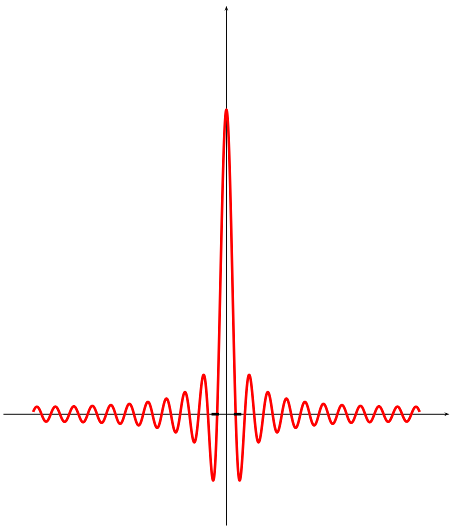

Mathematical Foundations of

Signal Reconstruction

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Introduction

One of the main problems in signal analysis is: Given the samples $f(t_k)$, $k \in \mathbb{Z}$, of a function $f \in L^2(\mathbb{R})$, determine f. If f is band-limited and if the samples are equally spaced and dense enough, then it is well-known that f can be uniquely recovered by the cardinal series. In practice, the sampling sequence $(t_k)_{k\in\mathbb{Z}}$ can be irregularly spaced. Several approaches have been made to recover f from irregular samples.

In this note we present the notions and results which are fundamental for both the regular and irregular, starting with the discrete sampling theory (cf. [1] and [2]).

Chapter 1

Discrete Periodic Functions

In this chapter we consider signals that are discrete and periodic. We first present the mathematical background and setting.

1.1 The Hilbert Space $\ell(\mathbb{Z}_N)$

Let $N \in \mathbb{N} := \{1, 2, 3 \ldots, \}$ be a positive integer and $a, b \in \mathbb{Z}$. Recall that a is said to be *congruent to* b *mod* N, notation $a \equiv b \pmod N$, if $N \mid (b - a)$. Clearly, the congruence is an equivalent relation on $\mathbb{Z}$. We will denote the *congruence class* or *residue class* of $a \in \mathbb{Z}$ by $\overline{a}$. Recall that

$$\begin{aligned} \overline{a} &= \{b \in \mathbb{Z} \mid a \equiv b \pmod N\} && (1.1) \\ &= a + N\mathbb{Z} = \{a + kN \mid k \in \mathbb{Z}\}. && (1.2) \end{aligned}$$

There are precisely N distinct congruence classes modulo N, namely $\overline{0}, \overline{1}, \ldots, \overline{N-1}$, defined by the possible remainders by division by N. These classes form a partition of $\mathbb{Z}$. The set of all congruence classes modulo N is usually denoted by

$$\mathbb{Z}/N\mathbb{Z} := \{\overline{0}, \overline{1}, \ldots, \overline{N-1}\}. \tag{1.3}$$

and called the *integers modulo* N. Recall that $\mathbb{Z}/N\mathbb{Z}$ forms a ring (with 1) under the operations defined by

$$\begin{aligned} \overline{a} + \overline{b} &:= \overline{a + b} && (1.4) \\ \overline{a} \cdot \overline{b} &:= \overline{a \cdot b}, && (1.5) \end{aligned}$$

A function $f : \mathbb{Z} \to \mathbb{C}$ is said to be N-*periodic* if $f(a + N) = f(a)$ for all $a \in \mathbb{Z}$. We denote the set of all N-periodic function on $\mathbb{Z}$ by $\ell_N(\mathbb{Z})$, i.e.

$$\ell_N(\mathbb{Z}) := \{f : \mathbb{Z} \to \mathbb{C} \mid f \text{ is } N\text{-periodic}\}. \tag{1.6}$$

Clearly, $\ell_N(\mathbb{Z})$ forms a $\mathbb{C}$-algebra under the operations defined by

$$\begin{aligned}(f+g)(a) &:= f(a)+g(a)\\ (\alpha f)(a) &:= \alpha f(a)\\ (f\cdot g)(a) &:= f(a)g(a)\end{aligned}$$

for $f, g \in \ell_N(\mathbb{Z})$ and $\alpha \in \mathbb{C}$. Let $\pi_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}\ a \mapsto \overline{a} = a + N\mathbb{Z}$ denote the canonical projection from $\mathbb{Z}$ onto the quotient ring $\mathbb{Z}/N\mathbb{Z}$. For each N-periodic function $f : \mathbb{Z} \to \mathbb{C}$ there exists a unique function $\overline{f} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ such that $f = \overline{f} \circ \pi_N$, i.e.

$$f(a) = \overline{f}(\overline{a})$$

for all $a \in \mathbb{Z}$. Let

$$\ell(\mathbb{Z}/N\mathbb{Z}) := \{\overline{f} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \mid \overline{f} \text{ is a map}\}$$

denote the set of all maps from $\mathbb{Z}/N\mathbb{Z}$ into $\mathbb{C}$. Clearly, the association $f \mapsto \overline{f}$ defines a canonical bijection $\Psi_N : \ell_N(\mathbb{Z}) \to \ell(\mathbb{Z}/N\mathbb{Z})$, as can be easily verified. Moreover, Ψ_n is an (algbra-) isomorphism if $\ell(\mathbb{Z}/N\mathbb{Z})$ is endowed with the operations induced by Ψ_n:

$$\begin{aligned}(\overline{f}+\overline{g})(\overline{a}) &= f(a)+g(a)\\ (\alpha\overline{f})(\overline{a}) &= \alpha f(a)\\ (\overline{f}\overline{g})(\overline{a}) &= f(a)g(a)\end{aligned}$$

for all $f, g \in \ell_N(\mathbb{Z})$ and for all $a \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$. Thus, $\ell(\mathbb{Z}/N\mathbb{Z})$ forms a $\mathbb{C}$-algebra which is isomorphic to $\ell_N(\mathbb{Z})$.

For convenience, we will usually denote the elements $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{N-1}$ of $\mathbb{Z}/N\mathbb{Z}$ by $0, 1, 2, \ldots, N-1$ and write for $\mathbb{Z}/N\mathbb{Z}$ simply

$$Z_N = \{0, 1, 2, \ldots, N-1\}$$

where addition and multiplication are reduced mod N. Note that the additive inverse of the element $a \in \mathbb{Z}_N$ is $N - a$ which we will denote by $-a$.

Similarly, we will usually denote the function $\overline{f} \in \ell(\mathbb{Z}/N\mathbb{Z})$ simply by f and abbreviate $\ell(\mathbb{Z}/N\mathbb{Z})$ by

$$\ell(\mathbb{Z}_N) = \{f : \mathbb{Z}_N \to \mathbb{C} \mid f \text{ is a map}\}.$$

We thus identify a function $f : \mathbb{Z}_N \to \mathbb{C}$ with its periodic extension which is justified by the isomorphism $\Psi_N : \ell_N(\mathbb{Z}) \to \ell(\mathbb{Z}/N\mathbb{Z})$ defined above. It is customary to call the functions $f : \mathbb{Z}_N \to \mathbb{C}$ *Nth order periodic sequences* on $\mathbb{C}$ or just *periodic sequences* if the reference to N is clear. Since $\mathbb{Z}_N$ is finite, the $\mathbb{C}$-algebra $\ell(\mathbb{Z}_N)$ can be considered as any of the ℓ^p-spaces on $\mathbb{Z}_N$.

Definition 1.1 (Even, Odd, Real, Imaginary Periodic Sequences)

1. A Nth-order periodic sequence f is called *even* if $f(-m) = f(m)$ for all $m \in \mathbb{Z}_N$, and *odd* if $f(-m) = -f(m)$ for all $m \in \mathbb{Z}_N$.

2. Moreover, a Nth-order periodic sequence f is said to be *real* if $\mathrm{im}(f) \subseteq \mathbb{R}$ and *imaginary* if $\mathrm{im}(f) \subseteq i\mathbb{R}$.

Definition 1.2
For convenience, we set

$$W_N := e^{-2\pi i/N}. \tag{1.7}$$

Some important properties of W_N are listed as

Lemma 1.1

1. *$W_N^k = 1$ if and only if $k \equiv 0 \pmod N$ (which is equivalent to $k \in N\mathbb{Z}$). In particular, $W_N \neq 1$ for all $N \in \mathbb{N}$.*

2. *For all $k \in \mathbb{Z}$*

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & \text{if } k \equiv 0 \pmod N \\ 0 & \text{else} \end{cases}$$

PROOF: (1) $W_N^k = 1$ if and only if $e^{-2\pi ik/N} = 1$ if and only if $-k/N \in \mathbb{Z}$ if and only if $k \in N\mathbb{Z}$.

(2) From the summation formula for geometric sequences

$$\sum_{n=0}^{N-1} r^n = \begin{cases} N & \text{if } r = 1 \\ \frac{1-r^N}{1-r} & \text{else,} \end{cases}$$

follows by (1) that

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & \text{if } k \equiv 0 \pmod N \\ \frac{1-W_N^{Nk}}{1-W_N^k} & \text{else,} \end{cases}$$

However, applying (1) a second time, we see that $W_N^{Nk} = 1$, which implies that

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & \text{if } k \equiv 0 \pmod N \\ 0 & \text{else,} \end{cases}$$

as claimed. □

Theorem 1.1 (Inner Products)
The maps $\langle \cdot, \cdot \rangle_0,\ \langle \cdot, \cdot \rangle_2 : \ell(\mathbb{Z}_N) \times \ell(\mathbb{Z}_N) \to \mathbb{C}$ defined by

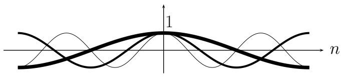
$$\langle f, g \rangle_0 := \frac{1}{N} \sum_{m=0}^{N-1} f(m)\overline{g(m)}$$

and

$$\langle f, g \rangle_2 := \sum_{m=0}^{N-1} f(m)\overline{g(m)}$$

for all $f, g \in \ell(\mathbb{Z}_N)$ constitute inner products on the space $\ell(\mathbb{Z}_N)$ and induce the norms

$$\|f\|_0 := \sqrt{\langle f, f\rangle_0} = \frac{1}{\sqrt{N}} \left(\sum_{m=0}^{N-1} |f(m)|^2 \right)^{1/2}$$

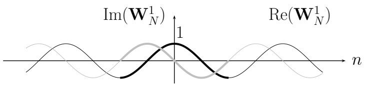


The figure shows the graphs of the real part of the functions $\mathbf{W}_N^1$ (heavy) $\mathbf{W}_N^2$ (medium) and $\mathbf{W}_N^3$ (light) with $N = 51$ plotted as a curve on the interval $[-26, 26]$.

and

$$\|f\|_2 := \sqrt{\langle f, f\rangle_2} = \left(\sum_{m=0}^{N-1} |f(m)|^2 \right)^{1/2} = \sqrt{N}\, \|f\|_0$$

for all $f \in \ell(\mathbb{Z}_N)$) on $\ell(\mathbb{Z}_N)$. The unitary spaces $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_2\big)$ and $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_0\big)$ represent the time and frequency space, respectively. They are complete with respect to the metrics induced by the norms and are therefore Hilbert spaces.



The figure shows the graphs of real part (black) and imaginary part (light gray) of the function $\mathbf{W}_N^1$ with $N = 19$ plotted as a curve on the interval $[-26, 26]$.

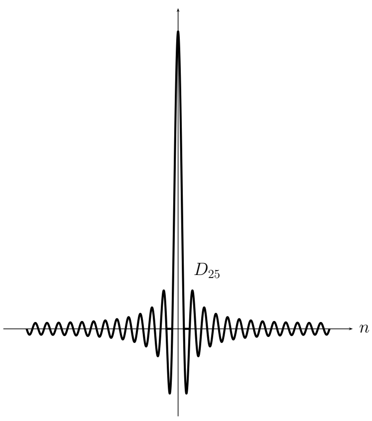
The follwing sequences will often be used to define basis for $\ell(\mathbb{Z}_N)$ or subspaces like the space $\mathcal{B}_M$ of all M-bandlimited sequences which well be introduced later.

Definition 1.3 (Dirichlet Kernel)

1. Let $\mathbf{W}_N^k : \mathbb{Z}_N \to \mathbb{C},\ m \mapsto W_N^{km} = e^{-2\pi i mk/N}$ for all $k \in \mathbb{Z}_N$.
2. Moreover, let $\mathbf{e}_k^{(N)} : \mathbb{Z}_N \to \mathbb{C},\ m \mapsto \delta_{km}$ for all $k \in \mathbb{Z}_N$, where δ_{km} denotes the Kronecker-delta as usual. The Nth-order periodic sequence $\mathbf{e}_k^{(N)}$ is the discrete version of the shifted Dirac delta function.
3. Finally, for $M \in \mathbb{N}$ and $M < N/2$, we define the function $D_M : \mathbb{Z}_N \to \mathbb{C}$ by

$$D_M := D_M^{(N)} := \sum_{k=-M}^{M} \mathbf{W}_N^{-k},$$

which is called the *discrete Dirichlet kernel* of order M.



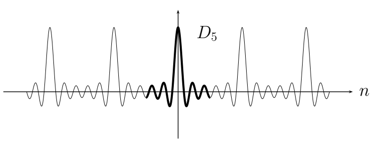
The figure shows the graph of the Dirichlet kernel function D_{25} with $N = 51$ plotted as a curve.

The functions defined in Definition 1.3 are clearly N-periodic and therefore elements of $\ell(\mathbb{Z}_N)$. The following identities follow directly from the definitions of the functions.

Lemma 1.2 (Closed Form Representation of Dirichlet Kernel)

1. *Clearly,* $\mathbf{W}_N^s \cdot \mathbf{W}_N^t = \mathbf{W}_N^{s+t}$ *for all* $s, t \in \mathbb{Z}_N$.
2. $\mathbf{e}_k^{(N)} f = \big(\delta_{km} f(k)\big)$ *for all* $f \in \ell(\mathbb{Z}_N)$ *and* $k \in \mathbb{Z}_N$.
3. *The Dirichlet kernel is a real-valued function. In particular,*

$$D_M(n) := \begin{cases} \dfrac{\sin\big((2M+1)\pi n/N\big)}{\sin\big(\pi n/N\big)}, & \text{if } n \neq 0 \\ 2M+1, & \text{else} \end{cases}$$



The figure shows the graph of the Dirichlet kernel function D_5 with $N = 11$ plotted as a curve on the interval $[-26, 26]$.

PROOF: (1) and (2) are immediate. To establish identity (3), recall the geometric series identity: For any $K, L \in \mathbb{Z}$ with $K \le L$ and $r \in \mathbb{C}$ we have

$$\sum_{k=K}^{L} r^k = \begin{cases} L-K+1, & \text{if } r = 1 \\ \dfrac{r^{L+1}-r^K}{r-1}, & \text{if } r \neq 1 \end{cases} \tag{1.8}$$

or equivalently

$$(r-1)\sum_{k=K}^{L} r^k = r^{L+1} - r^K. \tag{1.9}$$

Applying identity (1.9) to

$$D_M(n) = \sum_{k=-M}^{M} e^{\frac{2\pi i k n}{N}} = \sum_{k=-M}^{M} \left(e^{\frac{2\pi i n}{N}}\right)^k,$$

we first obtain

$$\big(e^{\frac{2\pi i n}{N}} - 1\big) D_M(n) = e^{\frac{2\pi i n}{N}(M+1)} - e^{-\frac{2\pi i n}{N}M},$$

and after multiplying this equation by $\frac{1}{2i}e^{-\frac{\pi i n}{N}}$

$$\left(\frac{e^{\frac{\pi i n}{N}} - e^{-\frac{\pi i n}{N}}}{2i}\right) D_M(n) = \frac{e^{\frac{2\pi i}{N}(M+\frac{1}{2})} - e^{-\frac{2\pi i}{N}(M+\frac{1}{2})}}{2i}$$

which is equivalent to

$$\begin{aligned} \sin\left(\frac{\pi n}{N}\right) D_M(n) &= \sin\left(\frac{2\pi n}{N}\big(M+\tfrac{1}{2}\big)\right) \\ &= \sin\left((2M+1)\frac{\pi n}{N}\right), \end{aligned}$$

and from which follows the assertion. □

Theorem 1.2 (Orthogonal Families)
Let the systems of N-th order periodic sequences be defined by

1. $\mathcal{O}_0^{(N)} := \big(\mathbf{W}_N^n \mid n \in \{0, \ldots, N-1\}\big)$
2. $\mathcal{O}_2^{(N)} := \left(\frac{1}{\sqrt{N}}\mathbf{W}_N^n \,\middle|\, n \in \{0, \ldots, N-1\}\right)$
3. $\mathcal{E}_0^{(N)} := \left(\sqrt{N}\mathbf{e}_k^{(N)} \,\middle|\, k \in \{0, \ldots, N-1\}\right)$
4. $\mathcal{E}_2^{(N)} := \left(\mathbf{e}_k^{(N)} \,\middle|\, k \in \{0, \ldots, N-1\}\right)$

Then $\mathcal{O}_0^{(N)}$, $\mathcal{E}_0^{(N)}$ and $\mathcal{O}_2^{(N)}$, $\mathcal{E}_2^{(N)}$ constitute orthonormal bases for the (frequency) space $\big(\ell(\mathbb{Z}_N, \langle\cdot,\cdot\rangle_0\big)$ and (time) space $\big(\ell(\mathbb{Z}_N, \langle\cdot,\cdot\rangle_2\big)$, respectively.

