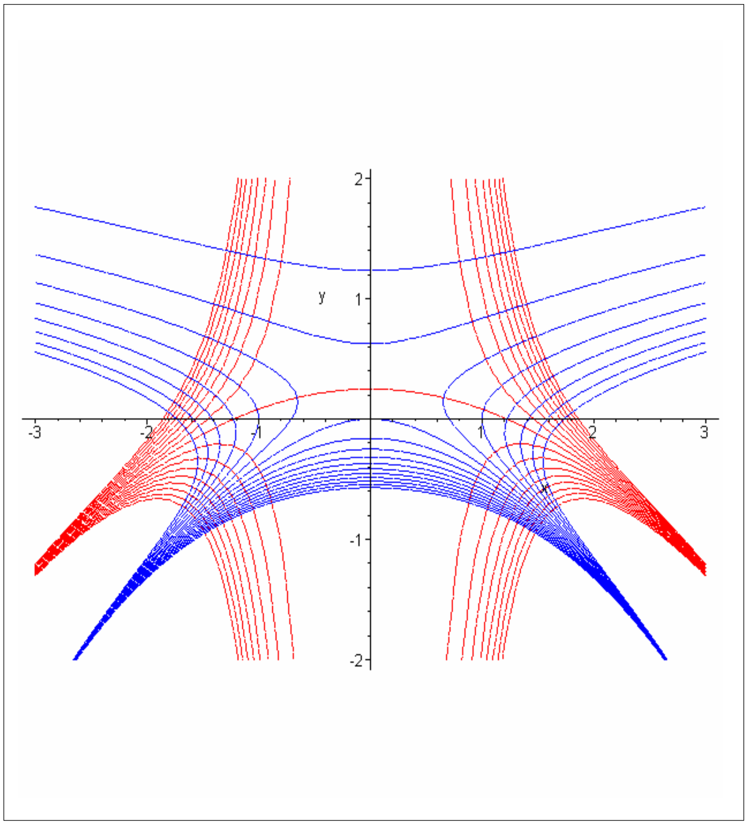
Chapter 2

Ordinary Differential Equations



Chapter 2 Ordinary Differential Equations

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2.1 Basic concepts, definitions, notations and classification

Engineering design focuses on the use of models in developing predictions of natural phenomena. These models are developed by determining relationships between key parameters of the problem. Usually, it is difficult to find immediately the functional dependence between needed quantities in the model; at the same time, often, it is easy to establish relationships for the rates of change of these quantities using empirical laws. For example, in heat transfer, directional heat flux is proportional to the temperature gradient (Fourier's Law)

$$q = -k\frac{dT}{dx}$$

where the coefficient of proportionality is called the coefficient of conductivity. Also, during light propagation in the absorbing media, the rate of change of intensity I with distance is proportional to itself (Lambert's Law)

$$\frac{dI}{ds} = -kI$$

where the coefficient of proportionality is called the absorptivity of the media.

In another example, if we are asked to derive the path $x(t)$ of a particle of mass m moving under a given time-dependent force $f(t)$, it is not easy to find it directly, however, Newton's second law (acceleration is proportional to the force) gives a differential equation describing this motion.

$$m\frac{d^2x(t)}{dt^2} = f(t)$$

The solution of which gives an opportunity to establish the dependence of path on the acting force.

The basic approach to deriving models is to apply conservation laws and empirical relations for control volumes. In most cases, the governing equation for a physical model can be derived in the form of a differential equation. The governing equations with one independent variable are called ordinary differential equations. Because of this, we will study the methods of solution of differential equations.

Differential equation

Definition 1 A differential equation is an equation, which includes at least one derivative of an unknown function.

Example 1:

a) $\dfrac{dy(x)}{dx} + 2xy(x) = e^x$

b) $y(y'')^2 + y' = \sin x$

c) $\dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0$

d) $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$

e) $\dfrac{\partial^2 u(x,t)}{\partial x^2} - v\dfrac{\partial u(x,t)}{\partial x} = 0$

If a differential equation (DE) only contains unknown functions of one variable and, consequently, only the ordinary derivatives of unknown functions, then this equation is said to be an **ordinary differential equation** (ODE); in a case where other variables are included in the differential equation, but not the derivatives with respect to these variables, the equation can again be treated as an ordinary differential equation in which other variables are considered to be parameters. Equations with partial derivatives are called **partial differential equations**

(PDE). In *Example 1,* equations a),b) and d) are ODE's, and equation c) is a PDE; equation e) can be considered an ordinary differential equation with the parameter t .

Differential operator D

It is often convenient to use a special notation when dealing with differential equations. This notation called differential operators, transforms functions into the associated derivatives. Consecutive application of the operator D transforms a differentiable function $f(x)$ into its derivatives of different orders:

$$\mathrm{D}f(x)=\frac{df(x)}{dx} \qquad \mathrm{D}: f \to f'$$

$$\mathrm{D}^2 f(x)=\frac{d^2 f(x)}{dx^2} \qquad \mathrm{D}^2 : f \to f''$$

$$\vdots$$

A single operator notation D can be used for application of combinations of operators; for example, the operator

$$D = a\mathrm{D}^n + b\mathrm{D}$$

implies

$$Df(x) = a\mathrm{D}^n f(x) + b\mathrm{D}f(x) = a\frac{d^n f(x)}{dx^n} + b\frac{df(x)}{dx}$$

Order of DE

The order of DE is the order of the highest derivative in the DE. It can be reflected as an index in the notation of the differential operator as

$$D_2 = a\mathrm{D}^2 + b\mathrm{D} + c$$

Then a differential equation of second order with this operator can be written in the compact form

$$D_2 y = F(x)$$

Linear operator

A differential operator D_n is **linear** if its application to a linear combination of n times differentiable functions $f(x)$ and $g(x)$ yields a linear combination

$$D_n(\alpha f + \beta g) = \alpha D_n f + \beta D_n g\ , \quad \alpha, \beta \in R$$

The most general form of a linear operator of n^{th} order may be written as

$$L_n \equiv a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$$

where the coefficients $a_i(x) \in C(\mathbb{R})$ are continuous functions.

Linear and non-linear DE

A DE is said to be **linear**, if the differential operator defining this equation is linear. This occurs when unknown functions and their derivatives appear as DE's of the first degree and not as products of functions combinations of other functions. A linear DE does not include terms, for example, like the following:

$$y^2\ ,\ (y')^3\ ,\ yy'\ ,\ \ln(y), \text{ etc.}$$

If they do, they are referred to as **non-linear DE's**.

A linear ODE of the n^{th} order has the form

$$L_n y(x) \equiv a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = F(x)$$

where the coefficients $a_i(x)$ and function $F(x)$ are, usually, continuous functions. The most general form of an n^{th} order non-linear ODE can be formally written as

$$F\left(x, y, y', ..., y^{(n)}\right) = 0$$

which does not necessarily explicitly include the variable x and unknown function y with all its derivatives of order less than n.

A homogeneous linear ODE includes only terms with unknown functions:

$$L_n y(x) = 0$$

A non-homogeneous linear ODE involves a free term (in general, a function of an independent variable):

$$L_n y(x) = F(x)$$

A normal form of an n^{th} order ODE is written explicitly for the n^{th} derivative:

$$y^{(n)} = f\left(x, y, y', ..., y^{(n-1)}\right)$$

Solution of DE

Definition 2 Any n times differentiable function $y(x)$ which satisfies a DE

$$F\left(x, y, y', ..., y^{(n)}\right) = 0$$

is called a **solution** of the DE, i.e. substitution of function $y(x)$ into the DE yields an identity.

"Satisfies" means that substitution of the solution into the equation turns it into an identity. This definition is constructive – we can use it as a trial method for finding a solution (guess a form of a solution (which in modern mathematics is often called **ansatz**), substitute it into the equation and force the equation to be an identity).

Example 2: Consider the ODE $y' + y = 0$ on $x \in I = (-\infty, \infty)$

Look for a solution in the form $y = e^{ax}$

Substitution into the equation yields

$$ae^{ax} + e^{ax} = 0$$

$$(a+1)e^{ax} = 0 \qquad \text{divide by } e^{ax} > 0$$

$$a + 1 = 0 \qquad \Rightarrow \quad a = -1$$

Therefore, the solution is $y = e^{-x}$.

But this solution is not necessarily a unique solution of the ODE.

The Solution of the ODE may be given by an explicit expression like in example 2 called the **explicit solution**; or by an implicit function (called the **implicit solution integral** of the differential equation)

$$g(x, y) = 0$$

If the solution is given by a zero function $y(x) \equiv 0$, then it is called to be a **trivial solution**. Note, that the ODE in example 2 posses also a trivial solution.

The **complete solution** of a DE is a set of all its solutions.

The **general solution** of an ODE is a solution which includes parameters, and variation of these parameters yields a complete solution.

Thus, $\left\{y = ce^{-x}, c \in R\right\}$ is a complete solution of the ODE in example 2.

The general solution of an n^{th} order ODE includes n independent parameters and symbolically can be written as

$$g(x, y, c_1, ..., c_n) = 0$$

The particular solution is any individual solution of the ODE. It can be obtained from a general solution with particular values of parameters. For example, e^{-x} is a particular solution of the ODE in example 2 with $c = 1$.

A solution curve is a graph of an explicit particular solution. An **integral curve** is defined by an implicit particular solution.

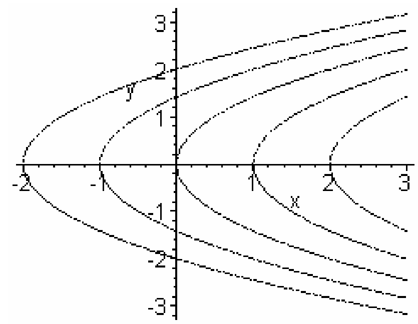
Example 3: The differential equation

$$yy' = 1$$

has a general solution

$$\frac{y^2}{2} = x + c$$

The integral curves are implicit graphs of the general solution for different values of the parameter c



To get a particular solution which describes the specified engineering model, the initial or boundary conditions for the differential equation should be set.

Initial Value Problem

An initial value problem (IVP) is a requirement to find a solution of n^{th} order ODE

$$F\left(x, y, y', ..., y^{(n)}\right) = 0 \text{ for } x \in I \subset \mathbb{R}$$

subject to n conditions on the solution $y(x)$ and its derivatives up to order *n-1* specified at one point $x_0 \in I$:

$$\begin{aligned} &y(x_0) = y_0 \\ &y'(x_0) = y_1 \\ &\vdots \\ &y^{(n-1)}(x_0) = y_{n-1} \end{aligned}$$

where $y_0, y_1, ..., y_{n-1} \in \mathbb{R}$.

Boundary Value Problem

In a boundary value problem (BVP), the values of the unknown function and/or its derivatives are specified at the boundaries of the domain (end points of the interval (possibly $\pm\infty$)).

