1}^{L_2} x y_n^2(x)dx$$

$$= \int_{L_1}^{L_2} x\left[\frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_1)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_1)}\right]^2 dx$$

$$= \frac{1}{J_\nu^2(\lambda_n L_1)}\int_{L_1}^{L_2} xJ_\nu^2(\lambda_n x)dx + \frac{1}{Y_\nu^2(\lambda_n L_1)}\int_{L_1}^{L_2} xY_\nu^2(\lambda_n x)dx -$$

$$- \frac{1}{J_\nu(\lambda_n L_1)Y_\nu(\lambda_n L_1)}\int_{L_1}^{L_2} xJ_\nu(\lambda_n x)Y_\nu(\lambda_n x)dx$$

<u>Summary:</u> For an annular domain with boundary conditions:

$$[y]_{x=L_1} = 0 \qquad \text{(Dirichlet)}$$

$$\left[\frac{dy}{dx}\right]_{x=L_2} = 0 \qquad \text{(Neumann)}$$

Eigenvalues λ_n are positive roots of the characteristic equation

$$[J_\nu(\lambda L_1)]\left[-\lambda Y_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2}Y_\nu(\lambda L_2)\right]$$

$$-[Y_\nu(\lambda L_1)]\left[-\lambda J_{\nu+1}(\lambda L_2) + \frac{\nu}{L_2}J_\nu(\lambda L_2)\right] = 0$$

The eigenfunctions are

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{J_\nu(\lambda_n L_1)} - \frac{Y_\nu(\lambda_n x)}{Y_\nu(\lambda_n L_1)}$$