PROOF: Suppose $k, n \in \mathbb{Z}_N$, then

$$\begin{aligned}\left\langle \frac{1}{\sqrt{N}}\mathbf{W}_N^k, \frac{1}{\sqrt{N}}\mathbf{W}_N^n \right\rangle_2 &= \frac{1}{N}\left\langle \mathbf{W}_N^k, \mathbf{W}_N^n \right\rangle_2 = \left\langle \mathbf{W}_N^k, \mathbf{W}_N^n \right\rangle_0 \\ &= \frac{1}{N}\sum_{m=0}^{N-1} W_N^{mk} W_N^{-mn} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} W_N^{m(k-n)} \\ &= \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Hence $\mathcal{O}_j^{(N)}$ $(j = 0, 2)$ are orthonormal and thus linearly independent. Since $\ell(\mathbb{Z}_N)$ is a N-dimensional vector space over $\mathbb{C}$, they constitute orthonormal basis for the spaces $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_0\big)$ and $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_2\big)$, respectively. The corresponding statement for $\mathcal{E}_j^{(N)}$ $(j = 0, 2)$ is obvious. □

1.2 The Convolution Product

Definition 1.4 (Convolution)
The convolution $f * g$ of two Nth-order periodic sequences $f, g \in \ell(\mathbb{Z}_N)$ is defined by

$$(f * g)(n) := \sum_{j=0}^{N-1} f(j)g(n-j) = \sum_{j\in\mathbb{Z}_N} f(j)g(n-j)$$

for all $n \in \mathbb{Z}_N$.

It can be easily verified that $f * g$ is N-periodic, i.e. $\ell(\mathbb{Z}_N)$ is closed under convolution. Note that because of the N-periodicity of f we have for all $m \in \mathbb{Z}_N$

$$\sum_{j=m}^{N-1+m} f(j) = \sum_{j=0}^{N-1} f(j). \tag{1.10}$$

In fact,

$$\sum_{j=m}^{N-1+m} f(j) = \sum_{j=m}^{N-1} f(j) + \sum_{j=N}^{N-1+m} f(j) \tag{1.11}$$

and with the index substitution $i := j - N$, noting that $f(i + N) = f(i)$, this equals

$$= \sum_{j=m}^{N-1} f(j) + \sum_{i=0}^{m-1} f(i+N) \tag{1.12}$$

$$= \sum_{j=m}^{N-1} f(j) + \sum_{i=0}^{m-1} f(i) = \sum_{j=0}^{N-1} f(j). \tag{1.13}$$

The convolution product has the following properties:

Lemma 1.3
For all $f, g, h \in \ell(\mathbb{Z}_N)$ *and* $\lambda \in \mathbb{C}$ *we have:*

1. $f * (g * h) = (f * g) * h$;
2. $f * g = g * f$;
3. $f * (g + h) = f * g + f * h$;
4. $f * (\lambda g) = (\lambda f) * g = \lambda(f * g)$;
5. $\|f * g\|_j \leq \|f\|_j \|g\|_j$ *for* $j \in \{0, 2\}$;
6. $\displaystyle\sum_{n=0}^{N-1} (f * g)(n) = \left(\sum_{n=0}^{N-1} f(n)\right)\left(\sum_{n=0}^{N-1} g(n)\right)$;
7. $\mathbf{W}_N^j * \mathbf{W}_N^k = \begin{cases} N\mathbf{W}_N^j, & \text{if } j \equiv k \pmod{N} \\ 0, & \text{else}; \end{cases}$
8. $\tau_s(D_M) * \tau_t(D_M) = N\tau_{s+t}(D_M)$;

PROOF: (2) Let $f, g \in \ell(\mathbb{Z}_N)$ and $n \in \mathbb{Z}_N$, then

$$(f * g)(n) = \sum_{j \in \mathbb{Z}_N} f(j)g(n - j)$$

and with the substitution $i := n - j$, noting that $n - \mathbb{Z}_N = \mathbb{Z}_N$

$$\begin{aligned} &= \sum_{i \in n - \mathbb{Z}_N} f(n-i)g(i) = \sum_{i \in \mathbb{Z}_N} g(i)f(n-i) \\ &= (g * f)(n). \end{aligned}$$

(7) By Definition 1.4 and Lemma 1.1(2), we have

$$\begin{aligned} \left(\mathbf{W}_N^j * \mathbf{W}_N^k\right)(n) &= \sum_{t \in \mathbb{Z}_N} W_N^{jt} W_N^{k(n-t)} = \sum_{t \in \mathbb{Z}_N} W_N^{kn} W_N^{(j-k)t} \\ &= W_N^{jn} \cdot \begin{cases} N & \text{if } j \equiv k \pmod{N} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Hence,

$$\mathbf{W}_N^j * \mathbf{W}_N^k = \begin{cases} N\mathbf{W}_N^j, & \text{if } j \equiv k \pmod{N} \\ 0, & \text{else.} \end{cases}$$

(8) With Definitions 1.4, 1.5 and Lemma 1.3(3),(7) follows immediately that

$$\begin{aligned}\tau_s(D_M) * \tau_t(D_M) &= \left(\sum_{k=-M}^{M} \mathbf{W}_N^k W_N^{-ks}\right) * \left(\sum_{\ell=-M}^{M} \mathbf{W}_N^\ell W_N^{-\ell t}\right) \\ &= \sum_{k=-M}^{M} \sum_{\ell=-M}^{M} (\mathbf{W}_N^k * \mathbf{W}_N^\ell) W_N^{-ks} W_N^{-\ell t} \\ &= \sum_{k=-M}^{M} N \mathbf{W}_N^k W_N^{-(s+t)} = N\tau_{s+t}(D_M).\end{aligned}$$

□

1.3 Translation, Reflection and Modulation

We now introduce the operations of translation, reflection and modulation (also known as frequency translation) on the space $\ell(\mathbb{Z}_N)$.

Definition 1.5 (Translation, Reflection, Modulation)
For $n_0 \in \mathbb{Z}_N$, we define the maps

$$\tau_{n_0}, \varrho, \mu_{n_0} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N)$$

by setting

$$\begin{aligned}\tau_{n_0}(f)(n) &:= f(n - n_0), \\ \varrho(f)(n) &:= f(-n) = f(N - n)\end{aligned}$$

and

$$\mu_{n_0}(f)(n) := \mathbf{W}_N^{n_0}(n) \cdot f(n).$$

for all $n \in \mathbb{Z}_N$. We call the maps τ_{n_0}, δ_{n_0}, and μ_{n_0}, *translation*, *reflection* and *modulation* (by n_0), respectively.

Lemma 1.4
The maps $\tau_{n_0}, \varrho, \mu_{n_0} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N)$ are (vector space) automorphisms on $\ell(\mathbb{Z}_N)$. τ_{n_0} and ϱ also preserve multiplication, i.e. they are $\mathbb{C}$-algebra automorphisms.

PROOF: The proof is straightforward. We just mention $\mu_{n_0}(f) = 0$ if and only if $f\mathbf{W}_N^{n_0} = 0$ if and only if $f(n)W_N^{n_0 n} = 0$ for all $n \in \mathbb{Z}_N$ if and only if $f(n) = 0$ for all $n \in \mathbb{Z}_n$, since $W_N^k \neq 0$ for all $k \in \mathbb{Z}_N$. □

1.4 The Discrete Fourier Transform

Definition 1.6 (Discrete Fourier Transform)
Given a Nth-order periodic sequence $f \in \ell(\mathbb{Z}_N)$, the *discrete Fourier transform* (DFT) of f is defined to be the function $F : \mathbb{Z}_N \to \mathbb{C}$, defined by

$$F(n) := \sum_{m=0}^{N-1} f(m)e^{-2\pi imn/N}$$

for all $n \in \mathbb{Z}_N$.

Since $e^{\pm 2\pi in} = 1$ for all $n \in \mathbb{Z}$, F is N-periodic. In fact

$$\begin{aligned} F(n) &= \sum_{m=0}^{N-1} f(m)e^{-2\pi imn/N} \\ &= \sum_{m=0}^{N-1} f(m)e^{-2\pi imn/N} \cdot e^{-2\pi in} \\ &= \sum_{m=0}^{N-1} f(m)e^{-2\pi im(n+N)/N} = F(n+N) \end{aligned}$$

for all $n \in \mathbb{Z}$.

The association $f \mapsto F$ defines an vector space automorphism $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N)$ as we will see below. It is often convenient to denote the DFT of f by $\hat{f}$. Using the basis functions $(\mathbf{W}_N^j \mid j = 0, \ldots, N-1)$, the DFT of f can be written compactly as

$$\hat{f} = \sum_{m \in \mathbb{Z}_N} f(m)\mathbf{W}_N^m$$

for all $n \in \mathbb{Z}_N$.

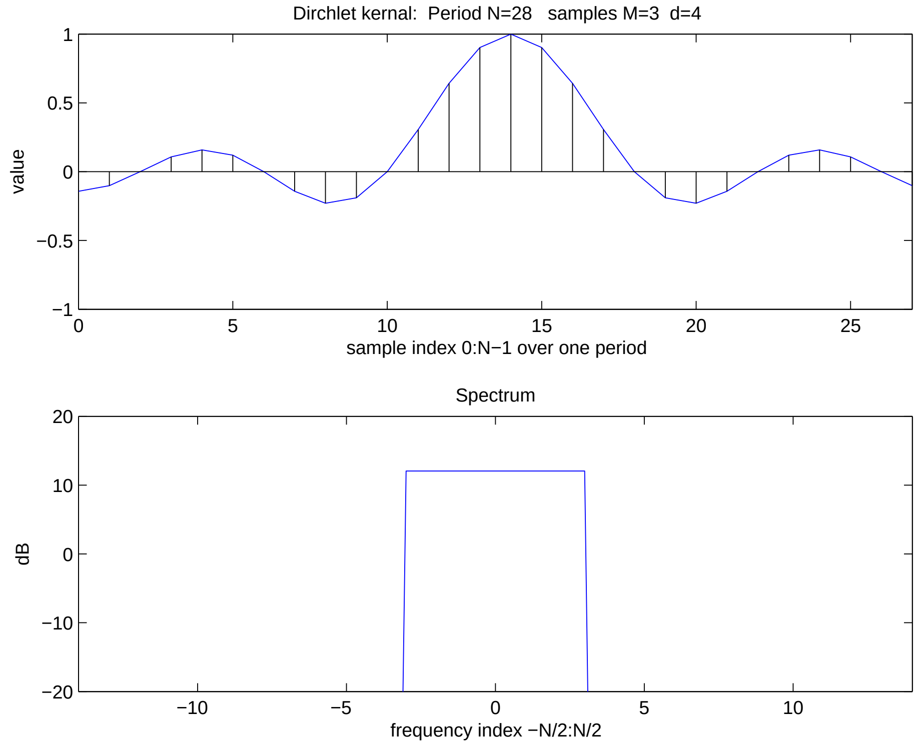
Theorem 1.3 (Inversion Formula for the DFT)
Let $N \in \mathbb{N}$ with $N > 1$ and suppose $f \in \ell(Z_N)$ has DFT F. Then

$$f = \frac{1}{N} \sum_{n \in \mathbb{Z}_N} F(n)\mathbf{W}_N^{-n}.$$

PROOF: For $0 \neq k \in \mathbb{Z}_N$, follows from Lemma 1.1 that

$$\sum_{n=0}^{N-1} W_N^{nk} = 0.$$

Figure 1.1: *The Discrete Fourier Transform*. The figure depicts the Dirichlet kernel and its Fourier transform. The domain of the tranformed function is centered at 0. Note that the Dirichlet kernel is "3-band-limited".



Hence

$$
\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} F(n)e^{2\pi imn/N} &= \frac{1}{N}\sum_{n=0}^{N-1} F(n)W_N^{-mn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{j=0}^{N-1} f(j)W_N^{jn}\right)W_N^{-mn} \\
&= \frac{1}{N}\sum_{j=0}^{N-1} f(j)\left(\sum_{n=0}^{N-1} W_N^{(j-m)n}\right) \\
&= \frac{1}{N}\sum_{j=0}^{N-1} f(j)\left(\begin{cases} N & \text{if } j=m \\ 0 & \text{else}\end{cases}\right) \\
&= \frac{1}{N}f(m)N = f(m).
\end{aligned}
$$

□

Suppose V and W are two finite-dimensional vector spaces over the field K with (ordered) bases $\mathcal{B} = (v_0, \ldots, v_n)$ and $\mathcal{C} = (w_0, \ldots, w_m)$, respectively. Let $\mathcal{E}^{(p)} = (e_0, \ldots, e_p)$ denote the canonical basis of K^p and let $\psi_{\mathcal{B}} : V \to K^n : v_j \mapsto e_j$ denote the canonical isomorphism which associates to each element $v \in V$ its coordinates with respect to the (ordered) basis $\mathcal{B}$. Moreover, for any homomorphism $f : V \to W$, let $M_{\mathcal{CB}}(f) \in K_{mn}$ denote the representation matrix of f with respect to the (ordered) basis $\mathcal{B}$ and $\mathcal{C}$, i.e.:

$$f(v) = \psi_{\mathcal{C}}^{-1}\Big(M_{\mathcal{CB}}(f)\psi_{\mathcal{B}}(v)\Big)$$

for all $v \in V$.

Lemma 1.5

1. *$\mathcal{F}\big(N^{-1/2}\mathbf{W}_N^k\big) = \sqrt{N}\,\mathbf{e}_{N-k}^{(N)}$ for all $k \in \mathbb{Z}_N$.*

2. *$\mathcal{F}\big(\mathbf{e}_k^{(N)}\big) = \mathbf{W}_N^k$ for all $k \in \mathbb{Z}_N$.*

3. *The representation matrix $M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F})$ of the DFT $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N)$ with respect to the orthonormal basis $\mathcal{E}_2^{(N)}$ and $\mathcal{E}_0^{(N)}$ in the time and frequency space, respectively, is given by*

$$\begin{aligned} M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F}) &= \left(\frac{1}{\sqrt{N}} W_N^{kn}\right)_{k,n\in\mathbb{Z}_N} \\ &= \frac{1}{\sqrt{N}}\begin{pmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{(N-1)2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix}. \end{aligned}$$

   Thus

$$\mathcal{F}(f) = \sqrt{N}\left(M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F})\right) f$$

   for all $f \in \ell(\mathbb{Z}_N)$.