For example, find the solution of $y'' + y = x^2$ on $x \in [a, b]$ satisfying boundary conditions:

$$\begin{aligned} &y(a) = y_a \\ &y(b) = y_b \end{aligned}$$

where $y_a, y_b \in \mathbb{R}$

The solution of IVP's or BVP's consists of determining parameters in the general solution of a DE for which the particular solution satisfies specified initial or boundary conditions.

Types of Boundary Conditions

I) a boundary condition of the Ist kind (**Dirichlet** boundary condition) specifies the value of the unknown function at the boundary $x = L$:

$$u\big|_{x=L} = f$$

II) a boundary condition of the IInd kind (**Neumann** boundary condition) specifies the value of the derivative of the unknown function at the boundary $x = L$ (flux):

$$\left.\frac{du}{dx}\right|_{x=L} = f$$

III) a boundary condition of the IIIrd kind (**Robin** boundary condition or **mixed** boundary condition) specifies the value of the combination of the unknown function with its derivative at the boundary $x = L$ (a convective type boundary condition)

$$\left[k\frac{du}{dx} + hu\right]\bigg|_{x=L} = f$$

Boundary value problems can be **well-posed** or **ill-posed**.

Uniqueness of solution

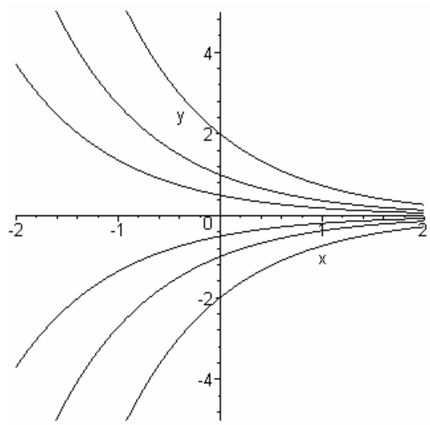
The solution of an ODE is **unique** at the point (x_0, y_0), if for all values of parameters in the general solution, there is only one integral curve which goes through this point. Such a point where the solution is not unique or does not exist is called a **singular point**.

The question of the existence and uniqueness of the solution of an ODE is very important for mathematical modeling in engineering. In some cases, it is possible to give a general answer to this question (as in the case of the first order ODE in the next section.)

Example 4: a) The general solution of the ODE in *Example 2* is

$$\left\{ y = ce^{-x}, c \in R \right\}$$

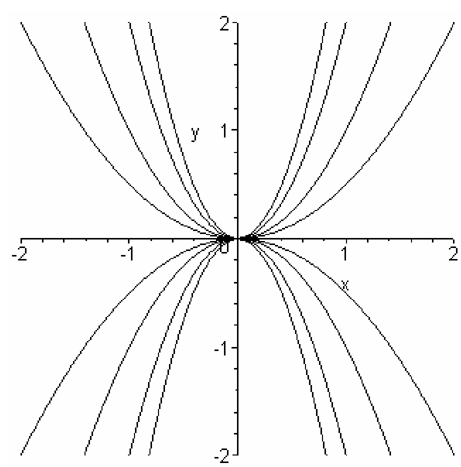
There exists a unique solution at any point in the plane



b) Consider the ODE $xy' - 2y = 0$

The general solution of this equation is $\left\{ y = cx^2, c \in R \right\}$

$(0,0)$ is a singular point for this ODE



2.2 First order ODE

In this section we will consider the first order ODE, the **general form** of which is given by

$$F(x, y, y') = 0$$

This equation may be linear or non-linear, but we restrict ourselves mostly to equations which can be written in **normal form** (solved with respect to the derivative of the unknown function):

normal form

$$y' = f(x, y)$$

or in the **standard differential form**:

standard differential form

$$M(x, y)dx + N(x, y)dy = 0$$

Note that the equation in standard form can be easily transformed to normal form and vice versa. If the equation initially was given in general form, then during transformation to normal or standard form operations (like division or root extraction) can eliminate some solutions, which are called **suppressed solutions**. Therefore, later we need to check for suppressed solutions.

Initial Value Problem

In an initial value problem (IVP) for a first order ODE, it is required to find a solution of

$$F(x, y, y') = 0 \quad \text{for } x \in I \subset R$$

subject to the initial condition at $x_0 \in I$:

$$y(x_0) = y_0, \quad y_0 \in R$$

Boundary value problems will differ only by fixing x_0 at the boundary of the region *I*.

The question of existence and uniqueness of the solution of an IVP for the first order ODE can be given in the form of sufficient conditions for equations in normal form by Picard's Theorem:

Picard's Theorem

Theorem *(existence and uniqueness of the solution of IVP)*

Let the domain *R* be a closed rectangle centered at the point $(x_0, y_0) \in R^2$:

$$R = \left\{ (x, y) \in R^2 : |x - x_0| \le a, |y - y_0| \le b \right\}$$

and let the function $f(x, y)$ be continuous and continuously differentiable in terms of the y function in the domain *R*:

$$f(x, y) \in C[R]$$
$$f_y(x, y) \in C[R]$$

and let the function $f(x, y)$ be bounded in *R*:

$$|f(x, y)| \le M \quad \text{for } (x, y) \in R .$$

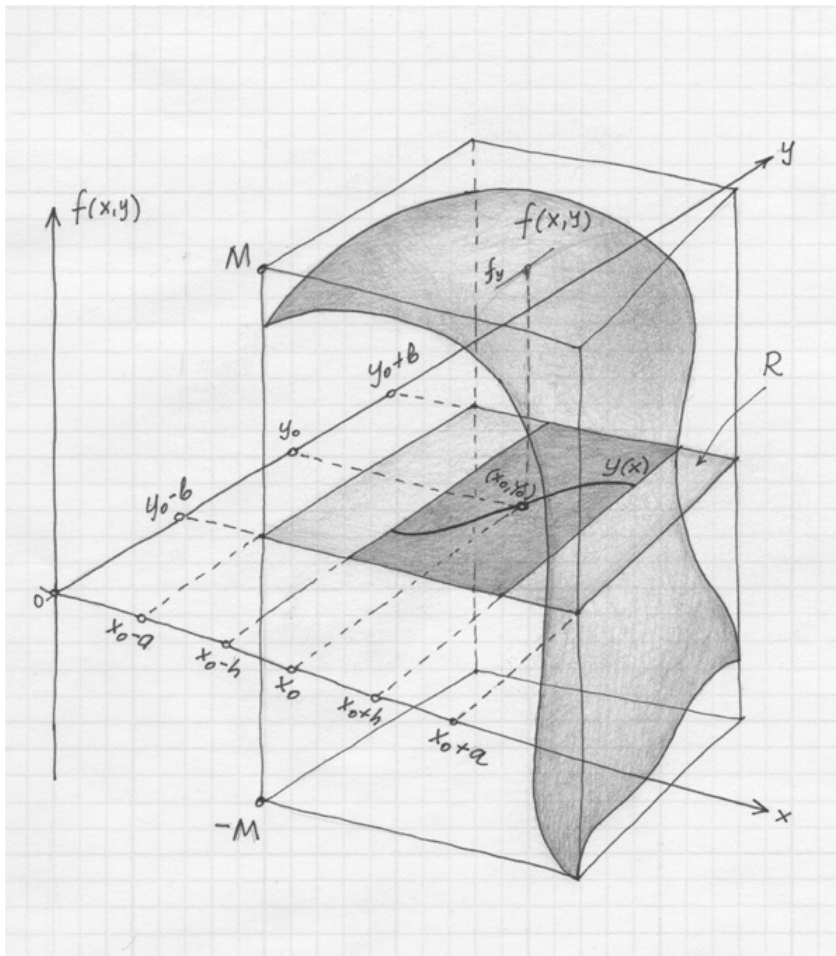
Then the initial value problem

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

has a unique solution $y(x)$ in the interval

$$I = \{ x : |x - x_0| \le h \}, \quad \text{where } h = \min\left\{ a, \frac{b}{M} \right\}$$

The proof of Picard's theorem will be given in the following chapters; it also can be found in Hartmann [ ], Perco [ ] etc. and it is based on Picard's successful approximations to the solution of IVP which we will consider later. This theorem guarantees that under given conditions there exists a unique solution of the IVP, but it does not claim that the solution does not exist if conditions of the theorem are violated. Now we will consider the most important methods of solution of the first order ODE


f(x,y)
M
f(x,y)
f_y
y
R
y_0+b
y_0
$y(x)$
(x_0,y_0)
y_0-b
0
x_0-a
x_0-h
x_0
x_0+h
x_0+a
x
-M

2.2.1 Exact ODE

Consider a first order ODE written in the **standard differential form**:

$$M(x,y)dx + N(x,y)dy = 0\,, \qquad (x,y)\in D\subset \mathbb{R}^2 \tag{1}$$

If there exists a differentiable function $f(x,y)$ such that

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \tag{2}$$

$$\frac{\partial f(x,y)}{\partial y} = N(x,y) \tag{3}$$

for all $(x,y)\in D$, then the left hand side of the equation is an **exact differential** of this function, namely

exact differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy$$

and the function $f(x,y)$ satisfying conditions (2) and (3) is said to be a **potential function** for equation (1). The equation in this case is called to be an **exact differential equation**, which can be written as

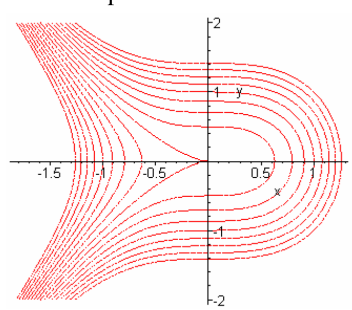
$$df(x,y) = 0 \tag{4}$$

direct integration of which yields a **general solution** of equation (1):

$$\boxed{f(x,y) = c} \tag{5}$$

where $c\in R$ is a constant of integration. The solution given implicitly defines integral curves of the ODE or the **level curves** of function $f(x,y)$.

Example 1 **The** First order ODE $3x^2dx + 2ydy = 0$ is an exact equation with the general solution $f(x,y) \equiv x^3 + y^2 = c$. Then the integral curves of this equation are



To recognize that a differential equation is an exact equation we can use a test given by the following theorem:

Test on exact differential

Theorem *1* *(Euler, 1739)*

Let functions $M(x,y)$ and $N(x,y)$ be continuously differentiable on $D\subset\mathbb{R}^2$, then the differential form

$$M(x,y)dx + N(x,y)dy \tag{6}$$

is an exact differential if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \quad \text{in } D\subset\mathbb{R}^2 \tag{7}$$

Proof: 1) Suppose that the differential form is exact. According to definition, it means that there exists a function $f(x,y)$ such that $\dfrac{\partial f(x,y)}{\partial x} = M(x,y)$ and $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$. Then differentiating the first of these equations with respect to y and the second one with respect to x, we get $\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$. Since the left hand sides of these equations are the same, it follows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

2) Suppose now that the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds for all $(x,y) \subset D$. To show that there exists a function $f(x,y)$ which produces an exact differential of the form (6), we will construct such a function. The same approach is used for finding a solution of an exact equation.

