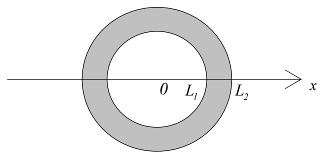
4 *Neumann-Neumann boundary conditions*



Consider BE in the annular domain

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right)y = 0, \quad x \in \left(L_1, L_2\right)$$

with homogeneous boundary conditions:

$$\left[\frac{dy}{dx}\right]_{x=L_1} = 0 \qquad \text{(Neumann)}$$

$$\left[\frac{dy}{dx}\right]_{x=L_2} = 0 \qquad \text{(Neumann)}$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

The derivative of the general solution (use chain rule and differential identities)

$$\frac{d}{dx} y(x) = c_1 \lambda \left[-J_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} J_\nu(\lambda x) \right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} Y_\nu(\lambda x) \right]$$

Substitute into boundary conditions:

$x = L_1$ $$c_1 \lambda \left[-J_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} J_\nu(\lambda L_1) \right] - c_2 \lambda \left[-Y_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} Y_\nu(\lambda L_1) \right]$$

$x = L_2$ $$c_1 \lambda \left[-J_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} J_\nu(\lambda L_2) \right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} Y_\nu(\lambda L_2) \right]$$

Denote:

$$a_{11} = \left[\lambda J_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) J_\nu(\lambda L_1) \right]$$

$$a_{12} = \left[\lambda Y_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) Y_\nu(\lambda L_1) \right]$$

$$a_{21} = \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(\frac{\nu}{L_2}\right) J_\nu(\lambda L_2) \right]$$

$$a_{22} = \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(\frac{\nu}{L_2}\right) Y_\nu(\lambda L_2) \right]$$

Then a system for coefficients has the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A necessary condition for a system to have a non-trivial solution is

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

it yields a characteristic equation for values of the parameter λ for which the BVP has a non-trivial solution:

Equation for eigenvalues λ_n:

$$\left[\lambda J_{\nu+1}(\lambda L_1)-\left(\frac{\nu}{L_1}\right)J_\nu(\lambda L_1)\right]\left[-\lambda Y_{\nu+1}(\lambda L_2)+\left(\frac{\nu}{L_2}\right)Y_\nu(\lambda L_2)\right]$$
$$-\left[\lambda Y_{\nu+1}(\lambda L_1)-\left(\frac{\nu}{L_1}\right)Y_\nu(\lambda L_1)\right]\left[-\lambda J_{\nu+1}(\lambda L_2)+\left(\frac{\nu}{L_2}\right)J_\nu(\lambda L_2)\right]=0$$

The positive roots of this equation provide an infinite set of eigenvalues λ_n (note for Neumann boundary conditions, $\lambda_n = 0$ is also an eigenvalue for $\nu = 0$). Then for the determined eigenvalues λ_n, coefficients $c_{1,n}$ and $c_{2,n}$ can be found from one of the equations of the system (choose the second one):

$a_{21}c_1 + a_{22}c_2 = 0$

One of the coefficients can be taken as a free parameter, choose

$c_1 = \dfrac{1}{a_{21}}$, then $c_2 = \dfrac{1}{a_{22}}$

With determined coefficients, solutions of the BVP $y_n(x)$ (eigenfunctions) have the form:

Eigenfunctions:

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{a_{21,n}} - \frac{Y_\nu(\lambda_n x)}{a_{22,n}} \qquad y_0(x) = 1 \;\; for\; \nu=0$$

$$= \frac{J_\nu(\lambda_n x)}{\left[-\lambda_n J_{\nu+1}(\lambda_n L_2)+\left(\frac{\nu}{L_2}\right)J_\nu(\lambda_n L_2)\right]} - \frac{Y_\nu(\lambda_n x)}{\left[-\lambda_n Y_{\nu+1}(\lambda_n L_2)+\left(\frac{\nu}{L_2}\right)Y_\nu(\lambda_n L_2)\right]}$$

The norm of the eigenfunctions is determined by the integral

$$N_{\nu,n}^2 = \int_{L_1}^{L_2} x y_n^2(x)dx, \qquad N_{0,0}^2 = \int_{L_1}^{L_2} x y_n^2(x)dx = \int_{L_1}^{L_2} x dx = \left[\frac{x^2}{2}\right]_{L_1}^{L_2} = \frac{L_2^2 - L_1^2}{2}$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \quad \text{and} \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n y_n(x) \;\; \text{for } \nu = 0$$

where
$$a_0 = \frac{\int_{L_1}^{L_2} x y_0^2(x) f(x)dx}{\int_{L_1}^{L_2} x y_0^2(x)dx} = \frac{\int_{L_1}^{L_2} x f(x)dx}{\int_{L_1}^{L_2} x dx} = \frac{\int_{L_1}^{L_2} x f(x)dx}{N_{0,0}^2} \quad \text{for } \nu = 0$$

and
$$a_n = \frac{\int_{L_1}^{L_2} x y_n(x) f(x)dx}{\int_{L_1}^{L_2} x y_n^2(x)dx} = \frac{\int_{L_1}^{L_2} x y_n(x) f(x)dx}{N_{\nu,n}^2}$$