4. *$\mathcal{D}_N := M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F})$ is unitary, i.e. $\mathcal{D}_N^{-1} = \overline{\mathcal{D}_N^t} = \overline{\mathcal{D}_N}$. $\mathcal{D}_N$ is called the DFT-matrix.*

5. *The representation matrix $M_{\mathcal{E}_2^{(N)}}(\mathcal{F}) := M_{\mathcal{E}_2^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F})$ of the DFT $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N)$ with respect to the orthonormal basis $\mathcal{E}_2^{(N)}$ in both the time and frequency space, is given by*

$$M_{\mathcal{E}_2^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F}) = \sqrt{N}\,\mathcal{D}_N$$

PROOF: (1) Let $k, n \in \mathbb{Z}_N$. then

$$\begin{aligned}\mathcal{F}\big(N^{-1/2}\mathbf{W}_N^k\big)(n) &= N^{-1/2}\sum_{m=0}^{N-1}\mathbf{W}_N^k(m)W_N^{mn}\\ &= N^{-1/2}\sum_{m=0}^{N-1}W_N^{(k+n)m}\end{aligned}$$

which, by Lemma 1.1, equals

$$= \begin{cases} N^{1/2} & \text{if } n \equiv -k \pmod N \\ 0 & \text{else,}\end{cases}$$

hence

$$\mathcal{F}\big(N^{-1/2}\mathbf{W}_N^k\big) = \sqrt{N}\,\mathbf{e}_{N-k}^{(N)}.$$

(2) Let $k, n \in \mathbb{Z}_N$. then

$$\mathcal{F}\big(\mathbf{e}_k^{(N)}\big)(n) = \sum_{m=0}^{N-1}\mathbf{e}_k(m)W_N^{mn} = W_N^{kn} = W_N^k(n).$$

(3) Since for $k \in \mathbb{Z}_N$ the image $\mathcal{F}\big(\mathbf{e}_k^{(N)}\big)$ of $\mathbf{e}_k^{(N)}$ under the DFT has the basis representation

$$\mathcal{F}\big(\mathbf{e}_k^{(N)}\big) = \mathbf{W}_N^k = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\mathbf{W}_N^k(n)\big(\sqrt{N}\mathbf{e}_n^{(N)}\big),$$

with respect to $\mathcal{E}_0^{(N)}$, it follows that

$$M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F}) = \left(\frac{1}{\sqrt{N}}W_N^{kn}\right)_{k,n\in\mathbb{Z}_N}.$$

Moreover, using the identification $\ell(\mathbb{Z}_N) = \mathbb{C}^N$ and setting $\mathcal{D}_N := M_{\mathcal{E}_0^{(N)}\mathcal{E}_2^{(N)}}(\mathcal{F})$, we see that the diagram

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\mathcal{F}} & \mathbb{C}^N \\ {\scriptstyle\psi_{\mathcal{E}_2}}\big\downarrow & & \big\downarrow{\scriptstyle\psi_{\mathcal{E}_0}} \\ \mathbb{C}^N & \xrightarrow[\mathcal{D}_N]{} & \mathbb{C}^N \end{array}$$

commutes. Note that $\psi_{\mathcal{E}_2} = \mathrm{id}\,\mathbb{C}$ and $\psi_{\mathcal{E}_0} = 1/\sqrt{N}\,\mathrm{id}\,\mathbb{C}$, thus $\psi_{\mathcal{E}_0}^{-1} = \sqrt{N}\,\mathrm{id}\,\mathbb{C}$. Hence

$$\begin{aligned}\mathcal{F}(f) &= \psi_{\mathcal{E}_0}^{-1}\mathcal{D}_N\psi_{\mathcal{E}_2}(f)\\ &= \sqrt{N}\,\mathcal{D}_N\,f.\end{aligned}$$

(5) Since by (2)

$$\mathcal{F}\big(\mathbf{e}_k^{(N)}\big) = \mathbf{W}_N^k = \sum_{n=0}^{N-1} \mathbf{W}_N^k(n)\, \mathbf{e}_n^{(N)},$$

we have

$$M_{\mathcal{E}_2\mathcal{E}_2}(\mathcal{F}) = \big(W_N^{kn}\big)_{k,n\in\mathbb{Z}_N} = \sqrt{N}\,\mathcal{D}_N.$$

□

Theorem 1.4 (Properties of the DFT)
For all $f, g \in \ell(\mathbb{Z}_N)$ and $n_0 \in \mathbb{Z}_N$, we have

1. *$\mathcal{F}$ is linear.*
2. *$\widehat{\tau_{n_0}(f)} = \mu_{n_0}(\hat{f})$.*
3. *$\widehat{\mu_{n_0}(f)} = \tau_{-n_0}(\hat{f})$.*
4. *$\mathcal{F} \circ \mathcal{F} = N\varrho$.*
5. *$\widehat{\varrho(f)} = \varrho(\hat{f})$.*

PROOF: (1) For $f, g \in \ell(\mathbb{Z}_N)$ and $\alpha, \beta \in \mathbb{R}$ follows immediately that

$$\begin{aligned}\mathcal{F}(\alpha f + \beta g) &= \sum_{m\in\mathbb{Z}_N} \big(\alpha f + \beta g\big)(m)\mathbf{W}_N^m = \sum_{m\in\mathbb{Z}_N} \big(\alpha f(m)\mathbf{W}_N^m + \beta g(m)\mathbf{W}_N^m\big) \\ &= \alpha \sum_{m\in\mathbb{Z}_N} f(m)\mathbf{W}_N^m + \beta \sum_{m\in\mathbb{Z}_N} g(m)\mathbf{W}_N^m = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)\end{aligned}$$

(2) Let $n_0 \in \mathbb{Z}_N$ and $f \in \ell(\mathbb{Z}_N)$, then follows

$$\widehat{\tau_{n_0}(f)} = \sum_{m\in\mathbb{Z}_N} \tau_{n_0}(f)(m)\mathbf{W}_N^m = \sum_{m\in\mathbb{Z}_N} f(m - n_0)\mathbf{W}_N^m$$

which equals, using the substitution $k := m - n_0$,

$$\begin{aligned}&= \sum_{k-n_0\in\mathbb{Z}_N} f(k)\mathbf{W}_N^{k+n_0} = \sum_{k\in\mathbb{Z}_N} f(k)\mathbf{W}_N^k\mathbf{W}_N^{n_0} \\ &= \hat{f}(N)\mathbf{W}_N^{n_0} = \mu_{n_0}(\hat{f}).\end{aligned}$$

(3) Again, let $n_0 \in \mathbb{Z}_N$ and $f \in \ell(\mathbb{Z}_N)$, then follows

$$\begin{aligned}\widehat{\mu_{n_0}(f)} &= \sum_{m\in\mathbb{Z}_N} \mu_{n_0}(f)(m)\mathbf{W}_N^m = \sum_{m\in\mathbb{Z}_N} f(m)\mathbf{W}_N^m\mathbf{W}_N^{n_0} \\ &= \sum_{m\in\mathbb{Z}_N} f(m)\mathbf{W}_N^{m+n_0} = \sum_{m\in\mathbb{Z}_N} f(m)\tau_{-n_0}\big(\mathbf{W}_N^{m_0}\big) \\ &= \tau_{-n_0}\left(\sum_{m\in\mathbb{Z}_N} f(m)\mathbf{W}_N^{m_0}\right) = \tau_{n_0}\big(\hat{f}\big).\end{aligned}$$

(4) For $f \in \ell(Z_N)$ follows

$$\big(\mathcal{F} \circ \mathcal{F}\big)(f) = \mathcal{F}\left(\sum_{m \in \mathbb{Z}_N} f(m)\mathbf{W}_N^m\right)$$

and using the linearity of $\mathcal{F}$

$$= \sum_{m \in \mathbb{Z}_N} f(m)\mathcal{F}(\mathbf{W}_N^m) = N \sum_{m \in \mathbb{Z}_N} f(m)\mathbf{e}_{N-m}^{(N)} = N\varrho(f).$$

(5) Applying the previous identity twice, we obtain:
$\varrho(\hat{f}) = 1/N\big(N\varrho\big(\mathcal{F}(f)\big)\big) = 1/N\mathcal{F}\big(\mathcal{F}(\mathcal{F}(f))\big) = 1/N\mathcal{F}\big(N\varrho(f)\big) = \mathcal{F}(\varrho(f)) = \widehat{\varrho(f)}$.

(6) If $f \in \ell(\mathbb{Z}_N)$ is odd, i.e. $f = -\varrho(f)$, then we obtain by applying the previous identity that $-\varrho(\hat{f}) = -\widehat{\varrho(f)} = -\widehat{(-f)} = \hat{f}$, i.e. $\hat{f}$ is also odd. The other case can be shown similarly. $\square$

Corollary 1.1
The map $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N) : f \mapsto F = \hat{f}$ is an automorphism.

PROOF: Trivial. $\square$

The discrete Fourier transforms interacts well with the convolution product.

Theorem 1.5 (DFT and Convolution)
For all $f, g \in \ell(\mathbb{Z}_N)$ and $k \in \mathbb{Z}_N$ holds

1. $\widehat{(f * g)} = \hat{f}\hat{g}$.
2. $\widehat{fg} = \frac{1}{N}\big(\hat{f} * \hat{g}\big)$.
3. $\tau_k(f) = \mathbf{e}_k^{(N)} * f$.
4. $\mathbf{W}_N^k * \mathbf{e}_\ell^{(N)} = W_N^{-k\ell}\mathbf{W}_N^k$.
5. $\widehat{\mu_k(f)} = \mathbf{e}_k^{(N)} * \hat{f}$.

PROOF: (1) Let $f, g \in \ell(\mathbb{Z}_N)$, then

$$\begin{aligned}
\mathcal{F}(f)\mathcal{F}(g) &= \left(\sum_{m=0}^{N-1} f(m)\mathbf{W}_N^m\right)\left(\sum_{k=0}^{N-1} g(k)\mathbf{W}_N^k\right) \\
&= \left[\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} f(m)g(k)\right]\mathbf{W}_N^{m+k}
\end{aligned}$$

substituting $j := m + k$ and using equation (1.10) we obtain

$$\begin{aligned}
&= \left[\sum_{m=0}^{N-1}\sum_{j=m}^{N-1+m} f(m)g(j-m)\right]\mathbf{W}_N^j \\
&= \left[\sum_{m=0}^{N-1}\sum_{j=0}^{N-1} f(m)g(j-m)\right]\mathbf{W}_N^j
\end{aligned}$$

which equals if we change the order of summation

$$\begin{aligned}
&= \sum_{j=0}^{N-1}\left[\sum_{m=0}^{N-1} f(m)g(j-m)\right]\mathbf{W}_N^j = \sum_{j=0}^{N-1}(f*g)(j)\mathbf{W}_N^j \\
&= \mathcal{F}(f*g)
\end{aligned}$$

(2) This statement also follows directly from Definition 1.4 and Lemma 1.3(3),(7):

$$\begin{aligned}
\tfrac{1}{N}\big(\hat{f} * \hat{g}\big) &= \tfrac{1}{N}\left(\sum_{m\in\mathbb{Z}_N} f(m)\mathbf{W}_N^m\right) * \left(\sum_{k\in\mathbb{Z}_N} g(k)\mathbf{W}_N^k\right) \\
&= \tfrac{1}{N}\sum_{m\in\mathbb{Z}_N} N f(m)g(m)\mathbf{W}_N^m = \widehat{fg}.
\end{aligned}$$

(3) Let $f \in \ell(\mathbb{Z}_N)$ and $k \in \mathbb{Z}_N$, then follows from Theorem 1.4(2) and the definition of μ_k that $\mathcal{F}\big(\tau_k(f)\big) = \mu_k\big(\hat{f}\big) = \mathbf{W}_N^k \cdot \hat{f}$. Since by Lemma 1.5(2) $\mathcal{F}\big(\mathbf{e}_k^{(N)}\big) = \mathbf{W}_N^k$ it follows that $\mathcal{F}\big(\tau_k(f)\big) = \mathcal{F}\big(\mathbf{e}_k^{(N)}\big)\mathcal{F}(f)$ and thus $\tau_k(f) = \mathbf{e}_k^{(N)} * f$.

(4) Let $n \in \mathbb{Z}_N$, then $\big(\mathbf{W}_N^k * \mathbf{e}_\ell^{(N)}\big)(n) = \tau_\ell(\mathbf{W}_N^k)(n) = \mathbf{W}_N^k(n-\ell) = W_N^{k(n-\ell)} = W_N^{kn}W_N^{-k\ell} = W_N^{-k\ell}\mathbf{W}_N^k(n)$.

(5) Again, let $f \in \ell(\mathbb{Z}_N)$ and $k \in \mathbb{Z}_N$. Then we conclude with Theorem 1.4(3) that $\mathcal{F}\big(\mu_k(f)\big) = \tau_k(\hat{f})$, which by Theorem 1.5(3) equals $\mathbf{e}_k^{(N)} * \hat{f}$, as claimed. □

Theorem 1.6 (Further Properties of the DFT)
If $f \in \ell(\mathbb{Z}_N)$ and $F = \mathcal{F}(f)$ denotes its discrete Fourier transform, then follows:

1. *If f is odd/even then F is odd/even.*
2. *If f is real then $F = \varrho(\overline{F})$. If F is real then $f = \varrho(\overline{f})$.*
3. *If f is imaginary then $F = -\varrho(\overline{F})$. If F is imaginary then $f = -\varrho(\overline{f})$.*
4. *If f is real and even then F is real and even.*
5. *If f is real and odd then F is imaginary and odd.*
6. *If f is imaginary and even then F is imaginary and even.*
7. *If f is imaginary and odd, then F is real and odd.*

Theorem 1.7 (Parseval's Formula)
The discrete Fourier transform $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N) : f \mapsto \hat{f}$ *is an isometric isomorphism from the (time) space* $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_2\big)$ *onto the (frequency) space* $\big(\ell(\mathbb{Z}_N), \langle\cdot,\cdot\rangle_0\big)$*, i.e.: if* $f, g \in \ell(\mathbb{Z}_N)$*, then* $\langle f, g\rangle_2 = \langle \hat{f}, \hat{g}\rangle_0$.

PROOF: Let $f, g \in \ell(\mathbb{Z}_N)$, then

$$\begin{aligned}
\langle f, g\rangle_2 &= \sum_{m=0}^{N-1} f(m)\overline{g(m)} \\
&= \sum_{m=0}^{N-1} \left[\left(\frac{1}{N} \sum_{n=0}^{N-1} \hat{f}(n) W_N^{mn} \right) \overline{\left(\frac{1}{N} \sum_{k=0}^{N-1} \hat{g}(k) W_N^{mk} \right)} \right] \\
&= \frac{1}{N}\frac{1}{N} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \hat{f}(n)\overline{\hat{g}(k)} W_N^{m(n-k)} \\
&= \frac{1}{N} \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \hat{f}(n)\overline{\hat{g}(k)} \frac{1}{N} \sum_{m=0}^{N-1} W_N^{m(n-k)} \\
&= \frac{1}{N} \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \hat{f}(n)\overline{\hat{g}(k)} \frac{1}{N} \begin{cases} N & \text{if } n = k \\ 0 & \text{else} \end{cases} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \hat{f}(n)\overline{\hat{g}(k)} \\
&= \langle \hat{f}, \hat{g}\rangle_0.
\end{aligned}$$

□

Corollary 1.2 (Parseval's Formula)
For all $f, g \in \ell(\mathbb{Z}_N)$ *holds*

1. $\displaystyle\sum_{m=0}^{N-1} f(m)\overline{g(m)} = \frac{1}{N}\sum_{n=0}^{N-1} \hat{f}(n)\overline{\hat{g}(n)}$
2. $\sqrt{N}\|f\|_2 = \|\hat{f}\|_2$
3. $\|f\|_2 = \|\hat{f}\|_0$.

PROOF: (1) This identity is just a restatement of $\langle f, g\rangle_2 = \langle \hat{f}, \hat{g}\rangle_0$. (2) Follows directly from (1) by choosing $g = f$. (3) Follows directly from (2) by the definition of the norms on $\ell(\mathbb{Z}_N)$ (see Theorem 1.1).