We are looking for a function $f(x,y)$, the differential form (6) of which is an exact differential. Then this function should satisfy conditions (2) and (3). Take the first of these conditions:

$$\frac{\partial f(x,y)}{\partial x} = M(x,y)$$

and integrate it formally over variable x, treating y as a constant, then

$$f(x,y) = \int M(x,y)dx + k(y) \tag{8}$$

where the constant of integration depends on y. Differentiate this equation with respect to y and set it equal to condition (3):

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + \frac{dk(y)}{dy} \quad = N(x,y)$$

Rearrange the equation as shown

$$\frac{dk(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

Then integration over the variable y yields:

$$k(y) = \int\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]dy + c_1$$

Substitute this result into equation (8) instead of $k(y)$

$$f(x,y) = \int M(x,y)dx + \int\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]dy + c_1 \tag{9}$$

To show that this function satisfies conditions (2) and (3), differentiate it with respect to x and y and use condition (7). Therefore, differential form (6) is an exact differential of the function $f(x,y)$ constructed in equation (9).
The other form of the function $f(x,y)$ can be obtained if we start first with condition (3) instead of condition (2):

$$f(x,y) = \int N(x,y)dy + \int\left[M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dx\right]dx + c_2 \tag{10}$$

Note, that condition (7) was not used for construction of functions (9) or (10), we applied it only to show that form (6) is an exact differential of these functions. ■

Then according to equation (5), a general solution of exact equation is given by an implicit equations:

$$\boxed{f(x,y) = \int M(x,y)dx + \int\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]dy = c} \tag{11}$$

or

$$\boxed{f(x,y) = \int N(x,y)dy + \int\left[M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dx\right]dx = c} \tag{12}$$

The other form of the general solution can be obtained by constructing a function with help of a definite integration involving an arbitrary point (x_0, y_0) in the region $D \subset \mathbb{R}^2$:

$$f(x,y) = \int_{x_0}^{x} M(t,y)dt + \int_{y_0}^{y} N(x_0,t)dt = c \tag{13}$$

$$f(x,y) = \int_{x_0}^{x} M(t,y_0)dt + \int_{y_0}^{y} N(x,t)dt = c \tag{14}$$

Formulas (1) and (12) or (13) and (14) are equivalent – they should produce the same solution set of differential equation (1), but actual integration may be more convenient for one of them.

Example 2 Find a complete solution of the following equation

$$(3y + x)dx + (y + 3x)dy = 0$$

Test for exactness:

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3 \qquad \Rightarrow \text{ the equation is exact}$$

We can apply eqns. 11-14, but in practice, usually, it is more convenient to use the same steps to find the function $f(x,y)$ as in the derivation of the solution.

Start with one of the conditions for the exact differential

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) = (3y + x)$$

Integrate it over x, treating y as a parameter (this produces a constant of integration $k(y)$ depending on y)

$$f(x,y) = 3yx + \frac{x^2}{2} + k(y)$$

Use the second condition for the exact differential

$$\frac{\partial f(x,y)}{\partial y} = N(x,y) = 3x + y$$

$$3x + \frac{\partial k(y)}{\partial y} = 3x + y$$

$$\frac{\partial k(y)}{\partial y} = y$$

Solve this equation for $k(y)$

$$k(y) = \frac{y^2}{2}$$

neglecting the constant of integration. The function is completely determined and the solution of the ODE is given by

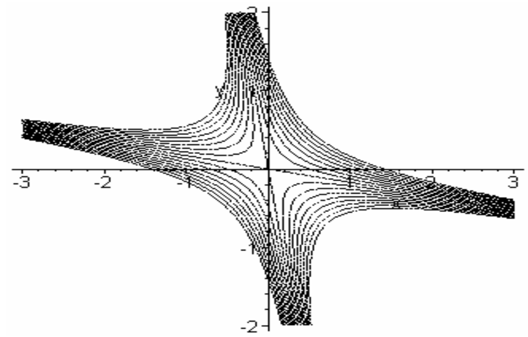
$$f(x,y) \equiv 3yx + \frac{x^2}{2} + \frac{y^2}{2} = c$$

or we can rewrite it as a general solution given by the implicit equation:

General solution:

$$x^2 + 6xy + y^2 = 0$$

Solution curves:



Note that at the point (0,0) the solution is not unique.
Where also conditions of Picard's theorem are violated?
The solution with help from equation 13:
Choose $x_0 = 0$, $y_0 = 0$, then

$$f(x,y) = \int_0^x (3y+t)\,dt + \int_0^y (0+t)\,dt = c$$

$$\left[3yt + \frac{t^2}{2}\right]_0^x + \left[\frac{t^2}{2}\right]_0^y = c$$

$$3yx + \frac{x^2}{2} + \frac{y^2}{2} = c$$

This is the same solution as in the first approach.

2.2.2 Equations Reducible to Exact - Integrating Factor

Integrating factor

In general, non-exact equations, which possess a solution, can be transformed to exact equations after multiplication by some nonzero function $\mu(x, y)$, which is called an **integrating factor** (existence of the integral factor was proved by Euler).

Theorem *2* The function $\mu(x, y)$ is an integrating factor of the differential equation $M(x, y)dx + N(x, y)dy = 0$ if and only if $\mu(x, y)$ satisfies the partial differential equation

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu$$

Proof*:* as an exercise ■

But it is not always easy to find this integrating factor. There are several special cases for which the integrating factor can be determined:

1) $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0$ The test for exactness. The integrating factor

$$\mu(x, y) = 1$$

2) $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ The test for exactness fails but the given ratio is a function of x only. Then the integrating factor is

$$\mu(x) = e^{\int f(x)dx}$$

3) $\dfrac{-\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial x}}{M}=g(y)$ The test for exactness fails but the given ratio is a function of y only. Then the integrating factor is
$$\mu(y)=e^{\int g(y)dy}$$

4) $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{yN-xM}=h(xy)$ The test for exactness fails but the given ratio is a function of the product of x and y. Then the integrating factor is
$$\mu(x,y)=\int h(xy)d(xy)$$

5) $\dfrac{y^2\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)}{xM+yN}=k\left(\dfrac{y}{x}\right)$ The test for exactness fails but the given ratio is a function of the ratio $\dfrac{x}{y}$. Then the integrating factor is
$$\mu(x,y)=\int k\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

6) $\dfrac{M(\lambda x,\lambda y)}{N(\lambda x,\lambda y)}=\dfrac{M(x,y)}{N(x,y)}$ The functions M and N are homogeneous functions of the same degree (see section). Then the integrating factor is
$$\mu(x,y)=\frac{1}{xM+yN}$$
providing $xM+yN\neq 0$.

Example 3 Find a complete solution of the following equation
$$\left(x+y^2\right)dx-2xydy=0$$

Test for exactness:

$\dfrac{\partial M}{\partial y}=2y \quad \dfrac{\partial N}{\partial x}=-2y$ $\Rightarrow$ equation is not exact

test for integrating factor:

$\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-(-2y)}{-2xy}=\dfrac{-2}{x}=f(x)$ $\Rightarrow$ int.factor by Eq. 2

$$\mu(x)=e^{\int f(x)dx}=e^{-2\int \frac{1}{x}dx}=e^{-2\ln x}=e^{\ln\frac{1}{x^2}}=\frac{1}{x^2}$$

$$\frac{\left(x+y^2\right)}{x^2}dx-\frac{2y}{x}dy=0$$

$$\frac{\partial f}{\partial x}=\frac{x+y^2}{x^2}=\frac{1}{x}+\frac{y^2}{x^2} \quad \Rightarrow \quad f=\ln|x|-\frac{y^2}{x}+k$$

$$\frac{\partial f}{\partial y}=-\frac{2y}{x}+\frac{\partial k}{\partial y}=\frac{-2y}{x} \quad \Rightarrow \quad k=c$$

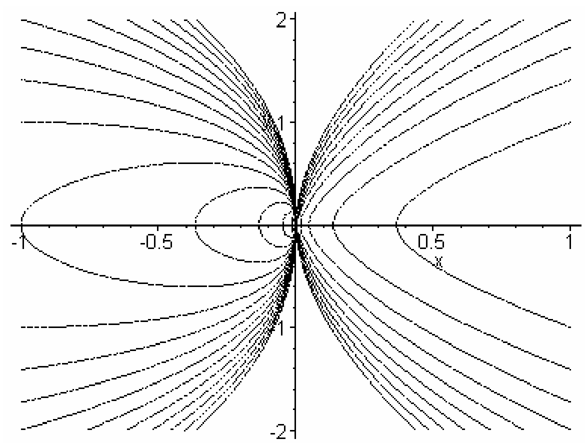
General solution:

$$-\frac{y^2}{x}+ln|x|=c \qquad \Rightarrow \qquad cx=e^{\frac{y^2}{x}}$$

Is $x=0$ a suppressed solution: $(x+y^2)\frac{dx}{dy}-2xy=0$ (yes)

Illustration of this problem with Maple:

```
> restart;
> with(plots):
> f:={seq(log(abs(x))-y^2/x=i,i=-10..10)}:
>implicitplot(f,x=1..1,y=2..2,numpoints=6000);
```



Solution with Maple:

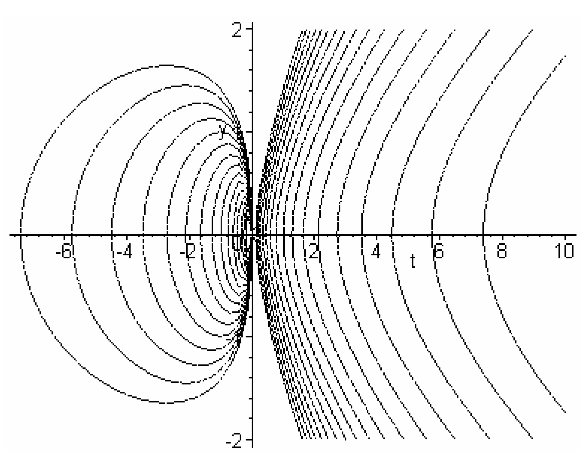
```
> restart;
> with(DEtools):
> DE:=diff(y(t),t)*2*y(t)*t=y(t)^2+t;
```

$$DE := 2\left(\frac{\partial}{\partial t}\mathrm{y}(t)\right)\mathrm{y}(t)\,t=\mathrm{y}(t)^2+t$$

```
> s:=dsolve(DE,y(t));
```

$$s := \mathrm{y}(t)=\sqrt{t\ln(t)+t\_C1},\ \mathrm{y}(t)=-\sqrt{t\ln(t)+t\_C1}$$

```
> restart;
> q:={seq(y(t)^2=t*ln(abs(t))+t*i/4,i=-8..8)}:
> with(plots):
> implicitplot(q,t=-10..10,y=2..2,numpoints=5000);
```



Supressed solutions

If the given differential equation is reduced to standard differential form

$$M(x, y)dx + N(x, y)dy = 0$$

with some algebraic operations, then zeros of the expressions involved in division can be solutions of the differential equation not included in the general solution. Such lost solutions are called **suppressed solutions**. If such operations were applied for the transformation of the differential equation, then the equation has to be checked for suppressed solutions.

