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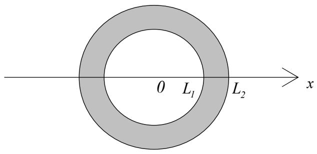
6 *Neumann-Robin boundary conditions*



Consider BE in the annular domain

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y = 0\,, \quad x \in \left(L_1, L_2\right)$$

with homogeneous boundary conditions:

$$\left[\frac{dy}{dx}\right]_{x=L_1} = 0 \qquad \text{(Neumann)}$$

$$\left[k_2 \frac{dy}{dx} + h_2 y\right]_{x=L_2} = 0 \qquad H_2 = \frac{h_2}{k_2} \qquad \text{(Robin)}$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

The derivative of the general solution (use chain rule and differential identities)

$$\frac{d}{dx} y(x) = c_1 \lambda \left[-J_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} J_\nu(\lambda x)\right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} Y_\nu(\lambda x)\right]$$

Substitute into boundary conditions:

$x = L_1$
$$c_1 \lambda \left[-J_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} J_\nu(\lambda L_1)\right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} Y_\nu(\lambda L_1)\right] = 0$$

$x = L_2$
$$c_1 \lambda \left[-J_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} J_\nu(\lambda L_2)\right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} Y_\nu(\lambda L_2)\right] + c_1 H_2 J_\nu(\lambda L_2) + c_2 H_2 Y_\nu(\lambda L_2) = 0$$

Collect terms

$x = L_1$
$$c_1 \left[\lambda J_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) J_\nu(\lambda L_1)\right] + c_2 \left[\lambda Y_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) Y_\nu(\lambda L_1)\right] = 0$$

$x = L_2$
$$c_1 \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2}\right) J_\nu(\lambda L_2)\right] + c_2 \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2}\right) Y_\nu(\lambda L_2)\right] = 0$$

Denote:

$$a_{11} = \left[\lambda J_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) J_\nu(\lambda L_1)\right]$$

$$a_{12} = \left[\lambda Y_{\nu+1}(\lambda L_1) - \left(\frac{\nu}{L_1}\right) Y_\nu(\lambda L_1)\right]$$

$$a_{21} = \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2}\right) J_\nu(\lambda L_2)\right]$$

$$a_{22} = \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2}\right) Y_\nu(\lambda L_2)\right]$$

Then a system for coefficients has the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A necessary condition for a system to have a non-trivial solution is

$$det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

it yields a characteristic equation for values of the parameter λ for which the BVP has a non-trivial solution:

Equation for eigenvalues λ_n :

$$\left[\lambda J_{\nu+1}(\lambda L_1)-\left(\frac{\nu}{L_1}\right)J_\nu(\lambda L_1)\right]\left[-\lambda Y_{\nu+1}(\lambda L_2)+\left(H_2+\frac{\nu}{L_2}\right)Y_\nu(\lambda L_2)\right]$$
$$-\left[\lambda Y_{\nu+1}(\lambda L_1)-\left(\frac{\nu}{L_1}\right)Y_\nu(\lambda L_1)\right]\left[-\lambda J_{\nu+1}(\lambda L_2)+\left(H_2-\frac{\nu}{L_2}\right)J_\nu(\lambda L_2)\right]=0$$

The positive roots of this equation provide an infinite set of eigenvalues λ_n . Then for the determined eigenvalues λ_n , coefficients $c_{1,n}$ and $c_{2,n}$ can be found from one of the equations of the system (choose the second one):

$a_{21}c_1 + a_{22}c_2 = 0$

One of the coefficients can be taken as a free parameter, choose

$c_1 = \frac{1}{a_{21}}$, then $c_2 = -\frac{1}{a_{22}}$

With determined coefficients, solutions of the BVP $y_n(x)$ (eigenfunctions) have the form:

Eigenfunctions:

$$y_n(x) = \frac{J_\nu(\lambda_n x)}{a_{21,n}} - \frac{Y_\nu(\lambda_n x)}{a_{22,n}}$$

$$= \frac{J_\nu(\lambda_n x)}{\left[-\lambda_n J_{\nu+1}(\lambda_n L_2)+\left(H_2+\frac{\nu}{L_2}\right)J_\nu(\lambda_n L_2)\right]} - \frac{Y_\nu(\lambda_n x)}{\left[-\lambda_n Y_{\nu+1}(\lambda_n L_2)+\left(H_2+\frac{\nu}{L_2}\right)Y_\nu(\lambda_n L_2)\right]}$$

The norm of the eigenfunctions is determined by the integral

$$N_{\nu,n}^2 = \int_{L_1}^{L_2} x y_n^2(x)\,dx$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

where $a_n = \frac{\int_{L_1}^{L_2} x y_n(x) f(x)\,dx}{\int_{L_1}^{L_2} x y_n^2(x)\,dx} = \frac{\int_{L_1}^{L_2} x y_n^2(x) f(x)\,dx}{N_{\nu,n}^2}$