□

1.5 Reconstruction from Regular Samples in Bandlimited Spaces

For $M \in \mathbb{N}$ and $0 < M < N/2$, we define the set of all band-limited N-th order periodic sequences of bandwith M by setting

$$\mathcal{B}_M := \{f \in \ell(\mathbb{Z}_N) \mid \hat{f}(n) = 0 \text{ for } |n| > M\},$$

which constitutes a subspace of $\ell(\mathbb{Z}_N)$ of dimension $2M+1$, since

$$\mathcal{B}_M = \mathcal{F}^{-1}\Big(\big\langle \sqrt{N}\mathbf{e}_k \mid |k| \leq M \big\rangle\Big).$$

For $S \subset \mathbb{Z}_N$ let χ_S denote the characteristic function of S, i.e.

$$\chi_S(n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \in \mathbb{Z}_N \setminus S \end{cases}.$$

The orthogonal projection P_M from $\ell(\mathbb{Z}_N)$ onto $\mathcal{B}_M$ is given by

$$\widehat{P_M(f)} = \chi_{\{-M,\ldots,M\}}\hat{f}. \tag{1.14}$$

Note that $\tau_k(D_M) = \mathbf{e}_k^{(N)} * D_M$. Also note that D_M is M-band-limited, in fact, we have the following

Lemma 1.6 (Properties of the Dirichlet Kernel)

1. *$\mathcal{F}\big(\tau_{n_0}(D_M)\big) = \mathbf{W}_N^{n_0} N\chi_{\{-M,\ldots,M\}}$. In particular, $\mathcal{F}(D_M) = N\chi_{\{-M,\ldots,M\}}$, and $\tau_{n_0}(D_M) \in \mathcal{B}_M$ for all $n_0 \in \mathbb{Z}_N$.*

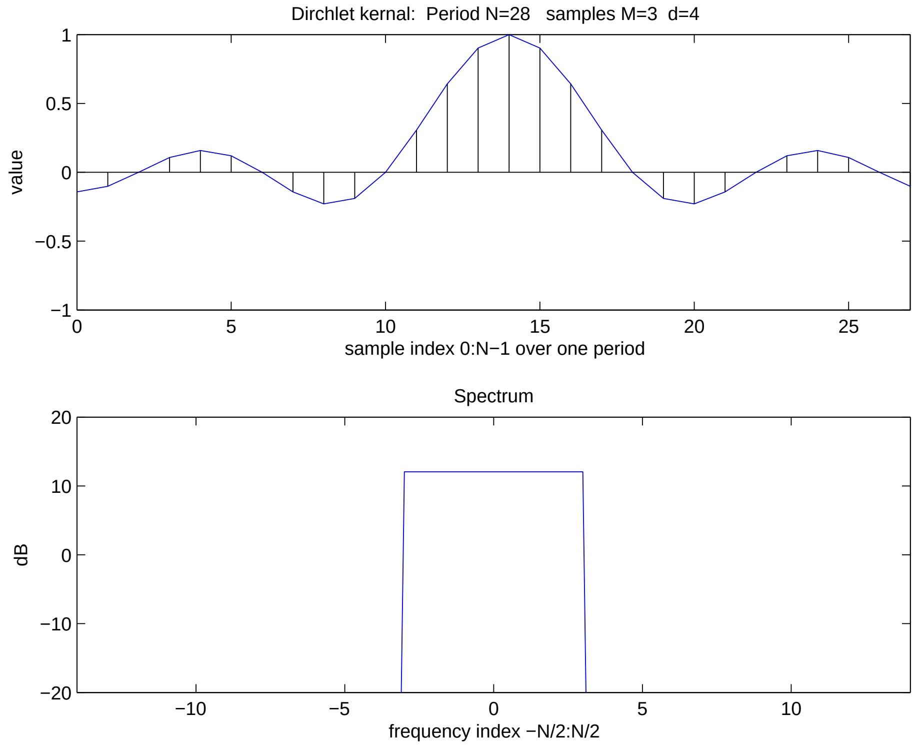
2. *$\langle \tau_{n_0}(D_M), \tau_{n_1}(D_M)\rangle_2 = ND_M(n_1 - n_2)$. Hence $\tau_{n_0}(D_M) \perp \tau_{n_1}(D_M)$ if and only if $(n_1 - n_0)(2M+1) = nN$ for some $n \in \mathbb{N}$.*

3. *$\|\tau_{n_0}(D_M)\|_2 = \sqrt{N(2M+1)}$ for all $n_0 \in \mathbb{Z}_N$.*

PROOF: (1) If $n_0 \in \mathbb{Z}_N$, then follows immediately from Lemma 1.4 that

$$\begin{aligned}
\mathcal{F}\big(\tau_{n_0}(D_M)\big) &= \mu_{n_0}\big(\mathcal{F}(D_M)\big) = \mu_{n_0}\left(\mathcal{F}\left(\sum_{k=-M}^{M} \mathbf{W}_N^{-k}\right)\right) \\
&= \mu_{n_0}\left(\sum_{k=-M}^{M} \mathcal{F}(\mathbf{W}_N^{-k})\right) = \mu_{n_0}\left(\sum_{k=-M}^{M} N\mathbf{e}_{N-k}^{(N)}\right) \\
&= N\mathbf{W}_N^{n_0} \sum_{k=-M}^{M} \mathbf{e}_k^{(N)} = N\mathbf{W}_N^{n_0}\chi_{\{-M\ldots,M\}}.
\end{aligned}$$

Figure 1.2: *The Dirichlet kernel*. The figure depicts the Dirichlet kernel and its spectrum.



(2) Let $n_0, n_1 \in \mathbb{Z}_N$, then follows with Parseval's Formula (Theorem 1.2), Theorem 1.4 and Lemma 1.6 that

$$\begin{aligned}
\langle \tau_{n_0}(D_M), \tau_{n_1}(D_M)\rangle_2 &= \langle \widehat{\tau_{n_0}(D_M)}, \widehat{\tau_{n_1}(D_M)}\rangle_0 \\
&= \langle \mu_{n_0}(\widehat{D_M}), \mu_{n_1}(\widehat{D_M})\rangle_0 \\
&= \langle \mathbf{W}_N^{n_0} N\chi_{\{-M,\dots,M\}}, \mathbf{W}_N^{n_1} N\chi_{\{-M,\dots,M\}}\rangle_0 \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{W}_N^{n_0}(n)N\chi_{\{-M,\dots,M\}} \overline{\mathbf{W}_N^{n_1}(n)N\chi_{\{-M,\dots,M\}}} \\
&= N\sum_{n=0}^{N-1} W_N^{-(n_1-n_0)n}\chi_{\{-M,\dots,M\}} \\
&= N\sum_{n=-M}^{M} W_N^{-(n_1-n_0)n}
\end{aligned}$$

which, by the definition of the Dirichlet kernel, equals

$$= ND_M(n_1 - n_0).$$

Thus $\langle \tau_{n_0}(D_M), \tau_{n_1}(D_M)\rangle_2 = 0$ if and only if $D_M(n_0 - n_1) = 0$ which holds if and only if there exists an integer $n \in \mathbb{Z}$, $n \neq 0$ such that $(2M+1)\pi(n_0 - n_1)/N = n\pi$ and thus if and only if $(n_0 - n_1)2(M+1) = nN$.

(3) Clearly, $\|\tau_{n_0}(D_M)\|_2 = \sqrt{\langle \tau_{n_0}(D_M), \tau_{n_0}(D_M)\rangle_2} = \sqrt{ND_M(n_0 - n_0)} = \sqrt{N(2M+1)}$. □

The following lemmas are crucial for the discrete sampling theorem which we will prove below.

Lemma 1.7
Suppose $0 < M < N/2$*, then*

1. $\tau_j \circ P_M = P_M \circ \tau_j$ *for any* $j \in \mathbb{Z}_N$*;*
2. $P_M(D_{M'}) = D_M$ *for any* $M \leq M' < N/2$*;*
3. $P_M\big(\tau_j(D_{M'})\big) = \tau_j(D_M)$*.*

PROOF: (1) Let $f \in \ell(\mathbb{Z}_N)$ and $j \in \mathbb{Z}_N$, then

$$\begin{aligned}(\widehat{\tau_j \circ P_M})(f) &= \widehat{\tau_j\big(P_M(f)\big)} = \mu_j(\widehat{P_M(f)}) = \mathbf{W}_N^j \cdot \big(\hat{f} \cdot \chi_{\{-M,\ldots,M\}}\big)\\ &= \big(\hat{f} \cdot \mathbf{W}_N^j\big) \cdot \chi_{\{-M,\ldots,M\}} = \mu_j(\hat{f})\chi_{\{-M,\ldots,M\}}\\ &= \widehat{\tau_j(f)}\chi_{\{-M,\ldots,M\}} = \widehat{P_M\big(\tau_j(f)\big)} = (\widehat{P_M \circ \tau_j})(f).\end{aligned}$$

Hence, $\tau_j \circ P_M = P_M \circ \tau_j$.

(2) From the definition of the orthogonal projection and Lemma 1.6 follows immediately

$$\begin{aligned}\widehat{P_M(D_{M'})} &= \widehat{D_{M'}}\chi_{\{-M,\ldots,M\}} = \big(N\chi_{\{-M,\ldots,M\}}\big)\chi_{\{-M,\ldots,M\}} = N\chi_{\{-M,\ldots,M\}}\\ &= \widehat{D_M}.\end{aligned}$$

Hence, $P_M(D_{M'}) = D_M$.

(3) This identity follow directly from (1) and (2). Indeed,

$$\begin{aligned}P_M\big(\tau_j(D_{M'})\big) &= (P_M \circ \tau_j)(D_{M'}) = (\tau_j \circ P_M)(D_{M'})\\ &= \tau_j\big(P_M(D_{M'})\big) = \tau_j(D_M)\end{aligned}$$

□

We now make some remarks on the topic of extending our sampling space $\ell(\mathbb{Z}_N)$. To this end let $N, N' \in \mathbb{N}$ with $N \leq N'$. We define the imbedding monomorphisms $i_0, i_2 : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_{N'})$ by $i_0(\mathbf{W}_N^j) := \mathbf{W}_{N'}^j$ and $i_2(\mathbf{e}_j^{(N)}) := \mathbf{e}_j^{(N')}$ for $j = 0, \ldots, N-1$, respectively. Then we can show the following

Lemma 1.8
Let $N, N' \in \mathbb{N}$ with $N \leq N'$ and let $M \in \mathbb{N}$ with $0 < M < N/2$. Then

1. *$\mathcal{F}_{N'} \circ i_2 = i_0 \circ \mathcal{F}_N$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \ell(\mathbb{Z}_N) & \xrightarrow{i_2} & \ell(\mathbb{Z}_{N'}) \\ {\scriptstyle \mathcal{F}_N}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_{N'}} \\ \ell(\mathbb{Z}_N) & \xrightarrow[i_0]{} & \ell(\mathbb{Z}_{N'}) \end{array}$$

2. *$i_0 \circ \tau_j^{(N)} = \tau_j^{(N')} \circ i_0$ for all $j = 0, \ldots, N-1$. In particular, $i_0\big(\mathbf{e}_j^{(N)} * f\big) = \mathbf{e}_j^{(N')} * i_0(f)$, and thus $i_0\big(\tau_j^{(N)}(D_M^{(N)})\big) = \tau_j\big(D_M^{(N')}\big)$ for all $j = 0, \ldots, N-1$.*

3. *$i_0\big(\mathcal{B}_M^{(N)}\big) = \mathcal{B}_M^{(N')}$.*

PROOF: (1) Using Lemma 1.5(2) and the definition of i_0, i_2, we obtain immediately that $\mathcal{F}_{N'}\big(i_2(\mathbf{e}_j^{(N)})\big) = \mathcal{F}_{N'}\big(\mathbf{e}_j^{(N')}\big) = \mathbf{W}_{N'}^j = i_0(\mathbf{W}_N^j) = i_0\big(\mathcal{F}_N(\mathbf{e}_J^{(N)}\big)$ for all $j = 0, \ldots, N-1$, which proves the assertion.

(2) By definition, we have for all $j, k = 0, \ldots, N-1$ and $n \in \mathbb{Z}_{N'}$ that $i_0\big(\tau_j^{(N)}(\mathbf{W}_N^k)\big)(n) = i_0(\mathbf{W}_N^k)(n - j) = \mathbf{W}_{N'}^{k(n-j)} = \tau_j^{(N')}(\mathbf{W}_{N'}^k)(n) = \tau_j^{(N')}\big(i_0(\mathbf{W}_N^k)\big)(n)$.

(3) By definition $\mathcal{F}\big(\mathcal{B}_M^{(N)}\big) = \langle \sqrt{N}\mathbf{e}_k^{(N)} \mid k \in \{-M, \ldots, M\}\rangle$. Hence, since $\mathcal{F}$ is an isomorphism,

$$\begin{aligned} \mathcal{B}_M^{(N)} &= \mathcal{F}^{-1}\Big(\langle \sqrt{N}\mathbf{e}_k^{(N)} \mid k \in \{-M, \ldots, M\}\rangle\Big) \\ &= \Big\langle \mathcal{F}^{-1}\big(\sqrt{N}\mathbf{e}_k^{(N)}\big) \mid k \in \{-M, \ldots, M\}\Big\rangle \end{aligned}$$

which equals by Lemma 1.5(1)

$$= \Big\langle \frac{1}{\sqrt{N}}\mathbf{W}_N^{-k} \mid k \in \{-M, \ldots, M\}\rangle = \langle \mathbf{W}_N^k \mid k \in \{-M, \ldots, M\}\Big\rangle.$$

Thus, utilizing that i_0 is a monomorphism,

$$i_0\big(\mathcal{B}_M^{(N)}\big) = \big\langle \mathbf{W}_{N'}^k \mid k \in \{-M, \ldots, M\}\big\rangle = \mathcal{B}_M^{(N')}.$$

$\square$

We now define a “right-inverse” $j_0 : \ell(\mathbb{Z}_{N'}) \to \ell(\mathbb{Z}_N)$ for i_0 by setting

$$j_0\big(\mathbf{W}_{N'}^j\big) := \begin{cases} \mathbf{W}_N^j & \text{if } j \in \{0, \ldots, N-1\} \\ 0 & \text{else.} \end{cases}$$

Then we can show

Lemma 1.9
With the assumption from above, we have

1. j_0 *is an epimorphism.*

2. $j_0 \circ i_0 = \mathrm{id}_{\ell(\mathbb{Z}_N)}$.

PROOF: (1) Clear from the definition. (2) Also clear, since by definition for all $j \in \{0, \ldots, N-1\}$ we have $j_0\big(i_0(\mathbf{W}_N^j\big) = j_0\big(\mathbf{W}_{N'}^j\big) = \mathbf{W}_N^j$. □

The following lemma presents a specialized form of the discrete classic sampling theorem.

Lemma 1.10 (Weak Form of the Discrete Sampling Theorem)
Suppose $0 < M < N/2$*,* $2(M+1)|N$ *and* $d := N/(2M+1)$*. Then*

1. *For all* $j, k = 0, 1, \ldots, 2M$ *we have*

$$\tau_{jd}(D_M)(kd) = \begin{cases} 0, & k \neq j \\ 2M+1, & k = j \end{cases}$$

2. $B_M := (\tau_{jd}(D_M) \mid j = 0, \ldots, 2M)$ *constitutes an orthogonal basis of* $\mathcal{B}_M$*;*

3. *Any* $f \in \mathcal{B}_M$ *has a basis representation of the form*

$$f = \sum_{j=0}^{2M} \frac{d}{N} f(jd) \tau_{jd}(D_M)$$

with respect to B_M*, i.e.*

$$f = \frac{d}{N}\Big(f_s * D_M\Big),$$

where $f_s := \sum_{j=0}^{2M} f(jd)\mathbf{e}_{jd}^{(N)} \in \ell(\mathbb{Z}_N)$.

PROOF: (1) Let $j, k = 0, 1, \ldots, 2M$, then follows by Lemma 1.2 that

$$\begin{aligned}\tau_{jd}(D_M)(kd) &= D_M(kd - jd) = D_M\big((k-j)d\big) \\ &= \begin{cases} \sin\left(\frac{(2M+1)\pi(k-j)d}{N}\right), & k \neq j \\ 2M+1, & k = j \end{cases}\end{aligned}$$

and since $(2M+1)d = N$

$$= \begin{cases} \sin\left(\pi(k-j)\right) = 0, & k \neq j \\ 2M+1, & k = j \end{cases}$$

(2) If $j_1, j_2 \in \{0, \ldots, 2M\}$, $j_1 \neq j_2$, then $(j_2 d - j_1 d)(2M+1) = (j_2 - j_1)N$, and by Lemma 1.6 $\langle \tau_{j_1 d}(D_M), \tau_{j_2 d}(D_M)\rangle_2 = 0$. Hence B_M is an orthogonal family consisting of $2M+1$ elements. Since $\dim \mathcal{B}_M = 2M+1$, B_M is a basis of $\mathcal{B}_M$.