To check if $y = a$ is a suppressed solution of Eq. 1, reduce the differential equation to normal form with y as a dependent variable

$$\frac{dy}{dx} = \frac{-M(x, y)}{N(x, y)}$$

and substitute $y = a$.

To check if $x = b$ is a suppressed solution of Eq. 1, reduce the differential equation to normal form with x as a dependent variable

$$\frac{dx}{dy} = \frac{-N(x, y)}{M(x, y)}$$

and substitute $x = b$.

Then the suppressed solutions should be added to the general solution.

Solid Geometry

2.2.3 Separable equations

Separable equation

Definition 1 A differential equation of the first order is called **separable** if it can be written in the following standard differential form:

$$M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0 \tag{1}$$

where $M_1(x), N_1(x)$ are functions of the variable x only and $M_2(y), N_2(y)$ are functions of the variable y only.

Assuming that $N_1(x) \neq 0$ and $M_2(y) \neq 0$ for all x and y in the range, variables in equation (1) can be separated by division with $M_2(y)N_1(x)$:

$$\frac{M_1(x)}{N_1(x)}dx + \frac{N_2(y)}{M_2(y)}dy = 0 \tag{2}$$

Then equation (2) can be formally integrated to obtain a general solution:

$$\boxed{\int \frac{M_1(x)}{N_1(x)}dx + \int \frac{N_2(y)}{M_2(y)}dy = c} \tag{3}$$

where $c \in R$ is an arbitrary constant.
Note, that separated equation (2) is exact - it can be obtained from equation (1) by multiplication by the integrating factor $\mu = \dfrac{1}{N_1(x)M_2(y)}$; the potential function for this equation

$$f(x,y) = \int \frac{M_1(x)}{N_1(x)}dx + \int \frac{N_2(y)}{M_2(y)}dy$$

Which yields the same general solution $f(x,y) = c$.
Because of division by $M_2(y)N_1(x)$, some solutions can be lost; therefore, equations should be checked for suppressed solutions. If $x = x_1$, where $x_1 \in R$ belongs to the domain and is a root of $N_1(x) = 0$, then the function $x = x_1$ is obviously a solution of differential equation (1). Similarly, if $y = y_1$ is a real root of $M_2(y) = 0$, then the function $y = y_1$ is also a solution. They both should be added to the general solution (3).

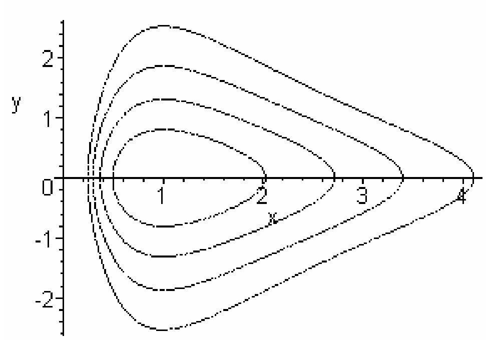
Example 1: Find a general solution of the following ODE:

$$xyy' + (y^2+1)\ln x = 0, \qquad x > 0$$

$$xydy + (y^2+1)\ln x dx = 0$$

$$\frac{\ln x}{x}dx + \frac{y}{y^2+1}dy = 0$$

$$(\ln x)^2 + \ln(y^2+1) = c \qquad \text{no suppressed solutions}$$



Example 6 Find a solution of the following ODE:

$$\left(x^2-4\right)y'-x\cot y=0$$

Solution: Separate variables:

$$\frac{xdx}{x^2-4}-\tan ydy=0$$

Integrate:

$$\ln\left(\cos y\right)^2+\ln\left|x^2-4\right|=\ln c$$

General solution:

$$\left(x^2-4\right)\cos^2 y=c$$

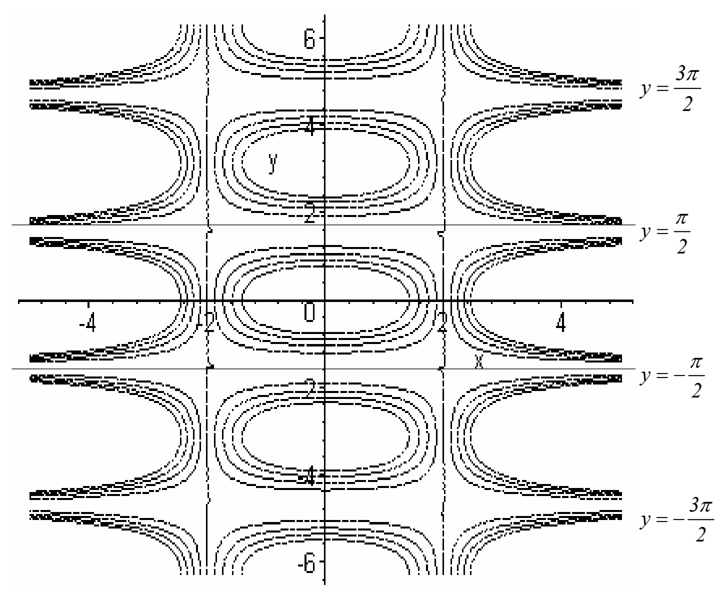
Check for suppressed solutions:

$y=\dfrac{\pi}{2}+n\pi$ are suppressed solutions.

$x=\pm 2$ are solutions of $\left(x^2-4\right)+\left(x\tan y\right)\dfrac{dx}{dy}=0$

if independent and dependent variables are reversed.

Then the family of solution curves is represented by



2.2.4 Homogeneous Equations

In this section, we will study a type of equations which can be reduced to a separable equation or to an exact equation.

Homogeneous function

Definition 1 Function $M(x,y)$ is homogeneous of degree *r*, if

$$M(\lambda x,\lambda y)=\lambda^r M(x,y) \text{ for any } \lambda\in R,\ \lambda>0$$

It means that after replacing x by λx and y by λy in the function $M(x,y)$, the parameter λ^r can be factored from the expression.

Examples 1:

a) *Homogeneous function of degree zero.*

Let $M(x,y)=\dfrac{x-y}{x+y}$, then

$$M(\lambda x,\lambda y)=\frac{\lambda x-\lambda y}{\lambda x+\lambda y}=\frac{x-y}{x+y}=\lambda^0 M(x,y)=M(x,y) \quad \text{for } \lambda>0$$

Therefore, $M(x,y)$ is homogeneous of degree zero.
If we divide the numerator and the denominator by *x,* then

$$M(x,y)=\frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

and we see that the function $M(x,y)$ depends on a single variable $\dfrac{y}{x}$.

It appears to be a fact for zero degree homogeneous functions:
the function $M(x,y)$ is homogeneous of degree zero if and only if it depends on a single variable $\dfrac{y}{x}$ [Goode, p.62]:

$$M(x,y)=f\left(\frac{y}{x}\right)$$

b) A more general fact: homogeneous functions of degree *r* can be written as

$$x^n M(x,y)=f\left(\frac{y}{x}\right) \text{ or}$$

$$y^n M(x,y)=g\left(\frac{x}{y}\right)$$

To show it, choose parameters of the form $\lambda=\begin{cases}\dfrac{1}{x} & x>0\\ \dfrac{-1}{x} & x<0\end{cases}$

c) Consider $M(x,y)=\sqrt{y^3-x^2y}$. Test on homogeneity yields

$$M(\lambda x,\lambda y)=\sqrt{(\lambda y)^3-(\lambda x)^2\lambda y}=\lambda^{2/3}\sqrt{y^3-x^2y}=\lambda^{2/3}M(x,y)$$

Therefore, the given function is homogeneous of degree $3/2$.

Homogeneous equation

Definition 2 A Differential equation written in standard differential form

$$M(x,y)dx + N(x,y)dy = 0$$

is called a **homogeneous differential equation** if functions $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree *r*.

Reduction to separable

If the equation written in standard differential form

$$M(x,y)dx + N(x,y)dy = 0$$

is homogeneous, then it can be reduced to a separable differential equation by the change of variable:

$$y = ux \qquad dy = xdu + udx$$

or

$$x = vy \qquad dx = vdy + ydv$$

Both approaches are equivalent, just because in standard differential form the variables are equivalent. But actual integration of the equation may be more convenient with one of them.

Justification: First apply the substitution to the differential equation $y = ux$

$$M(x,ux)dx + N(x,ux)xdu + N(x,ux)udx = 0$$

and divide it formally by $N(x,ux)dx$

$$\frac{M(x,ux)}{N(x,ux)} + x\frac{du}{dx} + u = 0$$

If the differential equation is homogeneous then the functions $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree *r* and, according to Example 1b), can be written as

$$M(x,y) = x^r f_1\left(\frac{y}{x}\right) = x^r f_1(u)$$

$$N(x,y) = x^r f_2\left(\frac{y}{x}\right) = x^r f_2(u)$$

Substitute them into the previous equation, then

$$\frac{f_1(u)}{f_2(u)} + x\frac{du}{dx} + u = 0$$

Now variables can be separated

$$\frac{dx}{x} + \frac{du}{\dfrac{f_1(u)}{f_2(u)} + u} = 0$$

Formally this equation can be integrated to a general solution

$$ln|x| + \int \frac{du}{\dfrac{f_1(u)}{f_2(u)} + u} = c$$

where *c* is a constant of integration. The solution of the original equation can be obtained by back substitution $u = \dfrac{y}{x}$.

Example 2: Solve the differential equation

$$\left(y^2 + 2x^2\right)dx + xydy = 0$$

M and N are homogeneous functions of degree 2.

