



9 Robin-Robin boundary conditions

Consider BE in the annular domain

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2)y = 0, \quad x \in (L_1, L_2)$$

with homogeneous boundary conditions:

$$\left[-k_1 \frac{dy}{dx} + h_1 y \right]_{x=L_1} = 0 \quad H_1 = \frac{h_1}{k_1} \quad (\text{Robin})$$

$$\left[k_2 \frac{dy}{dx} + h_2 y \right]_{x=L_2} = 0 \quad H_2 = \frac{h_2}{k_2} \quad (\text{Robin})$$

The general solution is given by

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

The derivative of the general solution (use chain rule and differential identities)

$$\frac{d}{dx} y(x) = c_1 \lambda \left[-J_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} J_\nu(\lambda x) \right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda x) + \frac{\nu}{\lambda x} Y_\nu(\lambda x) \right]$$

Substitute into boundary conditions:

$$x = L_1 \quad -c_1 \lambda \left[-J_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} J_\nu(\lambda L_1) \right] - c_2 \lambda \left[-Y_{\nu+1}(\lambda L_1) + \frac{\nu}{\lambda L_1} Y_\nu(\lambda L_1) \right] + c_1 H_1 J_\nu(\lambda L_1) + c_2 H_1 Y_\nu(\lambda L_1) = 0$$

$$x = L_2 \quad c_1 \lambda \left[-J_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} J_\nu(\lambda L_2) \right] + c_2 \lambda \left[-Y_{\nu+1}(\lambda L_2) + \frac{\nu}{\lambda L_2} Y_\nu(\lambda L_2) \right] + c_1 H_2 J_\nu(\lambda L_2) + c_2 H_2 Y_\nu(\lambda L_2) = 0$$

Collect terms

$$x = L_1 \quad c_1 \left[\lambda J_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) J_\nu(\lambda L_1) \right] + c_2 \left[\lambda Y_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) Y_\nu(\lambda L_1) \right] = 0$$

$$x = L_2 \quad c_1 \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2} \right) J_\nu(\lambda L_2) \right] + c_2 \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2} \right) Y_\nu(\lambda L_2) \right] = 0$$

Denote:

$$a_{11} = \left[\lambda J_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) J_\nu(\lambda L_1) \right]$$

$$a_{12} = \left[\lambda Y_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) Y_\nu(\lambda L_1) \right]$$

$$a_{21} = \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2} \right) J_\nu(\lambda L_2) \right]$$

$$a_{22} = \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2} \right) Y_\nu(\lambda L_2) \right]$$

Then a system for coefficients has the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A necessary condition for a system to have a non-trivial solution is

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

it yields a characteristic equation for values of the parameter λ for which the BVP has a non-trivial solution:

Equation for eigenvalues λ_n :

$$\left[\lambda J_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) J_{\nu}(\lambda L_1) \right] \left[-\lambda Y_{\nu+1}(\lambda L_2) + \left(H_2 + \frac{\nu}{L_2} \right) Y_{\nu}(\lambda L_2) \right] \\ - \left[\lambda Y_{\nu+1}(\lambda L_1) + \left(H_1 - \frac{\nu}{L_1} \right) Y_{\nu}(\lambda L_1) \right] \left[-\lambda J_{\nu+1}(\lambda L_2) + \left(H_2 - \frac{\nu}{L_2} \right) J_{\nu}(\lambda L_2) \right] = 0$$

The positive roots of this equation provide an infinite set of eigenvalues λ_n . Then for the determined eigenvalues λ_n , coefficients $c_{1,n}$ and $c_{2,n}$ can be found from one of the equations of the system (choose the second one):

$$a_{21}c_1 + a_{22}c_2 = 0$$

One of the coefficients can be taken as a free parameter, choose

$$c_1 = \frac{1}{a_{21}}, \text{ then } c_2 = \frac{1}{a_{22}}$$

With determined coefficients, solutions of the BVP $y_n(x)$ (eigenfunctions) have the form:

Eigenfunctions:

$$y_n(x) = \frac{J_{\nu}(\lambda_n x)}{a_{21,n}} - \frac{Y_{\nu}(\lambda_n x)}{a_{22,n}} \\ = \frac{J_{\nu}(\lambda_n x)}{\left[-\lambda_n J_{\nu+1}(\lambda_n L_2) + \left(H_2 + \frac{\nu}{L_2} \right) J_{\nu}(\lambda_n L_2) \right]} \\ - \frac{Y_{\nu}(\lambda_n x)}{\left[-\lambda_n Y_{\nu+1}(\lambda_n L_2) + \left(H_2 - \frac{\nu}{L_2} \right) Y_{\nu}(\lambda_n L_2) \right]}$$

The norm of the eigenfunctions is determined by the integral

$$N_{\nu,n}^2 = \int_{L_1}^{L_2} x y_n^2(x) dx$$

Fourier-Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

$$\text{where } a_n = \frac{\int_{L_1}^{L_2} x y_n(x) f(x) dx}{\int_{L_1}^{L_2} x y_n^2(x) dx} = \frac{\int_{L_1}^{L_2} x y_n^2(x) f(x) dx}{N_{\nu,n}^2}$$

Maple example:

$$\nu = 0 \quad \text{SF-AD-9-0.mws}$$

$$\nu = 1 \quad \text{SF-AD-9-1.mws}$$

$$L_1 = 2, \quad L_2 = 5$$

$$H_1 = 2, \quad H_2 = 3$$

$$f(x) = 1 - H(x - 3)$$