(3) Suppose $f \in \mathcal{B}_M$ is an arbitrary M-bandlimited function. Since B_M is a basis of $\mathcal{B}_M$ there exist unique coefficients $a_j \in \mathbb{C}$ such that

$$f = \sum_{j=0}^{2M} a_j \tau_{jd}(D_M).$$

Since B_M is an orthogonal basis, these coefficients are of the form

$$a_j = \frac{\langle f, \tau_{jd}(D_M)\rangle_2}{\langle \tau_{jd}(D_M), \tau_{jd}(D_M)\rangle_2}.$$

By Lemma 1.6, $\langle \tau_{jd}(D_M), \tau_{jd}(D_M)\rangle_2 = \|\tau_{jd}(D_M)\|_2^2 = N(2M+1)$. On the other hand, by Parseval's Theorem, we have

$$\langle f, \tau_{jd}(D_M)\rangle_2 = \langle \hat{f}, \widehat{\tau_{jd}(D_M)}\rangle_0$$

which by Lemma 1.6 equals

$$= \langle \hat{f}, \mathbf{W}_N^{jd} N \chi_{\{-M,\ldots,M\}}\rangle_0$$
$$= N\left(\frac{1}{N}\sum_{n=0}^{N-1} \hat{f}(n) \cdot \overline{W_N^{jdn}\chi_{\{-M,\ldots,M\}}}\right)$$

and, since f is M-bandlimited,

$$= N\left(\frac{1}{N}\sum_{n=0}^{N-1} \hat{f}(n)\mathbf{W}_N^{-n}(jd)\right)$$

which by Theorem 1.3 equals

$$= Nf(jd).$$

Hence, $a_j = Nf(jd)/\big(N(2M+1)\big) = f(jd)/(2M+1)$, which equals $d/Nf(jd)$, since $d = N/(2M+1)$. □

We can now formulate and prove the discrete Sampling Theorem:

Theorem 1.8 (Discrete Sampling Theorem)
If $0 < M < N/2$ and $d|N$ with $d \leq N/(2M+1)$ and if $r := N/d - 1$, then for all $f \in \mathcal{B}_M$

$$f = \sum_{j=0}^{r} \frac{d}{N} f(jd)\tau_{jd}(D_M)$$

with respect to B_M, i.e.

$$f = \frac{d}{N}\Big(f_s * D_M\Big),$$

where $f_s := \sum_{j=0}^{r} f(jd)\mathbf{e}_{jd}^{(N)} \in \ell(\mathbb{Z}_N)$.

Note that in this setting the family $(\tau_{jd}(D_M) \mid j = 0, \ldots, r)$ *in general is neither orthogonal nor linearly independent over* $\mathbb{C}$*!*

PROOF: Since $d|N$, there exists an integer $r + 1 \in \mathbb{N}$ such that $d(r + 1) = N$. Let $M' := 1/2r$ (r has to be even!), then $(2M' + 1)d = (r + 1)d = N$, in particular, $(2M' + 1)|N$. Moreover, since $d \leq N/(2M + 1)$, it follows that $2(M + 1)d \leq N = (2M' + 1)d$, hence $2M + 1 \leq 2M' + 1$ i.e. $M \leq M'$. Thus $\mathcal{B}_M \subseteq \mathcal{B}_{M'}$ is a subspace of $\mathcal{B}_{M'}$. Therefore, if $f \in \mathcal{B}_M$, then $f \in \mathcal{B}_{M'}$, and since $B_{M'} = \big(\tau_j(D_{M'}) \mid j = 0, \ldots, 2M' = r\big)$ by Lemma 1.10 is an orthogonal basis of $\mathcal{B}_{M'}$, f has the basis expansion

$$f = \sum_{j=0}^{r} \frac{N}{d} f(jd)\tau_{jd}(D_{M'}).$$

We now apply the orthogonal projection P_M from $\ell(Z_N)$ onto $\mathcal{B}_M$ to this equation and note that $P_M(f) = f$, since $f \in \mathcal{B}_M$. Then, follows by the linearity of P_M

$$f = P_M\left(\sum_{j=0}^{r} \frac{N}{d} f(jd)\tau_{jd}(D_{M'})\right) = \sum_{j=0}^{r} \frac{N}{d} f(jd)P_M\big(\tau_{jd}(D_{M'})\big)$$

and by Lemma 1.7(1)

$$= \sum_{j=0}^{r} \frac{N}{d} f(jd)\tau_{jd}\big(P_M(D_{M'})\big)$$

and by applying Lemma 1.7(2,) finally,

$$= \sum_{j=0}^{r} \frac{N}{d} f(jd)\tau_{jd}D_M.$$

□

1.6 Reconstruction from Regular Samples in Bandpass Spaces

The Sampling Theorem established in the previous section has a natural extension to bandpass spaces. We first consider the case of an asymmetric spectrum obtained by frequency shifting or modulation by a fixed “frequency” $k_0 \in \mathbb{Z}_N$, i.e. the associated frequency $\frac{2\pi k_0}{N}$.

As before, let $N, M \in \mathbb{N}$, $M < N/2$ and $d(2M+1) = N$. Moreover, let $k_0 \in \mathbb{Z}_N$. As shown above (cf. Lemma 1.4), modulation μ_{k_0} by the "frequency" k_0 constitutes an automorphishm on $\ell(\mathbb{Z}_N)$. Therefore, the image

$$\begin{aligned}\mathcal{B}_M^{k_0} :=& \mu_{k_0}(\mathcal{B}_M) = \{\mu_{k_0}(f) \in \ell(\mathbb{Z}_N) \mid \hat{f}(n) \neq 0 \text{ only for } n \in [-M, M]\} \\ =& \{f \in \ell(\mathbb{Z}_N) \mid \hat{f}(n) \neq 0 \text{ only for } n \in [-M-k_0, M-k_0]\}\end{aligned}$$

of the space $\mathcal{B}_M$ of all M-bandlimited functions in $\ell(\mathbb{Z}_N)$ under modulation by k_0 constitutes a vector space isomorphic to $\mathcal{B}_M$. Moreover, we can show

Theorem 1.9
Suppose $N, M \in \mathbb{N}$, $M < N/2$, $d(2M+1) = N$ *and* $k_0 \in \mathbb{Z}_N$. *Then*

$$\begin{aligned} g &= \sum_{j=0}^{2M} \frac{d}{N\mathbf{W}_N^{k_0}(jd)} g(jd)\big(\mu_{k_0} \circ \tau_{jd}\big)(D_M) \\ &= \sum_{j=0}^{2M} \frac{d}{N\mathbf{W}_N^{k_0}(jd)} g(jd)\tau_{jd}(D_M)\mathbf{W}_N^{k_0}\end{aligned}$$

for all $g \in \mathcal{B}_M^{k_0}$.

PROOF: Let $g \in \mathcal{B}_M^{k_0}$. Then there exists a unique $f \in \mathcal{B}_N$ such that $g = \mu_{k_0}(f)$. By Lemma 1.10(3), f has the basis expansion

$$f = \sum_{j=0}^{2M} \frac{d}{N} f(jd)\tau_{jd}(D_M)$$

and thus

$$\begin{aligned} g = \mu_{k_0}(f) &= \mu_{k_0}\left(\sum_{j=0}^{2M} \frac{d}{N} f(jd)\tau_{jd}(D_M)\right) \\ &= \sum_{j=0}^{2M} \frac{d}{N} f(jd)\mu_{k_0}\big(\tau_{jd}(D_M)\big)\end{aligned}$$

By Lemma 1.10(1)

$$\begin{aligned} g(kd) &= \sum_{j=0}^{2M} \frac{d}{N} f(jd)\mu_{k_0}\big(\tau_{jd}(D_M)(kd)\big) \\ &= \frac{d}{N} f(kd)\mathbf{W}_N^{k_0}(kd)(2M+1) = f(kd)\mathbf{W}_N^{k_0}(kd)\end{aligned}$$

Hence

$$f(kd) = \frac{g(kd)}{\mathbf{W}_N^{k_0}(kd)}$$

and thus

$$g = \sum_{j=0}^{2M} \frac{d}{N\mathbf{W}_N^{k_0}(jd)} g(jd)\mu_{k_0}\big(\tau_{jd}(D_M)\big)$$

as claimed. □

Again, let $N, M \in \mathbb{N}$, $M < N/2$ and $d(2M+1) = N$. We now pick $2M+1$ "frequencies" $k_0, k_1, \ldots, k_{2M} \in \mathbb{Z}_N$. Consider the homomorphism

$$\begin{aligned}\varphi := &\varphi_{(k_0,\ldots,k_{2M})} : \mathcal{B}_M \to \ell(\mathbb{Z}_N)\\ \tau_{jd}(D_M) \mapsto &\ \mu_{k_j}\big(\tau_{jd}(D_M)\big) = \tau_{jd}(D_M)\mathbf{W}_N^{k_j}\end{aligned}$$

for $j = 0, 1, \ldots, 2M$. Note that φ modulates each elements of the canonical orthogonal basis $B_M := \big(\tau_{jd}(D_M) \mid j = 0, \ldots, 2M\big)$ of $\mathcal{B}_M$ individually and in general is not an isomorphism. Thus the dimension of the vector space

$$\mathcal{B}' := \mathcal{B}(k_0, \ldots, k_{2M}) := \varphi(\mathcal{B}_M) = \langle \tau_{jd}(D_M)\mathbf{W}_N^{k_j} \mid j = 0, 1, \ldots, 2M\rangle$$

is less than or equal to $2M+1$. However, we can show

Theorem 1.10
Suppose that $N, M \in \mathbb{N}$, $M < N/2$ and $d(2M+1) = N$ and $k_0, k_1, \ldots, k_{2M} \in \mathbb{Z}_N$. Then

$$g = \sum_{j=0}^{2M} \frac{d}{N} g(jd) W_N^{-k_j jd} \tau_{jd}(D_M)\mathbf{W}_N^{k_j}$$

for all $g \in \mathcal{B}(k_0, \ldots, k_{2M})$.

PROOF: Suppose $g \in \mathcal{B}(k_0, \ldots, k_{2M})$. Then there exists a (not necessarily unique) function $f \in \mathcal{B}_M$ such that $g = \varphi(f)$. As above, we conclude with Lemma 1.10 and the linearity of φ that

$$\begin{aligned}g = \varphi(f) &= \varphi\left(\sum_{j=0}^{2M} \frac{d}{N} f(jd)\tau_{jd}(D_M)\right)\\ &= \sum_{j=0}^{2M} \frac{d}{N} f(jd)\tau_{jd}(D_M)\mathbf{W}_N^{k_j}\end{aligned}$$

which equals

$$= \sum_{j=0}^{2M} \frac{d}{N} g(jd) W_N^{-k_j jd} \tau_{jd}(D_M)\mathbf{W}_N^{k_j}$$

since

$$\begin{aligned} g(kd) &= \sum_{j=0}^{2M} \frac{d}{N} f(jd)\tau_{jd}(D_M)(kd)\mathbf{W}_N^{k_j}(kd) \\ &= \sum_{j=0}^{2M} \frac{d}{N} f(jd)\mathbf{W}_N^{k_j}(kd) \cdot \begin{cases} 0 & k \neq j \\ 2M+1 & k = j \end{cases} \\ &= \frac{d}{N} f(kd)\mathbf{W}_N^{k_k}(kd)(2M+1) = f(kd)W_N^{k_k kd} \end{aligned}$$

and thus

$$f(jd) = g(jd)W_N^{-k_j jd}.$$

□

1.7 Reconstruction from Irregular Samples in Bandlimited Spaces

We now turn to the problem of reconstructing a bandlimited function from irregularly spaced samples. The sampling map which assigns to each bandlimited function its sample vector at the (irregularly spaced) sampling location is an isomorphism from the vector space $\mathcal{B}_M$ of all M-bandlimited functions onto the space $\ell(\mathbb{Z}_{2M+1})$. This isomorphism can be used to transform irregular samples into regular samples from which the corresponding function can be recovered using Lemma 1.10.

As before, let $N, M \in \mathbb{N}$, $M < N/2$, $(2M+1) \mid N$ and $d := N/(2M+1)$. Moreover, let $B_M := \left(D_M * \mathbf{e}_{jd}^{(N)} \mid j = 0, 1, \ldots, 2M+1\right)$ denote the canonical orthogonal basis of $\mathcal{B}_M$. Finally, let $0 \leq n_0 < n_1 < n_2 < \cdots < n_{2M} \leq N-1$ be the (irregular) sampling locations where we assume the function values of the elements of $\mathcal{B}_M$ to be known.

Definition 1.7 (Sampling Homomorphism)
We call the unique homomorphism $s_M : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ defined by linear extension through the association

$$D_M * \mathbf{e}_{jd}^{(N)} \mapsto \begin{pmatrix} \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_0) \\ \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_1) \\ \vdots \\ \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_{2M}) \end{pmatrix}$$

$(j = 0, 1, \ldots, 2M)$ the *sampling homomorphism* of $\mathcal{B}_M$ with respect to $n_0, n_1, \ldots, n_{2M}$. The representation matrix

$$S_M = M_{\mathcal{E}_2^{(2M+1)}, B_M}(s_M) = \begin{pmatrix} \left(D_M * \mathbf{e}_0^{(N)}\right)(n_0) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)}\right)(n_0) \\ \vdots & & \vdots \\ \left(D_M * \mathbf{e}_0^{(N)}\right)(n_{2M}) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)}\right)(n_{2M}) \end{pmatrix}$$

is called the *sampling matrix* of $\mathcal{B}_M$ with respect to $n_0, n_1, \dots, n_{2M}$. Recall that $\mathcal{E}_2^{(2M+1)} = \big(\mathbf{e}_0^{(2M+1)}, \dots, \mathbf{e}_{2M}^{(2M+1)}\big)$ denotes the canonical basis of $\ell(\mathbb{Z}_{2M+1})$.

The *coordinate isomorphism* $\varphi_{B_M} : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ of $\mathcal{B}_M$ relative to B_M, defined by the association $D_M * \mathbf{e}_j^{(N)} \mapsto \mathbf{e}_j^{(2M+1)}$ for $j = 0, 1, \dots, 2M$, assigns to each function $f \in \mathcal{B}_M$ its coordinate vector $\varphi_{B_M}(f)$. We know from the Discrete Sampling Theorem (Theorem 1.8) that the coordinates of f relative to B_M are given by

$$\varphi_{B_M}(f)(j) = \frac{N}{d} f(jd)$$

for $j = 0, 1, \dots, 2M$, i.e. they are just the sampling values of f at the regular locations $0, d, 2d, 3d, \dots, 2Md$ scaled by $\frac{N}{d}$.

Theorem 1.11
The map $s_M : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ is an isomorphism.

$$\begin{tikzcd}
\mathcal{B}_M \arrow[r, "s_M"] \arrow[d, "\varphi_{B_M}"'] & \ell(\mathbb{Z}_{2M+1}) \\
\ell(Z_{2M+1}) \arrow[ur, "L_{S_M} : X \mapsto S_M \cdot X"']
\end{tikzcd}$$

PROOF: Clearly, s_M is linear by construction. Since $\dim \mathcal{B}_M = \dim \ell(\mathbb{Z}_{2M+1}) = 2M + 1$, we only have to show that $\ker(s_M) = \{0\}$. To this end, let $f \in \ker(s_M)$. We have to show that $f = 0$.