Use change of variable:

$$y = ux \qquad dy = xdu + udx$$

$$\left(u^2x^2 + 2x^2\right)dx + xux(xdu + udx) = 0$$

$$\left(u^2x^2 + 2x^2 + u^2x^2\right)dx + ux^3du = 0$$

$$2x^2\left(u^2 + 1\right)dx + ux^3du = 0 \quad \text{separable}$$

$$2\frac{dx}{x}+\frac{udu}{u^2+1}=0$$

$$2\frac{dx}{x}+\frac{1}{2}\frac{d\left(u^2+1\right)}{u^2+1}=0$$

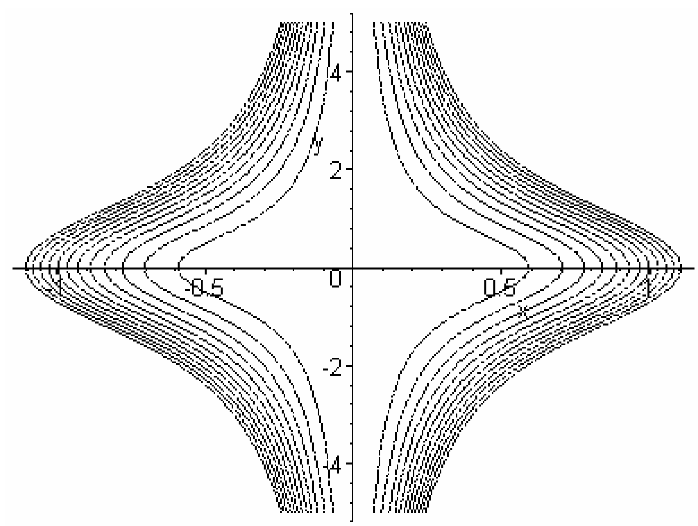
$$\ln x^4+\ln\left(u^2+1\right)=\ln c$$ general solution

$$x^4\left(u^2+1\right)=c$$ backsubstitution

$$x^2\left(y^2+x^2\right)=c$$

$$x=0$$ is also a solution

```
> f:={seq(x^2*(y^2+x^2)=i/8,i=0..12)}:
> implicitplot(f,x=-2..2,y=-5..5);
```



Reduction of homogeneous differential equation to a separable equation by transition to polar coordinates. This method is convenient when the solution is represented by complicated transcendental functions which are more suitable for representation in polar coordinates (ellipses, spirals, etc).
Conversion formulas from Cartesian to polar coordinates:

$$x=r\cos\theta \qquad x^2+y^2=r^2$$

$$y=r\sin\theta \qquad \frac{y}{x}=\tan\theta$$

$$dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial\theta}d\theta=\cos\theta dr-r\sin\theta d\theta$$

$$dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial\theta}d\theta=\sin\theta dr+r\cos\theta d\theta$$

Example 3: Solve the differential equation

$$\left(x-2y\right)dy-\left(2x-y\right)dx=0$$

It is a homogeneous equation of order 1.
Reduce it to a separable equation by transition to polar coordinates:

$$\left(y-2x\right)dx+\left(x-2y\right)dy=0$$

$$(r\sin\theta - 2r\cos\theta)(\cos\theta dr - r\sin\theta d\theta) + (r\cos\theta - 2r\sin\theta)(\sin\theta dr + r\cos\theta d\theta) = 0$$

$$2(\sin\theta\cos\theta - 1)dr + r(\cos^2\theta - \sin^2\theta)d\theta = 0$$ separable equation

$$2\frac{dr}{r} + \frac{(\cos^2\theta - \sin^2\theta)}{\sin\theta\cos\theta - 1}d\theta = 0$$

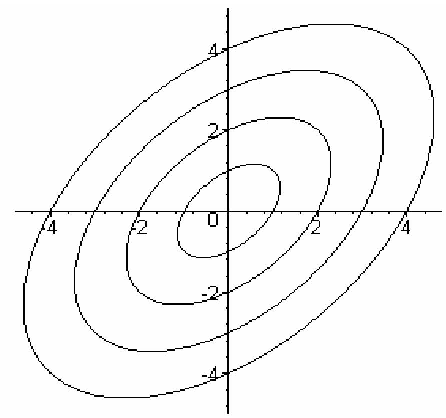
$$2\frac{dr}{r} + \frac{d(1-\sin\theta\cos\theta)}{1-\sin\theta\cos\theta} = 0$$

$$ln^{r^2} + ln|1-\sin\theta\cos\theta| = ln\,c$$ general solution

$$r = \frac{c}{\sqrt{1-\sin\theta\cos\theta}}$$ equation of ellipse in

polar coordinates

```
> f:={seq(i/sqrt(1-sin(r)*cos(r)),i=0..4)}:
> polarplot(f,r=0..2*Pi,y=-5..5);
```



Homogeneous functions in $\mathbb{R}^n$

Definition 3 A real valued function $f(x):\mathbb{R}^n \to \mathbb{R}$ defined in $\mathbb{R}^n$ is called homogeneous of degree r, if

$f(\lambda x) = \lambda^r f(x)$ $\lambda \in \mathbb{R}$, $\lambda > 0$

Theorem 1 *(Euler)*

Suppose $U \subseteq \mathbb{R}^n$ is a region in $\mathbb{R}^n$ and the function $f: U \to \mathbb{R}$ homogeneous of degree r, then

$$x\cdot\nabla f \equiv x_1\frac{\partial f(x)}{\partial x_1} + \cdots + x_n\frac{\partial f(x)}{\partial x_n} = rf(x)$$

Proof: Consider an identity following from the definition of the homogeneous function of degree r

$$f(x) = \frac{f(\lambda x)}{\lambda^r}$$

Differentiate it with respect to the parameter λ, using the chain rule

$$0 = -r\lambda^{-r-1}f(\lambda r) + \lambda^{-r}x\cdot\nabla f(\lambda x)$$

Choose $\lambda = 1$, then

$$0 = -rf(r) + x\cdot\nabla f(x)$$

from which follows the claimed result. ■

2.2.5 Linear 1st order ODE

The properties of a linear ODE of an arbitrary order will be established later.

Standard form

The general form of the first order linear differential equation is given by:

$$L_1 y \equiv a_0(x)y' + a_1(x)y = f(x) \qquad x \in D \subset \mathbb{R} \tag{1}$$

We can rewrite this equation in the **standard form**, if we divide it by $a_0(x)$

$$y' + \frac{a_1(x)}{a_0(x)}y = \frac{f(x)}{a_0(x)} \qquad a_0(x) \neq 0$$

Then for simplicity, coefficients may be renamed, and the equation becomes

$$\boxed{y' + P(x)y = Q(x)} \qquad \text{where } P(x) = \frac{a_1(x)}{a_0(x)} \text{ and } Q(x) = \frac{f(x)}{a_0(x)} \tag{2}$$

Initial value problem

For the first order o.d.e., an initial value problem (IVP) is formulated in the following way:

Solve the equation $\qquad y' + P(x)y = Q(x)$
subject to the condition $\qquad y(x_0) = y_0\,,\ x_0 \in D$

In other words, we need to find a particular solution of differential equation (2) which goes through the given point $(x_0, y_0) \in \mathbb{R}^2$. Picard's Theorem established conditions for existence and uniqueness of the solution of the IVP.

General solution

We will try to find a solution of the linear equation with a help from the methods which we have already studied (integrating factor) and to do that, we transform equation (2) into standard differential form

$$[P(x)y - Q(x)]dx + dy = 0 \tag{3}$$

from which we can identify the coefficients of the standard differential form as

$M(x, y) = P(x)y - Q(x)$ and $N(x, y) = 1$

Check this equation for exactness:

$\phi = \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = P(x), \qquad$ if $P(x) \neq 0$ then the equation is not exact.

From the test for an integrating factor

$\dfrac{\phi}{N} = P(x)$ (function of x only), it follows that the integrating factor is determined by the equation

$$\mu(x) = e^{\int P(x)dx} \tag{4}$$

Multiplication of our equation by the integrating factor $\mu(x)$ transforms it to an exact equation

$$\mu(x)[P(x)y - Q(x)]dx + \mu(x)dy = 0 \tag{5}$$

Following the known procedure, we can find a function $f(x, y)$ for which differential form (5) is an exact differential:

$$\frac{\partial f}{\partial x} = \mu(x)[P(x)y - Q(x)] \quad \Rightarrow \quad f = \int \mu(x)[P(x)y - Q(x)]dx + k(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int \mu(x)[P(x)y - Q(x)]dx + k'(y) = \mu(x)$$

$$\int \mu(x)P(x)dx + k'(y) = \mu(x)$$

$$k'(y) = \mu(x) - \int \mu(x)P(x)dx$$

$$k(y) = \mu(x)y - \int \mu(x)P(x)ydx$$

$$f = \int \mu(x)[P(x)y - Q(x)]dx + k(y)$$

$$f = \int \mu(x)[P(x)y - Q(x)]dx + \mu(x)y - \int \mu(x)P(x)ydx$$

$$f = \mu(x)y - \int \mu(x)Q(x)dx$$

$$\mu(x)y - \int \mu(x)Q(x)dx = c$$

Solving this equation with respect to y, we end up with the following general solution (division by $\mu(x)$ is permitted because $\mu(x)$ is an exponential function and never equals zero):

general solution

$$y = c\mu^{-1}(x) + \mu^{-1}(x)\int \mu(x)Q(x)dx \tag{6}$$

We see that the solution of a first order linear differential equation is given explicitly and may be obtained with this formula provided that integration can be performed.

The same result may be obtained, if we show first that the differential equation multiplied by the integrating factor may be written in the form

$$\frac{d}{dx}(\mu y) = \mu Q$$

then after direct integration (from inspection, μy is a function of x only) we end up with the same general solution.

In a case of an equation with constant coefficients, the integrating factor may be evaluated explicitly

$$\mu(x) = e^{\int P(x)dx} = e^{\int adx} = e^{ax}$$

And the solution becomes

$$y = ce^{-ax} + e^{-ax}\int e^{ax}Q(x)dx \tag{7}$$

Solution of IVP

Using initial condition $y(x_0) = y_0$, we can determine the constant of integration directly from the general solution.
In another more formal approach, we can check by inspection that

$$y = y_0\mu(x)\mu^{-1}(x) + \mu^{-1}(x)\int_{x_0}^{x} \mu(x)Q(x)dx \tag{8}$$

is a solution satisfying the initial condition.
For an equation with constant coefficients, the solution of the IVP is given by

$$y = y_0 e^{-a(x-x_0)} + e^{-ax}\int_{x_0}^{x} e^{ax}Q(x)\,dx \tag{9}$$

Example 1 *First order linear o.d.e. with variable coefficients*

Find a general solution of equation
$y' + (\cot x)y = \sin 2x$
and sketch the solution curves.

Solution: The integrating factor for this equation is

$$\mu(x) = e^{\int \cot x dx} = e^{\ln|\sin x|} = \sin x$$

then a general solution is

$$y = \frac{c}{\sin x} + \frac{1}{\sin x}\int \sin(x)\sin(2x)dx$$

$$= \frac{c}{\sin x} + \frac{2}{\sin x}\int \sin(x)\sin(x)\cos(x)dx$$

$$= \frac{c}{\sin x} + \frac{2}{\sin x}\int \sin^2(x)d\sin(x)$$

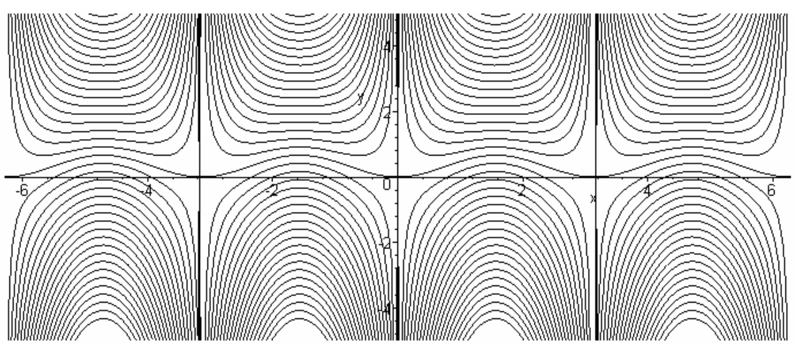
$$= \frac{c}{\sin x} + \frac{2\sin^2 x}{3}$$

In Maple, create a sequence of particular solutions by varying the constant c, and then plot the graph of solution curves:

```
> y(x):=2*sin(x)^2/3+c/sin(x);
```

$$y(x) := \frac{2}{3}\sin(x)^2 + \frac{c}{\sin(x)}$$

```
> f:={seq(subs(c=i/4,y(x)),i=-20..20)}:
> plot(f,x=-2*Pi..2*Pi,y=-5..5);
```



$x = -\pi$ $x = 0$ $x = \pi$

Example 2 *An Initial value problem for an equation with constant coefficients*

Solve the equation

$y' + y = sin\,x$ subject to the initial condition: $y(0) = 1$

Solution: Applying equation (9), we obtain the solution of the IVP:

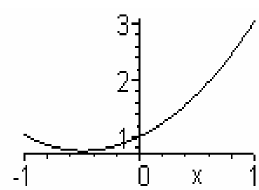
$$
\begin{aligned}
y &= y_0 e^{a(x-x_0)} + e^{-ax}\int_{x_0}^{x} e^{ax} Q(x)dx \\
&= 1 \cdot e^{1\cdot(x-0)} + e^{-1\cdot x}\int_{0}^{x} e^{1\cdot x}\,sin(x)dx \\
&= e^{x} + e^{-x}\int_{0}^{x} e^{x}\,sin(x)dx \\
&= e^{x} + e^{-x}\left[\frac{1}{2} + \frac{1}{2}e^{x}\,sin\,x - \frac{1}{2}e^{x}\,cos\,x\right] \\
&= e^{x} + \frac{1}{2}e^{-x} + \frac{sin\,x - cos\,x}{2}
\end{aligned}
$$

Use Maple to sketch the graph of the solution:

```
> y := exp(x)+exp(-x)/2+(sin(x)-cos(x))/2;
```

$$y := \mathrm{e}^{x} + \frac{1}{2}\mathrm{e}^{(-x)} + \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)$$

```
> plot(y,x=-1..1,color=black);
```



2.2.6 Special Equations

Some first order non-linear ODE's which do not fall into one of the abovementioned types can be solved with the help of special substitution. These equations arise as a mathematical model of specific physical phenomena, and they carry the names of mathematicians who first investigated these problems.

1. Bernoulli Equation

Definition 1 The differential equation which can be written in the form

$$\boxed{y' + P(x)y = Q(x)y^n} \tag{1}$$

where $n \in \mathbb{R}$ is a real number is called a Bernoulli equation.

If $n=0$ or $n=1$, then the equation is linear and it can be solved by a corresponding method, otherwise the Bernoulli equation is a non-linear differential equation.
By the change of dependent variable

$$y = z^{\frac{1}{1-n}} \tag{2}$$

The non-linear Bernoulli equation ($n \neq 1$) can be reduced to a linear first order ODE. Indeed, the derivative of the function y can be expressed as

$$\frac{dy}{dx} = \frac{d}{dx}\left[z^{\frac{1}{1-n}}\right] = \frac{1}{1-n} z^{\frac{1}{1-n}-1} \frac{dz}{dx} = \frac{1}{1-n} z^{\frac{n}{1-n}} z' \tag{3}$$

Substitution of (2) and (3) into equation (1) yields

$$\frac{1}{1-n} z^{\frac{n}{1-n}} z' + P(x) z^{\frac{1}{1-n}} = Q(x) z^{\frac{n}{1-n}}$$

Dividing this equation by $z^{\frac{n}{1-n}}$ and multiplying by $1-n$, we end up with

$$\boxed{z' + (1-n)P(x)z = (1-n)Q(x)} \tag{4}$$

Equation (4) is a linear ODE, the general solution of which can be found with a known method (see section 4). Then solution of the Bernoulli equation is determined by back substitution

$$z = y^{1-n} \tag{5}$$

It is easy to see that the Bernoulli equation possesses also a trivial solution $y = 0$ when n is positive.

Example 1 *Bernoulli equation*

Find a general solution of the equation

$$y' + y = xy^{\frac{2}{3}}$$

Solution: Use a change of variable $y = z^{\frac{1}{1-n}} = z^{\frac{1}{1-2/3}} = z^3$ which yields a linear equation

$$z' + \frac{1}{3} z = \frac{x}{3}$$

The integrating factor for this equation is $\mu = e^{\frac{x}{3}}$, and then the general solution is

$$z = ce^{\frac{-x}{3}} + e^{\frac{-x}{3}} \int e^{\frac{x}{3}} \frac{x}{3} dx = ce^{\frac{-x}{3}} + x - 3$$

Back substitution results in the general solution of the initial equation

$$z = y^{\frac{1}{3}} = ce^{\frac{-x}{3}} + x - 3$$

or in explicit form, the general solution is determined by the following equation

$$y = \left(ce^{\frac{-x}{3}} + x - 3 \right)^3$$

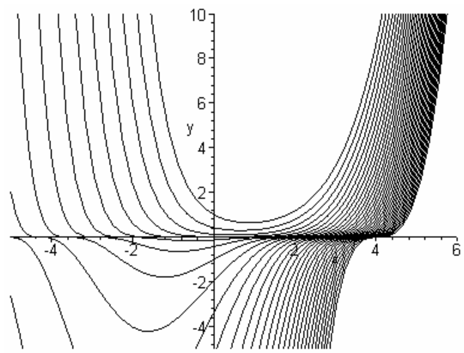
One more solution of the given equation is a trivial solution $y = 0$.

Use Maple to sketch the solution curves:

```
> y(x):=(c*exp(-x/3)+x-3)^3;
```

$$y(x) := (c\, e^{(-1/3\, x)} + x - 3)^3$$

```
> f:={seq(subs(c=i/4,y(x)),i=-16..16)}:
> plot(f,x=-5..6,y=-5..10,color=black);
```



2. Ricatti equation

Definition 2 A differential equation which can be written in the form

$$y' = P(x)y^2 + Q(x)y + R(x) \tag{6}$$

is called a **Ricatti** equation.

If one particular solution of Ricatti equation is known, then as it was first shown by Euler, it can be reduced to a first order linear ODE:

Theorem 1 Suppose that $P(x), Q(x), R(x) \in C[D]$, $D \in \mathbb{R}$ are continuous functions on *D*. Then if the function $u(x)$, $x \in D$ is a solution of the Ricatti equation (6) in *D*, then the substitution

$$y(x) = u(x) + \frac{1}{z(x)} \tag{7}$$

for all $x \in D$ for which $z(x) \neq 0$ transforms the Ricatti equation (6) into the first order ODE:

$$z' + [2P(x)u(x) + Q(x)]z + P(x) = 0 \tag{8}$$

Proof: Suppose the function $u(x): D \to R$ solves the Ricatti equation, then

$$y' = u'(x) - \frac{z'(x)}{(z(x))^2}$$

and, substituting the expression on the right into the Ricatti equation, we first obtain

$$u'(x)-\frac{z'(x)}{(z(x))^2} = P(x)\left[u(x)+\frac{1}{z(x)}\right]^2 + Q(x)\left[u(x)+\frac{1}{z(x)}\right]+R(x)$$

$$= P(x)\left[u^2(x)+2u(x)\frac{1}{z(x)}+\frac{1}{z^2(x)}\right]+Q(x)u(x)+Q(x)\frac{1}{z(x)}+R(x)$$

which, using the fact that (because $u(x)$ is a solution of (6))

$u'(x)-P(x)u^2(x)-Q(x)u(x)-R(x)=0$

simplifies to

$$0 = u'(x)-P(x)u^2(x)-Q(x)u(x)-R(x)$$

$$= \frac{z'(x)}{z^2(x)}+[2P(x)u(x)+Q(x)]\frac{1}{z(x)}+P(x)\frac{1}{z^2(x)}$$

Multiplication of this equation by $z^2(x)$, finally yields the claimed linear first order equation

$$z'+[2P(x)u(x)+Q(x)]z+P(x)=0$$

■

Remarks: - It does not matter how simple the particular solution $u(x)$ is;
- For an equation with constant coefficients, this particular solution can be found as a constant (steady state solution).

By the other substitution, the Ricatti equation can be reduced to a linear ODE of the second order:

Theorem 2 Suppose that $P(x), Q(x), R(x) \in C[D]$, $D \in \mathbb{R}$ are continuous functions on *D*. Then the substitution

$$y(x) = -\frac{w'(x)}{P(x)w(x)} \tag{9}$$

for all $x \in D$ for which $P(x) \neq 0$ and $w(x) \neq 0$ transforms the Ricatti equation (4) into a second order ODE:

$$w''-\left[\frac{P'(x)}{P(x)}+Q(x)\right]w'+R(x)P(x)w=0 \tag{10}$$

Proof: Differentiate equation (9)

$$y' = \frac{-w''}{Pw}+\frac{w'}{(Pw)^2}(P'w+Pw')$$

$$= \frac{-w''}{Pw}+\frac{P'w'}{P^2w}+\frac{(w')^2}{Pw^2}$$

and substitute it together with equation (9) into the Riccati equation (4). It yields the linear equation (10)

■

Example 2 *Riccati equation with a known particular solution*

Find a general solution of the equation

$y' = y^2 - 2y - 3$

Solution: Given that the equation has two obvious particular solutions:

$y=-1$ and $y=3$

Choose the first one of them for substitution (7):

$$y = \frac{1}{z} - 1$$

Identify coefficients of the Riccati equation:

$P = 1$
$Q = -1$
$R = -3$

Then the corresponding linear equation (8) is

$$z' - 4z = -1$$

The general solution of this first order linear ODE is

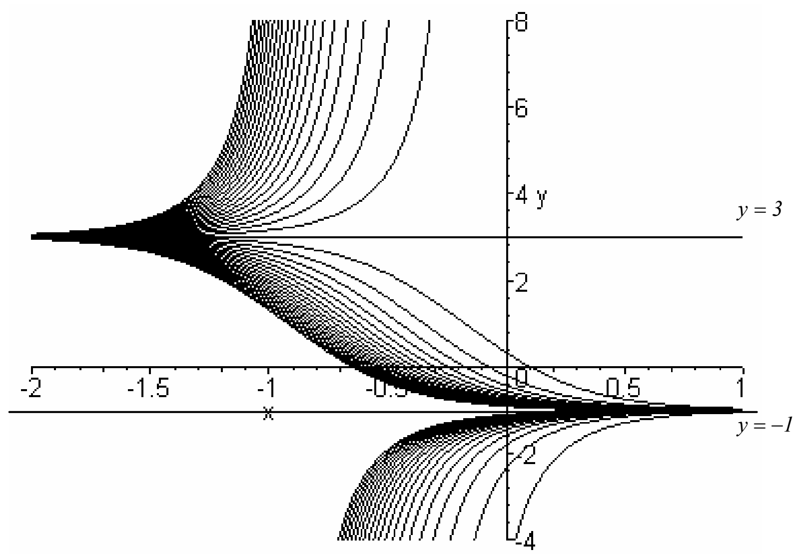
$$z = ce^{4x} + e^{4x}\int e^{-4x}(-1)dx = ce^{4x} + \frac{1}{4}$$

Then the solution of the given Riccati equation becomes

$$y = \frac{1}{z} - 1 = \frac{1}{ce^{4x} + \frac{1}{4}} - 1$$

Use Maple to sketch the solution curves:

```
> p:={seq(1/(i*exp(4*x)/2+1/4)-1,i=-20..20)}:
> plot(p,x=-2..1,y=-4..8,color=black,discont=true);
```



Special case of Riccati equation [Walas, p.13]

3. Clairaut equation

Definition 3 A differential equation which can be written in the form

$$\boxed{y = xy' + f(y')} \tag{11}$$

is called a **Clairaut** equation.