$\big(\mathbf{e}_k^{(N)} \mid k \in \{-M, \dots, M\}\big)$ is a basis of $\mathcal{F}\big(\mathcal{B}_M\big)$. Since $\mathcal{F}$ is an isomorphism and $\mathcal{F}^{-1}\big((e)_k^{(N)}\big) = \frac{1}{N}\mathbf{W}_N^{-k}$, the family $\mathcal{O}_2^{(2M+1)} := (\mathbf{W}_N^k \mid k \in \{-M, \dots, M\})$ is a basis of $\mathcal{B}_M$. Therefore, there exist unique coefficients $x_{-M}, \dots, x_M \in \mathbb{C}$ such that

$$f = \sum_{n=-M}^{M} x_n \mathbf{W}_N^n.$$

Since $f \in \ker(s_M)$ the coefficient vector $X := (x_{-M}, \dots, x_M)^t \in \mathbb{C}^{2M+1}$ of f has to satisfy the equations

$$0 = f(n_k) = \sum_{n=-M}^{M} W_N^{n_k n} x_n$$

for $k = 0, \dots 2M$ or equivalently, the matrix equation

$$0 = AX,$$

where

$$A := \big(W_N^{n_k n}\big) = \begin{pmatrix} W_N^{n_0(-M)} & \cdots & W_N^{n_0 M} \\ W_N^{n_1(-M)} & \cdots & W_N^{n_1 M} \\ \vdots & & \vdots \\ W_N^{n_{2M}(-M)} & \cdots & W_N^{n_{2M} M} \end{pmatrix}.$$

If we set $\alpha_k := W_N^{n_k}$, then the entries $a_{kn} := W_N^{n_k n}$ of A can be written as

$$a_{kn} = (\alpha_k)^n$$

and the matrix A as

$$A = \begin{pmatrix} \alpha_0^{-M} & \cdots & a_0^{-1} & 1 & a_0 & \cdots & a_0^M \\ \alpha_1^{-M} & \cdots & a_1^{-1} & 1 & a_1 & \cdots & a_1^M \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{2M}^{-M} & \cdots & a_{2M}^{-1} & 1 & a_{2M} & \cdots & a_{2M}^M \end{pmatrix}$$

$$= \operatorname{diag}\bigl(\alpha_0^{-M}, \ldots, \alpha_{2M}^{-M}\bigr) \cdot V(\alpha_0, \ldots, \alpha_{2M})$$

its first factor being the diagonal matrix

$$\operatorname{diag}\bigl(\alpha_0^{-M}, \ldots, \alpha_{2M}^{-M}\bigr) = \begin{pmatrix} a_0^{-M} & 0 & 0 & \cdots & 0 \\ 0 & a_1^{-M} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & & \cdots & a_{2M}^{-M} \end{pmatrix}$$

and the second factor, the Vandermonde matrix

$$V(\alpha_0, \ldots, \alpha_{2M}) = \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^{2M} \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{2M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{2M} & \alpha_{2M}^2 & \cdots & \alpha_{2M}^{2M} \end{pmatrix}.$$

In order to show that $f = 0$, we need to verify that $\ker(A) = \{0\}$ which is equivalent to $\det(A) \neq 0$. However this can now be easily established:

$$\begin{aligned} \det(A) &= \det\Bigl(\operatorname{diag}\bigl(\alpha_0^{-M}, \ldots, \alpha_{2M}^{-M}\bigr) \cdot V(\alpha_0, \ldots, \alpha_{2M})\Bigr) \\ &= \det\Bigl(\operatorname{diag}\bigl(\alpha_0^{-M}, \ldots, \alpha_{2M}^{-M}\bigr)\Bigr) \det\bigl(V(\alpha_0, \ldots, \alpha_{2M})\bigr) \\ &= \prod_{k=0}^{2M} \alpha_k^{-M} \prod_{0 \le s < t \le 2M} (\alpha_t - \alpha_s). \end{aligned}$$

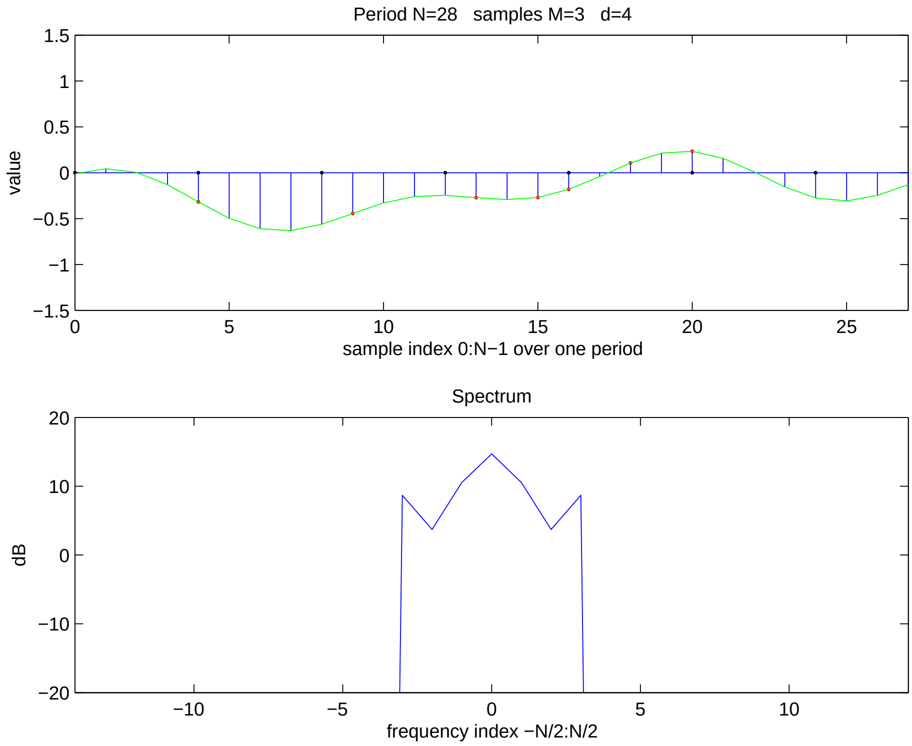
Since by definition, $\alpha_k = W_N^{n_k} \neq 0$ for all $k = 0, \ldots, 2M$, the first product is unequal zero. Moreover, $\alpha_t - \alpha_s = 0$ if and only if $W_N^{n_s} - W_N^{n_t} = 0$ which is equivalent to $W_N^{n_t - n_s} = 1$ and thus $n_t - n_s \in N\mathbb{Z}$. By assumption $0 \le n_0 < n_1 < \cdots < n_{2M} \le N - 1$, thus $n_t - n_s \in N\mathbb{Z}$ implies that $s = t$. Therefore, the second product is also unequal to zero and $\det(A) \neq 0$ as claimed. $\square$

Corollary 1.3 (Transforming Irregular Samples into Regular Samples)
If $X_f \in \ell(\mathbb{Z}_N)$ is the sampling vector of a bandlimited function $f \in \mathcal{B}_M$ with respect to the (irregularly spaced) locations $0 \le n_0 < n_1 < \cdots < n_{2M} \le N - 1$ then

$$Y_f := S_M^{-1} \cdot X_f$$

is the sampling vector of f with respect to the regular locations $0, d, 2d, \ldots, 2Md$ scaled by $\frac{N}{d}$.

Figure 1.3: *Reconstruction from irregular samples.* The red dots in the upper graph are the irregular samples of the signal. The black dots on the horizontal axis mark the locations of the regular samples into which the irregular samples are transformed by $(S_M)^{-1}$. The dark-blue graph in the lower diagram depicts the spectrum of the signal.



PROOF: Since all the maps $s_M, \varphi_{\mathcal{B}_M}$ and L_{S_M} are isomorphisms and $s_M = L_{S_M} \circ \varphi_{\mathcal{B}_M}$, it follow that $s_M \circ \varphi_{\mathcal{B}_M}^{-1} = L_{S_M}$ and thus

$$\varphi_{\mathcal{B}_M} \circ s_M^{-1} = L_{S_M}^{-1} = L_{(S_M)^{-1}},$$

Where L_{S_M} denotes the linear map defined by the matrix S_M, i.e. $L_{S_M}(X) = S_M \cdot X$. Hence

$$\varphi_{\mathcal{B}_M}(f) = \varphi_{\mathcal{B}_M}\left(s_M^{-1}(X_f)\right) = S_M^{-1} \cdot X_f.$$

Note that $\varphi_{\mathcal{B}_M}(f)$ is the coordinate vector of f relative to the basis B_M, i.e.

$$\frac{N}{d}\begin{pmatrix} f(0) \\ f(d) \\ f(2d) \\ \vdots \\ f(2Md) \end{pmatrix} = \varphi_{\mathcal{B}_M}(f) = S_M^{-1} \cdot X_f.$$

□

In order to transform irregular samples into regular ones, we first have to invert the sampling matrix S_M defined above. This can be facilitated by the following factorization obtained from suitable change of base for the space $\mathcal{B}_M$.

Theorem 1.12
The sampling matrix S_M has the factorization

$$S_M = HDV$$

where

$$H = \begin{pmatrix} 1 & W_N^{Md} & W_N^{M(2d)} & \cdots & W_N^{M2Md} \\ 1 & W_N^{(M-1)d} & W_N^{(M-1)(2d)} & \cdots & W_N^{(M-1)2Md} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(-M)d} & W_N^{(-M)(2d)} & \cdots & W_N^{(-M)2Md} \end{pmatrix}$$

$$D = \operatorname{diag}\Big(W_N^{-n_0 M}, W_N^{-n_1 M}, \ldots, W_N^{-n_{2M} M}\Big)$$

$$V = \begin{pmatrix} 1 & W_N^{n_0} & W_N^{n_0 2} & W_N^{n_0 3} & \cdots & W_N^{n_0 2M} \\ 1 & W_N^{n_1} & W_N^{n_1 2} & W_N^{n_1 3} & \cdots & W_N^{n_1 2M} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 1 & W_N^{n_{2M}} & W_N^{n_{2M} 2} & W_N^{n_{2M} 3} & \cdots & W_N^{n_{2M} 2M} \end{pmatrix}.$$

Moreover, the matrix $\frac{1}{\sqrt{2M+1}} H$ is hermitian, i.e. $H^{-1} = \frac{1}{2M+1}\left(\overline{H}\right)^t$, the diagonal matrix D has all nonzero entries and V is a Vandermonde matrix.

$$\begin{tikzcd} \mathcal{B}_M \arrow[r, "id"] \arrow[d, "\varphi_{B_M}", "B_M"'] & \mathcal{B}_M \arrow[r, "s_M"] \arrow[d, "\varphi_{\mathcal{O}_2}", "\mathcal{O}_2"'] & \ell(\mathbb{Z}_{2M+1}) \arrow[d, "id", "\mathcal{E}_2"'] \\ \ell(Z_{2M+1}) \arrow[r, "L_H"'] & \ell(Z_{2M+1}) \arrow[r, "L_{V'}"'] & \ell(Z_{2M+1}) \end{tikzcd}$$

PROOF: As above, let $B_M = \big(D_M * \mathbf{e}_{jd}^{(N)} \mid k = 0, \ldots, 2M\big)$ denote the canonical, orthogonal basis of $\mathcal{B}_M$ and $\mathcal{O}_2^{(2M+1)} = \big(\mathbf{W}_N^k \mid k \in \{-M, \ldots, M\}\big)$ the basis of $\mathcal{B}_M$ introduced in the proof of Theorem 1.11. We first compute the representation matrices $M_{\mathcal{O}_2, B_M}(\mathrm{id})$ and $M_{\mathcal{E}_2, \mathcal{O}_2}(s_M)$ of the identity map id relative to the basis B_M and $\mathcal{O}_2$ (base change) and the sampling homomorphism s_M relative to $\mathcal{O}_2$ and $\mathcal{E}_@$, respectively. We begin with the first matrix and compute the basis representation for the image $\mathrm{id}\big(D_M * \mathbf{e}_{jd}^{(N)}\big)$ of each basis vector $D_M * \mathbf{e}_{jd}^{(N)}$ of $\mathcal{B}_M$ under id relative to the basis $\mathcal{O}_2$. Clearly,

$$\begin{aligned} \mathrm{id}\Big(D_M * \mathbf{e}_{jd}^{(N)}\Big) = D_M * \mathbf{e}_{jd}^{(N)} &= \left(\sum_{k=-M}^{M} \mathbf{W}_N^{-k}\right) * \mathbf{e}_{jd}^{(N)} \\ &= \sum_{k=-M}^{M} \Big(\mathbf{W}_N^k * \mathbf{e}_{jd}^{(N)}\Big) = \sum_{k=-M}^{M} W_N^{-kjd} \mathbf{W}_N^k \end{aligned}$$

using Theorem 1.5(4). Thus the representation matrix of id relative to B_M and $\mathbf{O}_2$ is

$$H := M_{\mathcal{O}_2,B_M}(\mathrm{id}) = \begin{pmatrix} 1 & W_N^{Md} & W_N^{M(2d)} & \cdots & W_N^{M2Md} \\ 1 & W_N^{(M-1)d} & W_N^{(M-1)(2d)} & \cdots & W_N^{(M-1)2Md} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(-M)d} & W_N^{(-M)(2d)} & \cdots & W_N^{(-M)2Md} \end{pmatrix}.$$

Since

$$s_M(\mathbf{W}_N^k) = \begin{pmatrix} \mathbf{W}_N^k(n_0) \\ \mathbf{W}_N^k(n_0) \\ \vdots \\ \mathbf{W}_N^k(n_{2M} \end{pmatrix} = \sum_{j=-M}^{M} W_N^k n_j \mathbf{e}_j^{(N)},$$

the representation matrix of s_M relative to $\mathcal{O}_2$ and $\mathcal{E}_2$ is

$$V' := M_{\mathcal{E}_2,\mathcal{O}_2}(s_M) = \begin{pmatrix} W_N^{n_0(-M)} & W_N^{n_0(-M+1)} & \cdots & W_N^{n_0 M} \\ W_N^{n_1(-M)} & W_N^{n_1(-M+1)} & \cdots & W_N^{n_1 M} \\ \vdots & \vdots & \cdots & \vdots \\ W_N^{n_{2M}(-M)} & W_N^{n_{2M}(-M+1)} & \cdots & W_N^{n_{2M} M} \end{pmatrix}.$$

Since

$$V' = \mathrm{diag}(W_N^{-n_0 M}, \ldots, W_N^{n_{2M} M}) \cdot \begin{pmatrix} 1 & W_N^{n_0} & \cdots & W_N^{n_0 2M} \\ \vdots & \vdots & & \vdots \\ 1 & W_N^{n_{2M}} & \cdots & W_N^{n_{2M}} 2M \end{pmatrix},$$

we finally obtain

$$\begin{aligned} S_M &= M_{\mathcal{E}_2,B_M}(s_M) = M_{\mathcal{E}_2,B_M}(s_M \circ \mathrm{id}) \\ &= M_{\mathcal{E}_2,\mathcal{O}_2}(s_M) \cdot M_{\mathcal{O}_2,B_M}(\mathrm{id}) = H \cdot V' = HDV. \end{aligned}$$

We finish the proof by showing that $\frac{1}{\sqrt{2M+1}} H$ is hermitian.

$$\begin{aligned} H \cdot \frac{1}{2M+1} (\overline{H})^t &= \frac{1}{2M+1} \left(\mathbf{W}_N^{-kjd}\right)_{k,j} \cdot \left(\mathbf{W}_N^{j\ell d}\right)_{j\ell} \\ &= \frac{1}{2M+1} \left(\sum_{j=0}^{2M} \mathbf{W}_N^{-kjd} \mathbf{W}_N^{jkd} \right)_{k,\ell} \\ &= \frac{1}{2M+1} \left(\begin{cases} 2M+1 & \text{if } k = \ell \\ \dfrac{1 - W_N^{-(k-\ell)d(2M+1)}}{1 - W^{-(k-\ell)d}} & \text{if } k \neq \ell \end{cases} \right)_{k,\ell} \\ &= \frac{1}{2M+1} \left(\begin{cases} 2M+1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases} \right)_{k,\ell} = I_{2M+1} \end{aligned}$$

since by assumption $d(2M+1) = N$ and, by Lemma 1.1, $W_N^{-(k-\ell)N} = \left(W_N^N\right)^{-(k-\ell)} = 1$ if $k \neq \ell \pmod N$. $\square$

Corollary 1.4
The inverse of the sampling matrix S_M is given by

$$S_M^{-1} = \frac{1}{2M+1} V^{-1} \tilde{H}$$

where

$$\tilde{H} = \begin{pmatrix} W_N^{n_0 M} & W_N^{n_0 M} & \cdots & W_N^{n_0 M} \\ W_N^{n_1 M - Md} & W_N^{n_1 M-(M-1)d} & \cdots & W_N^{n_1 M+Md} \\ W_N^{n_2 M - M(2d)} & W_N^{n_2 M-(M-1)(2d)} & \cdots & W_N^{n_2 M+M(2d)} \\ \vdots & \vdots & & \vdots \\ W_N^{n_{2M} M - M(2Md)} & W_N^{n_{2M} M-(M-1)(2Md)} & \cdots & W_N^{n_{2M} M+M(2Md)} \end{pmatrix}$$

and

$$V^{-1} = \left(\frac{1}{A_j} \sum_{\substack{j \notin U \subseteq \{0,\ldots,2M\} \\ \#U = 2M+1-m}} (-1)^{2M+1-m} W_N^{\sum_{i \in U} n_i} \right)_{\substack{m=1,\ldots,2M+1 \\ j=0,\ldots,2M}}$$

and where

$$A_j = \prod_{\substack{k=0 \\ k \neq j}}^{2M} \left(W_N^j - W_N^k \right).$$

PROOF: We have

$$\begin{aligned} S_M^{-1} &= (HDV)^{-1} = V^{-1} D^{-1} H^{-1} \\ &= \frac{1}{2M+1} V^{-1} \cdot \operatorname{diag}\left(W^{n_0 M}, \ldots, W_N^{n_{2M} M}\right) \cdot (\overline{H})^t \\ &= \frac{1}{2M+1} V^{-1} \cdot \tilde{H}. \end{aligned}$$

the closed form representation of V^{-1} follows from Lemma 1.11. $\square$

Lemma 1.11
Let $\alpha_0, \ldots, \alpha_0$ be distinct elements of an arbitrary field $\mathbb{F}$ and $n \in \mathbb{N}$. Then the Vandermonde matrix

$$V := \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^{n-1} \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \end{pmatrix}$$

is invertible and its inverse is given by

$$V^{-1} = \left(\frac{1}{A_j} \sum_{\substack{j \notin U \subseteq \{0,\ldots,n-1\} \\ \#U = n-m}} (-1)^{n-m} \prod_{i \in U} \alpha_i \right)_{\substack{m=1,\ldots,n \\ j=0,\ldots,n-1}}$$

where

$$A_j = \prod_{\substack{k=0 \\ k \neq j}}^{n-1} (\alpha_j - \alpha_k).$$

PROOF: The proof follows an idea from J.J. Rushanon. Let $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{F}$ distinct elements of the field $\mathbb{F}$, $f(x) := (x - \alpha_0)(x - \alpha_1) \cdots (x - \alpha_{n-1}) \in \mathbb{F}[x]$ and let $R := \mathbb{F}[x]/f(x)$, which is isomorphic (as an algebra) to $\{h \in \mathbb{F}[x] \mid \deg(h) \leq n-1\}$. Let $\mathcal{X}^{(n)} = (1, x, x^2, \ldots, x^{n-1})$ for R denote the canonical basis for R and $\mathcal{E}^{(n)} := (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ the canonical basis of $\mathbb{F}^n$. Let $\varphi : R \to \mathbb{F}^n$ be defined by

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\varphi_X"'] & \mathbb{F}^n \\ \mathbb{F}^n \arrow[ur, "L_V : X \mapsto V \cdot X"'] & \end{tikzcd}$$

$$\varphi(x^j) := \begin{pmatrix} \alpha_0^j \\ \alpha_1^j \\ \vdots \\ \alpha_{n-1}^j \end{pmatrix}$$

for $j = 0, \ldots, n-1$ and let $V := M_{\mathcal{E}^{(n)}\mathcal{X}^{(n)}}(\varphi)$ denote the representation matrix of φ relative to the basis $\mathcal{X}^{(n)}$ and $\mathcal{E}^{(n)}$. Clearly,

$$V = \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^{n-1} \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \end{pmatrix}$$

is the Vandermonde matrix defined by the numbers $\alpha_0, \ldots, \alpha_{n-1}$. Since, by assumption, the numbers $\alpha_0, \ldots, \alpha_{n-1}$ are all distinct,

$$\det(V) = \prod_{0 \leq i < j \leq n-1} (a_j - \alpha_i) \neq 0,$$

i.e. V is invertible and φ an isomorphism. For $j = 0, 1, \ldots, n-1$, let

$$W^{(j)} = \begin{pmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{n,n-1} \end{pmatrix}$$

denote the j-th column of the inverse matrix V^{-1} of V. By definition, $W^{(j)}$ satisfies the equation

$$VW^{(j)} = \mathbf{e}_{j+1}$$

for $j = 0, 1, \ldots, n-1$. Using the isomorphism φ, we can restate this condition in terms of polynomials. To this end, let

$$p_j(x) := \varphi_{\mathcal{X}^{(n)}}^{-1}\big(W^{(j)}\big) \in R$$

for $j = 0, \ldots, n-1$. Then follows immediately (see diagram)

$$\begin{pmatrix} p_j(\alpha_0) \\ p_j(\alpha_1) \\ \vdots \\ p_j(\alpha_{n-1}) \end{pmatrix} = L_V\Big(\varphi_{\mathcal{X}^{(n)}}\big(p_j(x)\big)\Big) = VW^{(j)} = \mathbf{e}_{j+1}$$

for $j = 0, \ldots, n-1$. I.e $p_j(x)$ is the unique polynomial in R satisfying the condition

$$p_j(\alpha_i) = \delta_{ij}$$

for $i, j = 0, \ldots, n-1$. Thus $p_j(x)$ is the j-th Lagrange interpolation polynomial

$$\begin{aligned} p_j(x) = L_j(x) &= \prod_{\substack{k=0\\k\neq j}}^{n-1} \frac{(x-\alpha_k)}{(\alpha_j-\alpha_k)} \\ &= \frac{1}{A_j}\sum_{k=0}^{n-1}\left(\sum_{\substack{j\notin U\subseteq\{0,\ldots,n-1\}\\ \#U=k}} (-1)^k \prod_{i\in U}\alpha_i\right) x^{n-1-k}, \end{aligned}$$

where

$$A_j = \prod_{\substack{k=0\\k\neq j}}^{n-1} (\alpha_j - \alpha_k).$$

Hence, since $W^{(j)} = \varphi_{\mathcal{X}^{(n)}}\big((p_j(x)\big)$,

$$w_{m,j} = \frac{1}{A_j} \sum_{\substack{j\notin U\subseteq\{0,\ldots,n-1\}\\ \#U=n-m}} (-1)^{n-m} \prod_{i\in U}\alpha_i$$

for $m = 1, \ldots, n$ and $j = 0, \ldots, n-1$. □

1.8 Reconstruction from Aperture-filtered Samples

In this section we address the problem of reconstructing a bandlimited function from its aperture-filtered samples.

Let $M, N, r \in \mathbb{N}$, $M < N/2$, $(2M+1)|N$ and $d := N/(2M+1)$. Moreover, let $B_M := \big(D_M * \mathbf{e}_{jd}^{(N)} \mid j = 0, 1, \ldots, 2M\big)$ denote the canonical orthogonal basis of $\mathcal{B}_M$. Moreover, let $\mathfrak{v} := (v_0, \ldots, v_r)$ denote the sequence of aperture functions $v_0, \ldots, v_r \in$

$\ell(\mathbb{Z}_N)$ and let $\mathfrak{n} := (n_0, \ldots, n_r)$ denote the sequence of (irregular) sampling locations $0 \leq n_0 < n_1 < \cdots < n_r \leq N-1$, where we assume that the function values of the bandlimited functions $f \in \mathcal{B}_M$ are known relative to the apertures $\mathfrak{v}$. More precisely, we assume that we know the values

$$(f * v_0)(n_0), \ldots, (f * v_{2M})(n_r).$$

Definition 1.8 (Sampling Homomorphism)
As above, we call the unique homomorphism $s_M^{\mathfrak{v}} : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ defined by linear extension through the association

$$D_M * \mathbf{e}_{jd}^{(N)} \mapsto \begin{pmatrix} \big(D_M * \mathbf{e}_{jd}^{(N)} * v_0\big)(n_0) \\ \big(D_M * \mathbf{e}_{jd}^{(N)} * v_1\big)(n_1) \\ \vdots \\ \big(D_M * \mathbf{e}_{jd}^{(N)} * v_r\big)(n_r) \end{pmatrix}$$

$(j = 0, \ldots, 2M)$ the *sampling homomorphism of* $\mathcal{B}_M$ *with respect to* $\mathfrak{n}$ *relative to the apertures* $\mathfrak{v}$. The representation matrix

$$\begin{aligned} S_M^{\mathfrak{v}} &= M_{\mathcal{E}_2^{(2M+1)}, B_m} \\ &= \begin{pmatrix} \big(D_M * \mathbf{e}_0^{(N)} * v_0\big)(n_0) & \cdots & \big(D_M * \mathbf{e}_{2Md}^{(N)} * v_0\big)(n_0) \\ \vdots & & \vdots \\ \big(D_M * \mathbf{e}_0^{(N)} * v_r\big)(n_r) & \cdots & \big(D_M * \mathbf{e}_{2Md}^{(N)} * v_r\big)(n_r) \end{pmatrix} \end{aligned}$$

is called the *sampling matrix of* $\mathcal{B}_M$ *with respect to* $\mathfrak{n}$ *relative to the apertures* $\mathfrak{v}$. Moreover, let $\mathcal{B}_M(\mathfrak{v})$ denote the set of all bandlimited functions $f \in \mathcal{B}_M$ that can be reconstructed from their samples at $\mathfrak{n}$ relative to the apertures $\mathfrak{v}$.

Recall that $\mathcal{E}_2^{(2M+1)} = \big(\mathbf{e}_0^{(2M+1)}, \ldots, \mathbf{e}_{2M}^{(2M+1)}\big)$ denotes the canonical basis of $\ell(\mathbb{Z}_{2M+1})$. As above, $\varphi_{B_M} : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ denotes the coordinate isomorphism relative to B_M, which assigns to each function $f \in \mathcal{B}_M$ its coordinate vector $\varphi_{B_M}(f)$. We know from the Discrete Sampling Theorem (Theorem 1.8) that the coordinates of f relative to B_M are given by

$$\varphi_{B_M}(f)(j) = \frac{N}{d} f(jd)$$

for $j = 0, 1, \ldots, 2M$, i.e. they are just the sampling values of f at the regular locations $0, d, 2d, 3d, \ldots, 2Md$ scaled by $\frac{N}{d}$.

We first consider the special case where $r = 2M$ and $s_M^{\mathfrak{v}}$ is invertible. Then follows from the diagram that $\varphi_{B_M} \circ (s_M^{\mathfrak{v}})^{-1} = L_{S_M^{\mathfrak{v}}}^{-1} = L_{(S_M^{\mathfrak{v}})^{-1}}$ and thus we can easily recover the coordinates of f relative to the canonical basis B_M:

$$\begin{aligned} \varphi_{B_M}(f) &= \varphi_{B_M}\Big((s_M^{\mathfrak{v}})^{-1}(X_f^v)\Big) = \big(\varphi_{B_M} \circ (S_M^{\mathfrak{v}})^{-1}\big)(X_f^v) \\ &= L_{(S_M^{\mathfrak{v}})^{-1}}(X_f^v) = (S_M^{\mathfrak{v}})^{-1} \cdot X_f^v. \end{aligned}$$

$$\begin{tikzcd} \mathcal{B}_M \arrow[r, "s_M^v"] \arrow[d, "\varphi_{B_M}"'] & \ell(\mathbb{Z}_{2M+1}) \\ \ell(Z_{2M+1}) \arrow[ur, "L_{S_M^v} : X \longmapsto S_M^v \cdot X"'] & \end{tikzcd}$$

Hence, we have the following

Theorem 1.13
Suppose $X_f^v = \big((f * v_0)(n_0), \ldots, (f * v_{2M})(n_{2M})\big)^t$ *is the vector of the aperture-filtered samples of a bandlimited function* $f \in \mathcal{B}_M$ *with respect to the (irregularly spaced) locations* $0 \leq n_0 < \cdots < n_{2M} \leq N-1$ *and the apertures* $v_0, \ldots, v_{2M}$. *If the sampling homomorphism is an isomorphism, then* f *can be uniquely reconstructed from* X_f^v. *In fact*

$$Y_f^v = \left(S_M^{\mathfrak{v}}\right)^{-1} \cdot X_f^v$$

is the coordinate vector of f *relative to* B_M*, i.e.*

$$f = \varphi_{B_M}^{-1}\Big(\left(S_M^{\mathfrak{v}}\right)^{-1} \cdot X_f^v\Big) = \sum_{j=0}^{2M}\Big(\left(S_M^{\mathfrak{v}}\right)^{-1} \cdot X_f^v\Big)_j \tau_{jd}(D_M).$$

We will now investigate under which conditions on the aperture functions the homomorphism $s_M^{\mathfrak{v}}$ is invertible. We begin with the trivial case where all apertures are the same and where $r = 2M$.

$$\begin{tikzcd}
\mathcal{B}_M \arrow[r, "s_M^v"] \arrow[d, "c_v"'] & \ell(\mathbb{Z}_{2M+1}) \\
\mathcal{B}_M \arrow[ur, "s_M"'] &
\end{tikzcd}$$

Theorem 1.14
Suppose $\mathfrak{v} = (v, \ldots, v)$. *Then* $s_M^{\mathfrak{v}}$ *is an isomorphism if and only if* $\hat{v}(n) \neq 0$ *for all* $|n| \leq M$.

PROOF: Consider the function $c_v : \mathcal{B}_M \to \mathcal{B}_M$ defined by $f \mapsto f * v$. By Lemma 1.3, c_v is an homomorphism. We have to show that $\ker(c_v) = \{0\} \Leftrightarrow \hat{v}(n) \neq 0$ for all $|n| \leq M$. The surjectivity of c_v follows since its domain and codomain have the same dimension. Clearly, $f * v = 0$ is equivalent to $\hat{f}\hat{v} = 0$. Thus $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}_N$ (since f is M-bandlimited, we only have to worry about $|n| \leq M$!) if and only if $\hat{v}(n) \neq 0$ for all $|n| \leq M$. Hence c_v is an isomorphism and, since $s_M^{\mathfrak{v}} = s_M \circ c_v$ (cf. diagram), s_M^v, as composition of isomorphism, is an isomorphism. $\square$

Theorem 1.15
Suppose $\mathfrak{v} = (v, \ldots, v)$ *and the aperture* v *has the basis representation*

$$v = \sum_{\ell \in \mathbb{Z}_N} a_\ell \mathbf{W}_N^\ell$$

relative to the basis $\mathcal{O}_2 = (\mathbf{W}_N^\ell \mid \ell \in \mathbb{Z}_N)$. *Then*

$$S_M^{\mathfrak{v}} = N S_M \operatorname{diag}(a_{-M}, \ldots, a_M)$$

and if $a_\ell \neq 0$ *for* $\ell = -M, \ldots, M$*, then the inverse matrix exists and is given by*

$$\left(S_M^{\mathfrak{v}}\right)^{-1} = N^{-1} \operatorname{diag}(a_{-M}^{-1}, \ldots, a_M^{-1}) S_M^{-1}.$$

PROOF: Let $\mathcal{O}_2^{(2M+1)} = (\mathbf{W}_N^k \mid k \in \{-M, \ldots, M\})$ and $\mathcal{O}_2^{(N)} = (\mathbf{W}_N^k \mid k \in \mathbb{Z}_N)$. $\mathcal{O}_2^{(2M+1)}$ is a basis of $\mathcal{B}_M$ and $\mathcal{O}_2^{(N)}$ a basis of $\ell(\mathbb{Z}_N)$. Moreover, let

$$v = \sum_{\ell \in \mathbb{Z}_N} a_\ell \mathbf{W}_N^\ell$$

where $a_\ell \in \mathbb{C}$. We will compute the representation matrices $M_{\mathcal{O}_2^{(2M+1)},B_M}(c_v)$ and $M_{B_M,\mathcal{O}_2^{(2M+1)}}(\mathrm{id})$ and begin with first one. Since

$$\begin{aligned} c_v\big(D_M * \mathbf{e}_{jd}^{(N)}\big) &= D_M * \mathbf{e}_{jd}^{(N)} * v \\ &= \left(\sum_{k=-M}^{M} \mathbf{W}_N^k\right) * \left(\sum_{\ell\in\mathbb{Z}_N} a_\ell \mathbf{W}_N^\ell\right) * \mathbf{e}_{jd}^{(N)} \end{aligned}$$

and by Lemma 1.3(7) and Theorem 1.5(4)

$$= \left(\sum_{k=-M}^{M} a_k N \mathbf{W}_N^k\right) * \mathbf{e}_{jd}^{(N)} = \sum_{k=-M}^{M} N a_k W_N^{-jdk} \mathbf{W}_N^k,$$

the representation matrix of c_v relative to the basis B_M and $\mathcal{O}_2^{(2M+1)}$ is given by $M_{\mathcal{O}_2^{(2M+1)},B_M}(c_v) = \big(N a_k W_N^{-jdk}\big)_{k,j} = NH\,\mathrm{diag}(a_{-M},\dots,a_M)$, where

$$H = \begin{pmatrix} 1 & W_N^{Md} & W_N^{M(2d)} & \cdots & W_N^{M2Md} \\ 1 & W_N^{(M-1)d} & W_N^{(M-1)(2d)} & \cdots & W_N^{(M-1)2Md} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(-M)d} & W_N^{(-M)(2d)} & \cdots & W_N^{(-M)2Md} \end{pmatrix}.$$

$$\begin{tikzcd} B_M\ \mathcal{B}_M \arrow[r, "s_M^v"] \arrow[d, "c_v"] & \ell(\mathbb{Z}_{2M+1}) \\ \mathcal{O}_2\ \mathcal{B}_M \arrow[d, "\mathrm{id}"] & \\ B_M\ \mathcal{B}_M \arrow[uur, "s_M"'] & \end{tikzcd}$$

We now compute first compute $M_{\mathcal{O}_2^{(2M+1)},B_M}(\mathrm{id})$: Since

$$\begin{aligned} \mathrm{id}\big(D_M * \mathbf{e}_{jd}^{(N)}\big) &= D_M * \mathbf{e}_{jd}^{(N)} = \left(\sum_{k=-M}^{M} \mathbf{W}_N^k\right) * \mathbf{e}_{jd}^{(N)} \\ &= \sum_{k=-M}^{M} W_N^{-jdk} \mathbf{W}_N^k, \end{aligned}$$

we conclude that the base change matrix $M_{\mathcal{O}_2^{(2M+1)},B_M}(\mathrm{id}) = \big(W_N^{-jdk}\big)_{k,j} = H$ and therefore $M_{B_M,\mathcal{O}_2^{(2M+1)}}(\mathrm{id}) = H^{-1}$. Thus

$$\begin{aligned} S_M^{\mathfrak{v}} &= M_{\mathcal{E}_2^{(2M+1)},B_M}(s_M^{\mathfrak{v}}) \\ &= S_M M_{B_M,\mathcal{O}_2^{(2M+1)}}(\mathrm{id}) M_{\mathcal{O}_2^{(2M+1)},B_M}(c_v) \\ &= S_M H^{-1} N H\,\mathrm{diag}(a_{-M},\dots,a_M) \\ &= N S_M\,\mathrm{diag}(a_{-M},\dots,a_M). \end{aligned}$$

By Lemma 1.5(1)

$$\mathcal{F}(v) = \{ \left(\sum_{\ell\in\mathbb{Z}_N} a_\ell \mathbf{W}_N^\ell\right) = \sum_{\ell\in\mathbb{Z}_N} a_\ell N \mathbf{e}_{-\ell}^{(N)}.$$

In particular, $\hat{v}(n) = Na_{-n}$. Note that $\hat{v}(n) \neq 0$ for all $|n| \leq M$ if and only if $a_n \neq 0$ for all $|n| \leq M$.

If $a_\ell \neq 0$ for all $\ell \in \{-M, \ldots, M\}$ then $\operatorname{diag}(a_{-M}, \ldots, a_M)$ exists and is given by $\big(\operatorname{diag}(a_{-M}, \ldots, a_M)\big)^{-1} = \operatorname{diag}(a_{-M}^{-1}, \ldots, a_M^{-1})$. Hence,

$$\begin{aligned}\big(S_M^{\mathfrak{v}}\big)^{-1} &= \big(NS_M \operatorname{diag}(a_{-M}, \ldots, a_M)\big)^{-1} \\ &= N^{-1} \operatorname{diag}(a_{-M}^{-1}, \ldots, a_M^{-1}) S_M^{-1}\end{aligned}$$

as claimed. ☐

We now return to the general case $r \geq 2M$ and first formulate some general results.

Theorem 1.16
Suppose $M, N, r \in \mathbb{N}$, $M < N/2$, $(2M+1) \mid N$, $d := \frac{N}{2M+1}$ and $s_M^{\mathfrak{v}} : \mathcal{B}_M \to \ell(\mathbb{Z}_N)$ is the sampling homomorphism with respect to the samples $\mathfrak{n} = (n_0, \ldots, n_r)$ relative to the apertures $\mathfrak{v} = (v_0, \ldots, v_r)$. Then

$$\begin{tikzcd} \mathcal{B}_M \arrow[r, "s_M^{\mathfrak{v}}"] \arrow[d, "\pi"'] & \operatorname{im}\big(s_M^{\mathfrak{v}}\big) \arrow[r, "\iota"] & \ell(\mathbb{Z}_{2M+1}) \\ \mathcal{B}_M / \ker\big(s_M^{\mathfrak{v}}\big) \arrow[ur, "\sim", "\overline{s_M^{\mathfrak{v}}}"'] & & \end{tikzcd}$$

1. *$Y \in \ell(\mathbb{Z}_N)$ is the $\mathfrak{v}$-aperture-filtered sample of a band-limited function $f \in \mathcal{B}_M$ if and only if $Y \in \operatorname{im}\big(s_M^{\mathfrak{v}}\big)$.*

2. *The band-limited functions $f_1, f_2 \in \mathcal{B}_M$ have the sample $\mathfrak{v}$-aperture-filtered sample if and only if $f_2 - f_1 \in \ker\big(s_M^{\mathfrak{v}}\big)$.*

3. *The space of all M-band-limited functions that can be reconstructed from their samples at $\mathfrak{n}$ relative to the apertures $\mathfrak{v}$ is isomorphic to $\mathcal{B}_M / \ker\big(s_M^{\mathfrak{v}}\big)$, i.e. $\mathcal{B}_M(\mathfrak{v}) \cong \mathcal{B}_M / \ker\big(s_M^{\mathfrak{v}}\big)$. Moreover, $\dim \mathcal{B}_M(\mathfrak{v}) = \operatorname{rank}\big(s_M^{\mathfrak{v}}\big)$.*

PROOF: These statements follow directly from the Homomorphism Theorem, see the diagram in the margin. ☐

Remark 1.1
If $\ker\big(s_M^{\mathfrak{v}}\big) \neq \{0\}$, then $\mathcal{B}_M(\mathfrak{v})$ is a proper subspace of $\mathcal{B}_M$. $\mathcal{B}_M(\mathfrak{v})$ is not necessarily of the form $\mathcal{B}_{M'}$ for some $0 \leq M' \leq M$. We will consider the question, under which condition only the higher frequency terms are missing later-on.

If $r = 2M$ then we can easily formulate the conditions on the apertures under which we can reconstruct all elements of $\mathcal{B}_M$.

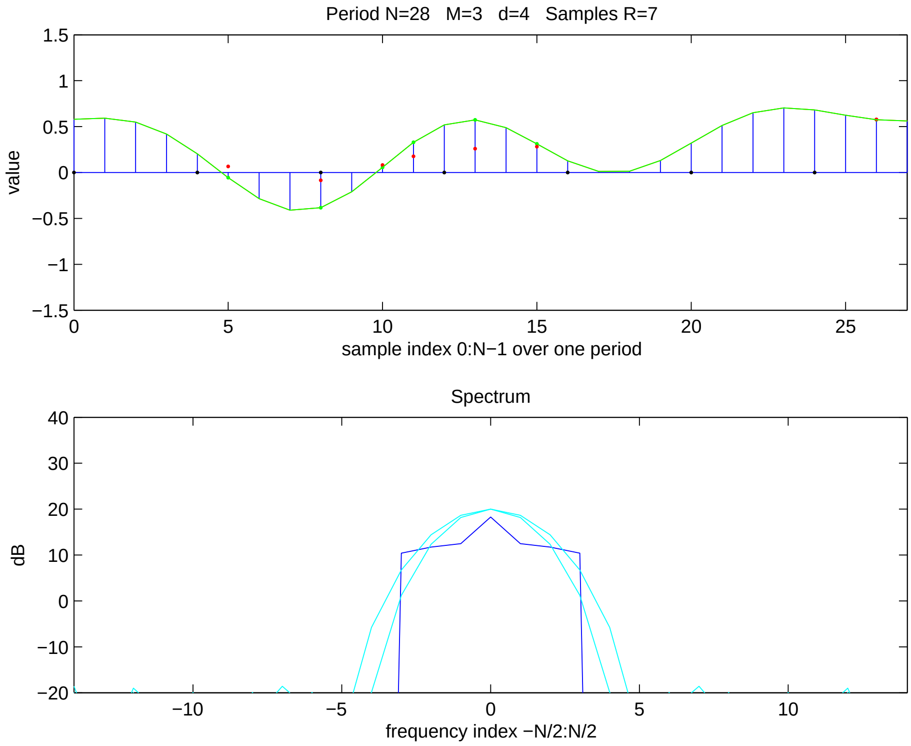
Theorem 1.17
Suppose $M, N \in \mathbb{N}$, $M < N/2$, $(2M+1) \mid N$, $d := \frac{N}{2M+1}$ and $s_M^{\mathfrak{v}} : \mathcal{B}_M \to \ell(\mathbb{Z}_N)$ is the sampling homomorphism with respect to the samples $\mathfrak{n} = (n_0, \ldots, n_{2M})$ relative to the apertures $\mathfrak{v} = (v_0, \ldots, v_{2M})$. Then the following statements are equivalent

1. *$s_M^{\mathfrak{v}}$ is an isomorphism.*

2. *$\mathcal{B}_M(\mathfrak{v}) = \mathcal{B}_M$*

3. *$\operatorname{rank}\big(S_M^{\mathfrak{v}}\big) = 2M + 1$*

Figure 1.4: *Reconstruction from aperture-filtered samples.* The samples are plotted in red. The light-blue graphs in the lower diagram depict the spectra of the two aperture functions used in the sampling. The dark-blue graph depicts the spectrum of the signal. The dark dots on the horizontal axis mark the locations of the regular samples into which the aperture-filtered samples are transformed by $(S_M^{\mathbf{v}})^{-1}$.



4. $\left(\hat{v}_k \mathbf{W}_N^{n_k} \chi_{\{-M,\dots,M\}} \mid k = 0,\dots,2M\right)$ *is linearly independent.*

5. $\left(P_M\left(v_k * \mathbf{e}_{-n_k}^{(N)}\right) \mid k = 0,\dots,2M\right)$ *is linearly independent.*

6. *The matrix*

$$\begin{pmatrix} \hat{v}_0(-M)W_N^{n_0(-M)} & \cdots & \hat{v}_{2M}(-M)W_N^{n_{2M}(-M)} \\ \vdots & & \vdots \\ \hat{v}_0(M)W_N^{n_0 M} & \cdots & \hat{v}_{2M}(M)W_N^{n_{2M} M} \end{pmatrix}$$

has rank $2M+1$.

PROOF: Conditions (1), (2) and (3) are clearly equivalent (Homomorphism theorem). Moreover, condition (6) is just the matrix formulation of condition (4). Conditions (4) and (5) are equivalent since the system in (4) is the isomorphic image of the system in (5) under the discrete Fourier transform (see also 1.14). We can therefore confine ourselves to establishing the equivalence of the conditions (3) and (4).

Consider the sampling matrix

$$\begin{aligned} S_M^{\mathfrak{v}} &= M_{\mathcal{E}_2^{(2M+1)},B_m} \\ &= \begin{pmatrix} \big(D_M * \mathbf{e}_0^{(N)} * v_0\big)(n_0) & \cdots & \big(D_M * \mathbf{e}_{2Md}^{(N)} * v_0\big)(n_0) \\ \vdots & & \vdots \\ \big(D_M * \mathbf{e}_0^{(N)} * v_{2M}\big)(n_{2M}) & \cdots & \big(D_M * \mathbf{e}_{2Md}^{(N)} * v_{2M}\big)(n_{2M}) \end{pmatrix} \end{aligned}$$

of $\mathcal{B}_M$ with respect to $\mathfrak{n}$ relative to $\mathfrak{v}$. Clearly, $\operatorname{rank}\big(S_M^{\mathfrak{v}}\big) = 2M+1$ if and only if the system

$$\Big(\big(S_M^{\mathfrak{v}}\big)^{(k)} \;\Big|\; k = 0, \ldots, 2M\Big) \tag{1.15}$$

of all row vectors

$$\begin{aligned} \big(S_M^{\mathfrak{v}}\big)^{(k)} &= \begin{pmatrix} \big(D_M * \mathbf{e}_0 * v_k\big)(n_k) \\ \vdots \\ \big(D_M * \mathbf{e}_{2Md} * v_k\big)(n_k) \end{pmatrix} = \begin{pmatrix} \big((D_M * v_k) * \mathbf{e}_0\big)(n_k) \\ \vdots \\ \big((D_M * v_k) * \mathbf{e}_{2Md}\big)(n_k) \end{pmatrix} \\ &= \begin{pmatrix} (D_M * v_k)(n_k) \\ \vdots \\ (D_M * v_k)(n_k - 2Md) \end{pmatrix} \end{aligned}$$

of $S_M^{\mathfrak{v}}$ is linearly independent. Note that for $k = 0, \ldots, 2M$, the vector $\big(S_M^{\mathfrak{v}}\big)^{(k)}$ contains $2M+1$ distinct samples of $D_M * v_k \in \mathcal{B}_M$. Therefore, suppose that for certain coefficients $x_k \in \mathbb{C}$ we have the relation

$$0 = \sum_{k=0}^{2M} x_k \big(S_M^{\mathfrak{v}}\big)^{(k)} = \sum_{k=0}^{2M} x_k \begin{pmatrix} (D_M * v_k)(n_k) \\ \vdots \\ (D_M * v_k)(n_k - 2Md) \end{pmatrix} \tag{1.16}$$

$$= \sum_{k=0}^{2M} x_k \begin{pmatrix} (D_M * v_k * \mathbf{e}_{-n_k})(0) \\ \vdots \\ (D_M * v_k * \mathbf{e}_{-n_k})(-2Md) \end{pmatrix} \tag{1.17}$$

This is equivalent to

$$0 = \left(\sum_{k=0}^{2M} x_k \big(D_M * v_k * \mathbf{e}_{-n_k}\big)\right)(jd) \tag{1.18}$$

for all $j = 0, \ldots, 2M$. Since $D_M * v_k * \mathbf{e}_{-n_k} \in \mathcal{B}_M$ also $\sum_{k=0}^{2M} x_k \big(D_M * v_k * \mathbf{e}_{-n_k}\big) \in \mathcal{B}_M$. Therefore, Equation (1.18) is equivalent to

$$0 = \left(\sum_{k=0}^{2M} x_k \big(D_M * v_k * \mathbf{e}_{-n_k}\big)\right)(n) \tag{1.19}$$

by the Discrete Sampling Theorem (Theorem 1.8), and thus also to the equation

$$0 = \sum_{k=0}^{2M} x_k \big(D_M * v_k * \mathbf{e}_{-n_k}\big). \tag{1.20}$$

Applying the discrete Fourier transform to Equation (1.20), we obtain the equivalent condition

$$0 = \sum_{k=0}^{2M} x_k N \chi_{\{-M,\dots,M\}} \hat{v}_k \mathbf{W}_N^{n_k}. \tag{1.21}$$

Thus we see that the system (1.15) is linearly independent if and only if the system

$$\big(\chi_{\{-M,\dots,M\}} \hat{v}_k \mathbf{W}_N^{n_k} \mid k = 0, \dots, 2M\big) \tag{1.22}$$

is linearly independent. □

Remark 1.2
From the proof of Theorem 1.17 follows that the association

$$\big(S_M^{\mathfrak{v}}\big)^{(k)} \leftrightarrow P_M\Big(v_k * \mathbf{e}_{-n_k}^{(N)}\Big) \leftrightarrow \hat{v}_k \mathbf{W}_N^{n_k} \chi_{\{-M,\dots,M\}}$$

defines canonical isomorphisms between the row space of the sampling matrix $S_M^{\mathfrak{v}}$*,* $\langle P_M\big(v_k * \mathbf{e}_{-n_k}\big) \mid k = 0, \dots, 2M\rangle$*,* $\langle \hat{v}, \mathbf{W}_N^{n_k} \chi_{\{-M,\dots,M\}} \mid k = 0, \dots, 2M\rangle$ *and the column space of the matrix*

$$\begin{pmatrix} \hat{v}_0(-M) W_N^{n_0(-M)} & \cdots & \hat{v}_{2M}(-M) W_N^{n_{2M}(-M)} \\ \vdots & & \vdots \\ \hat{v}_0(M) W_N^{n_0 M} & \cdots & \hat{v}_{2M}(M) W_N^{n_{2M} M} \end{pmatrix}.$$

Chapter 2

Discrete Aperiodic Functions

Chapter 3

Continuous Periodic Functions

Chapter 4

Continous Aperiodic Functions

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