The general solution of a Clairaut equation is given by:

$$y = cx + f(c) \tag{12}$$

This can be confirmed by a direct substitution into the Clairaut equation.

The Clairaut equation additionally may include a particular solution given in parametric form:

$$x = -f'(t)$$
$$y = f(t) - tf'(t) \tag{13}$$

Example 3 Solve

$$y = xy' - (y')^2$$

This equation belongs to the Clairaut type. Therefore, the general solution of the equation is given by the one-parameter family

$$y = cx - c^2$$

Check if the parametric solution (13) is also a solution of this equation:

$$x = 2t$$

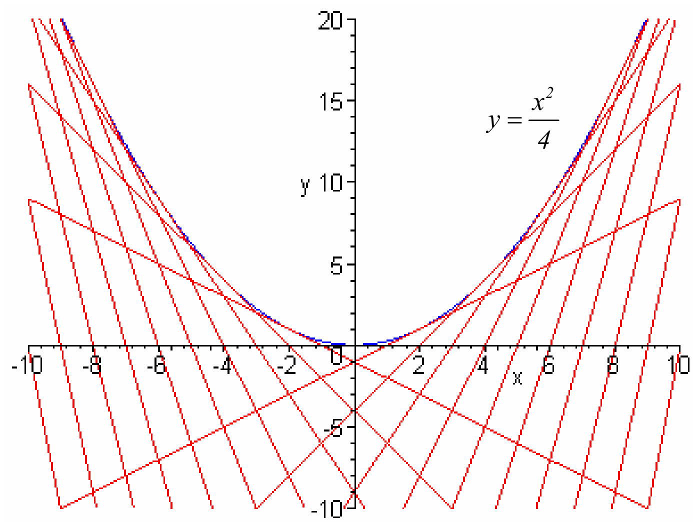
$$y = -t^2 + 2t^2 = t^2$$

Which can be reduced to an explicit equation by the solution of the first equation for t and substitution into the second equation:

$$y = \frac{x^2}{4}$$

This solution defines a (limiting curve) for the family of curves from the general solution:

```
> p:={seq(c*x-c^2,c=-20..20)}:
> g1:=plot(p,x=-10..10,y=-10..20,color=red):
> g2:=plot(x^2/4,x=-10..10,y=10..20,color=blue):
> display({g1,g2});
```



4. Lagrange equation

Definition 4 A differential equation which can be written in the form

$$\boxed{y = xg(y') + f(y')} \qquad (14)$$

is called a Lagrange equation.

Note, that the Clairaut equation is a particular case of a Lagrange equation when $g(y') = y'$.

Apply the substitution $v = y'$

$$y = xg(v) + f(v) \qquad (15)$$

Differentiate the equation w.r.t x $\quad y' \equiv v = g(v) + xg'(v)\dfrac{dv}{dx} + f'(v)\dfrac{dv}{dx}$

Solve this equation for $\dfrac{dv}{dx}$ $\quad \dfrac{dv}{dx} = \dfrac{v - g(v)}{xg'(v) + f'(v)}$

Invert the variables: $\quad \dfrac{dx}{dv} = \dfrac{xg'(v) + f'(v)}{v - g(v)}$

This equation is a linear equation for $x(v)$ as a function of an independent variable v

$$\frac{dx}{dv} - \frac{g'(v)}{v - g(v)}x = \frac{f'(v)}{v - g(v)}$$

The general solution can be obtained by integration to determine

$$x = F(v, c), \quad c \in \mathbb{R} \tag{16}$$

To determine a general solution of the Lagrange equation (14), use equation (15) to eliminate v (if possible) from equation (16) to get

$$\varphi(x, y, c) = 0 \tag{17}$$

Otherwise, the variable v can be used as a parameter in the parametric solution organized from equation (16) and equation (15) which is replaced from equation (16):

$$\begin{aligned} x &= F(v, c) && c \in \mathbb{R} \\ y &= F(v, c)g(v) + f(v) && v \in Z \end{aligned} \tag{18}$$

Example 5 *Lagrange equation*

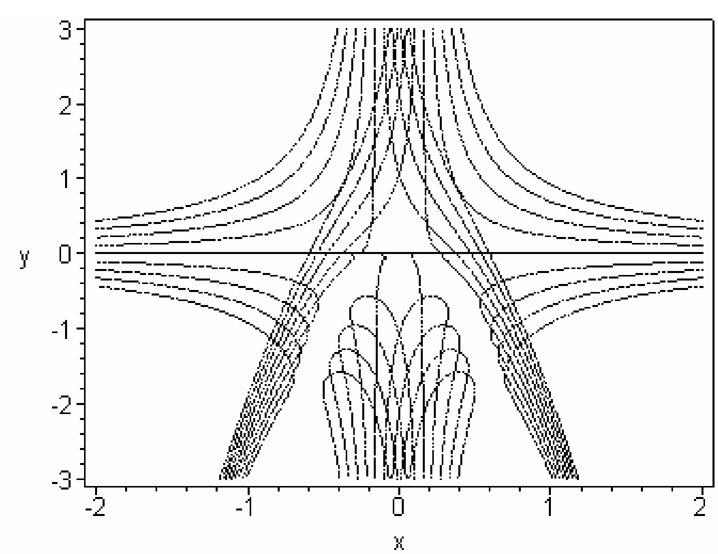
Find a general solution of $\quad y = 2x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$

General solution:

$$(7xy - 3c)^2 = y(2y + 14x^2) - 2x(7xy - 3c)(2y + 14x^2)$$

Plot the solution curves with Maple:

```
> p:={seq((7*x*y-3*i/2)^2=y*(2*y+14*x^2)-2*x*(7*x*y-
                                  3*i/2)*(2*y+14*x^2),i=-4..4)}:
> implicitplot(p,x=-2..2,y=-3..3,numpoints=10000,color=black);
```



5. Equations solvable for *y*

$$\boxed{y = f(x, y')} \tag{19}$$

This type of equations is a further generalization of Clairaut and Lagrange equations.

Apply the substitution $\quad v = y'$

$$y = f(x, v)$$

differentiate with respect to x

$$y' = \varphi\left(x, v, \frac{dv}{dx}\right) \quad \text{or}$$

$$v = \varphi\left(x, v, \frac{dv}{dx}\right)$$

this equation may be solvable for $\frac{dv}{dx}$ or $\frac{dx}{dv}$ to get a general solution

$$F(x, v, c) = 0$$

Then if from the two equations

$$\begin{aligned} &y = f(x, v) \\ &F(x, v, c) = 0 \end{aligned} \tag{20}$$

v can be eliminated, then it yields an explicit general solution

$$y = y(x, c)$$

and if v cannot be eliminated, then the system of equations (20) can be considered as a parametric solution of equation (19) with parameter v for fixed values of the constant of integration c.

Additional reading :

History of special equations: [D.Richards, p.629]
Tricky substitutions, Lagrange equation: [J.Davis, p.71]
Euler equations [Birkhoff, p.17] $\quad \left(1 - x^2\right) y'^2 = 1 - y^2$

(solution curves are conics)

2.2.7 Applications of first order ODE's

1. Orthogonal trajectories

There are many mathematical models of engineering processes where families of orthogonal curves appear. The most typical are: *isotherms* (curves of constant temperature) and *adiabats* (heat flow curves) in planar heat transfer systems; *streamlines* (lines tangent to the velocity vector) and *potential lines* of the incompressible flow of irrotational fluid; magnetic field … ; level curves and lines of steepest descent; …

Family of trajectories

A one-parameter family of planar curves is defined, in general, by the implicit equation

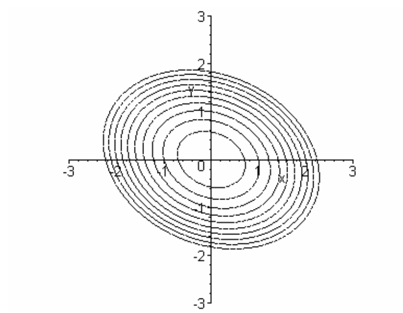
$$F(x,y,c)=0 \qquad x,y\in\mathbb{R}^2 \qquad c\in\mathbb{R} \tag{1}$$

For each value of the parameter *c,* there corresponds one particular curve (a *trajectory*). For example, equation

$$2x^2+xy+3y^2=c$$

describes the family of ellipses shown in the figure

```
> g:={seq(2*x^2+x*y+3*y^2=i,i=-10..10)}:
> implicitplot(g,x=-3..3,y=-3..3,numpoints=2000,
  scaling=constrained,view=[-3..3,-3.
```



Slope of tangent line

At each point of the curve, we can define a slope or tangent line to the curve by differentiation of equation (1) w.r.t. *x* and solving it for its derivative

$$\frac{\partial}{\partial x}F(x,y,c)=0 \qquad \Rightarrow \qquad y'=f(x,y,c) \tag{2}$$

Orthogonal lines

Lemma *(slope of orthogonal lines)*

Let two lines L_1 and L_2 be defined by equations

L_1: $\quad y=m_1x+b_1 \quad m_1\neq 0$

L_2: $\quad y=m_2x+b_2 \quad m_2\neq 0$

Then line L_1 is orthogonal to line L_2 if and only if

$$\boxed{m_1=-\frac{1}{m_2}} \tag{3}$$

Proof:

Define two lines l_1 and l_2 which are parallel to lines L_1 and L_2, but go through the origin, and define vectors on these lines:

$l_1:\quad y=m_1x \qquad \vec{u}_1=(1,m_1)$

$l_2:\quad y=m_2x \qquad \vec{u}_2=(1,m_2)$

If lines L_1 and L_2 are orthogonal, then lines l_1 and l_2 are also orthogonal, and, therefore vectors $\vec{u}_1$ and $\vec{u}_2$ are orthogonal. Two vectors are orthogonal if and only if their scalar product is equal to zero:

$$\vec{u}_1\cdot\vec{u}_2=(1,m_1)\cdot(1,m_2)=1+m_1m_2=0$$

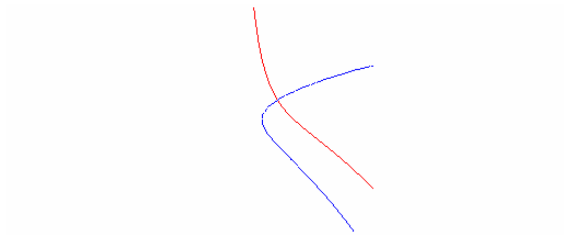
From this equation, it follows that

$$m_1=-\frac{1}{m_2} \qquad \blacksquare$$

Orthogonal trajectories

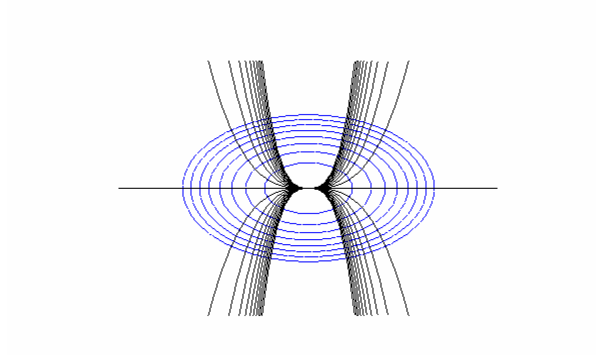
Definition 1 *(orthogonal curves)*

Two curves are orthogonal at the point of intersection if the tangent lines to the curves at this point are orthogonal



Definition 2 *(orthogonal families of curves)*

Two families of curves are called orthogonal families, if the curves from the different families are orthogonal at any point of their intersection



Algorithm

The following algorithm can be applied for finding the family of curves F_2 orthogonal to the given family of curves F_1 (shown with an example):
Let $F_1: \; 4y + x^2 + 1 + ce^{2y} = 0$, $c \in \mathbb{R}$. Find the orthogonal family F_2

1) Find the slope of the tangent lines to curves from F_1:

Differentiate $\frac{\partial}{\partial x} F_1: \; 4y' + 2x + 2cy'e^{2y} = 0$ and solve it for y' (if c appears in the equation, replace it by the solution of equation $F_1(x, y, c) = 0$ for c,
$c = -e^{-2y}\left(4y + x^2 + 1\right))$

$$y' = \frac{-x}{2 + ce^{2y}} = \frac{-x}{2 - e^{-2y}\left(4y + x^2 + 1\right)e^{2y}} = \frac{x}{x^2 + 4y - 1}$$

2) Determine the equation for the orthogonal slope as the negative reciprocals to the previous equation:

$$y' = \frac{x^2 + 4y - 1}{-x} = -x - \frac{4y}{x} + \frac{1}{x}$$

3) Solve the differential equation (the general solution will define an orthogonal family):

Rewrite the equation in the standard form of a linear equation

$$y' + \frac{4}{x} y = \frac{1}{x} - x$$

Find the integrating factor

$$\mu = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4$$

Then the general solution is:

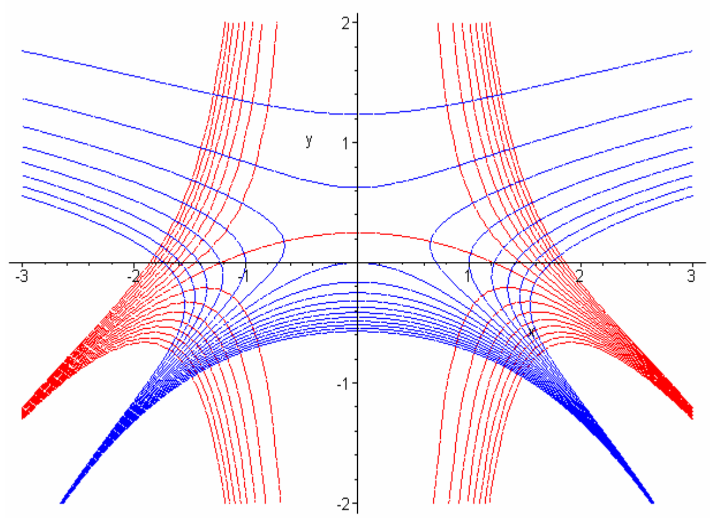
$$y = \frac{k}{x^4} + \frac{1}{x^4} \int x^4 \left(\frac{1}{x} - x \right) dx = \frac{k}{x^4} - \frac{x^6}{6} + \frac{1}{4}$$

4) Answer: F_1 : $4y + x^2 + 1 + ce^{2y} = 0 \qquad c \in \mathbb{R}$

F_2 : $y = \frac{k}{x^4} - \frac{x^2}{6} + \frac{1}{4} \qquad k \in \mathbb{R}$

Use Maple to sketch the graph of the curves (*1-14e01.mws*):

```
> restart;
> with(plots):
> F1:={seq(4*y+x^2+1+(i/2)*exp(2*y)=0,i=-8..8)}:
> p1:=implicitplot(F1,x=-3..3,y=-2..2,
                 color=blue,scaling=constrained,numpoints=2000):
> F2:={seq((j/2)/x^4-x^2/6+1/4=y,j=-8..8)}:
> p2:=implicitplot(F2,x=-3..3,y=-2..2,
                 color=red,scaling=constrained,numpoints=2000):
> display({p1,p2});
```



2.2.8 Approximate and Numerical methods for 1st order ODE's

1. Direction field

Consider a first order ODE written in normal form:

$$y' = f(x, y) \qquad (1)$$

Suppose that this equation satisfies conditions of Picard's Theorem in some domain $D \subset \mathbb{R}^2$. Then for any point $(x, y) \in \tilde{D} \subseteq D$ there exists only one solution curve which goes through this point; and equation (1) defines the slope of a tangent line to the solution curve at this point:

So equation (1) gives us a way to determine the direction of tangent lines to solution curves even without solving the equation. We can use it for visualization of the solution curves of the differential equation.

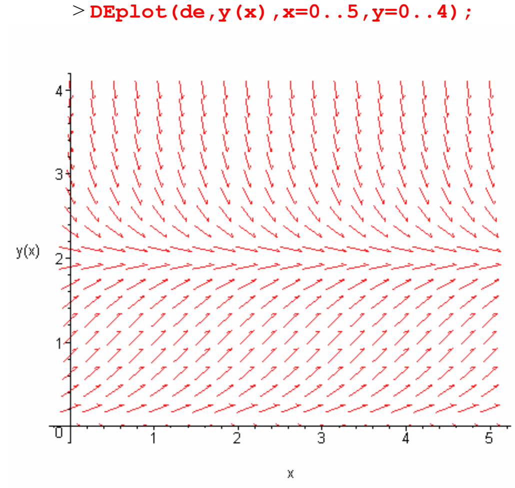
Create a grid in D as a set of points (x, y). At each point of the grid sketch a small segment with a slope given by equation (1). The obtained picture is called a **direction field** (or **slope field**) of the ODE. It gives us a general view on the qualitative behavior of solution curves of the ODE.

In Maple, the direction field of an ODE is generated by the command DEplot in the package DEtools:

```
> de:=diff(y(x),x)=2*y(x)-s*y(x)^2;
```

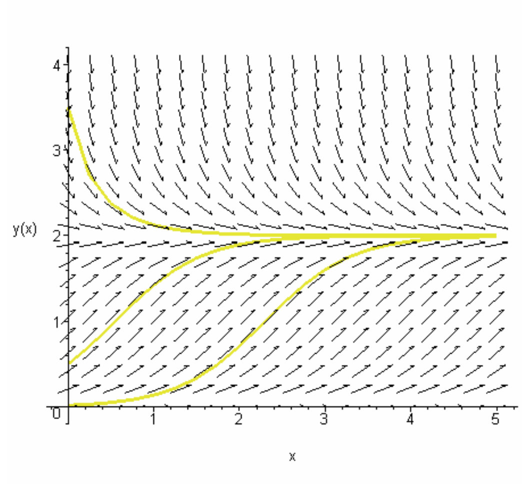
$$de := \frac{d}{dx}\,y(x) = 2\,y(x) - y(x)^2$$

```
> DEplot(de,y(x),x=0..5,y=0..4);
```



actual solution curves can be added by specifying the initial conditions:

```
> DEplot(de,y(x),x=0..5,{[0,0.02],[0,0.5],[0,3.5]},y=0..4);
```

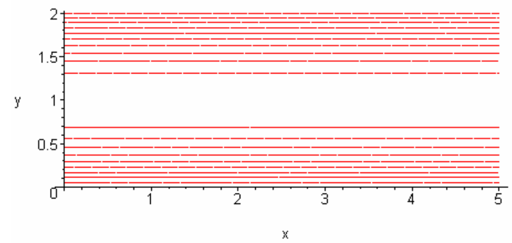


Isoclines

Isoclines of equation (1) are curves at each point of which, the slope of the solution curves is constant

$$c = f(x, y), \quad c \in \mathbb{R} \tag{2}$$

So, isoclines are curves defined by the implicit equation (2). In the previous example of the logistic equation, the function does not depend on *x,* and isoclines are the straight lines parallel to the *x*-axis:



The approximate method of solution based on application of the direction field is called the ***method of isoclines***. It consists in the construction of a direction field using isoclines and then drawing approximate solution curves following the direction segments.

2. Euler method

The direction field concept helps us to understand the idea of the Euler method, in which we use equation (1) to determine the slope of tangent lines to the solution curve step by step and construct an approximate solution curve of the IVP:

$$y' = f(x, y) \qquad\qquad y(x_0) = y_0$$

The solution is calculated at discrete points x_k, $k = 0,1,2,\dots$

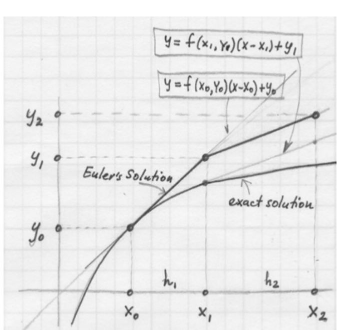
For the grid with step size h_k, the nodes are determined by

$$x_k = x_{k-1} + h_k, \quad k = 1,2,\dots$$

At the point x_0 the solution is given by the initial condition

$$y_0 = y(x_0)$$

Then we calculate the slope of the tangent line to the solution curve at the point (x_0, y_0) and draw a tangent line $y = f(x_0, y_0)(x - x_0) + y_0$. If we consider it to be an approximate solution for the interval $[x_0, x_1]$, then at the next point $x = x_1$, the approximation is given by



$$y_1 = f(x_0, y_0)(x_1 - x_0) + y_0$$

$$\boxed{y_1 = y_0 + h_1 f(x_0, y_0)}$$

Now the approximate value y_1 is known, we can calculate the slope of the tangent to the solution at the point (x_1, y_1) and draw a tangent $y = f(x_1, y_1)(x - x_1) + y_1$ from which the next approximation can be determined

$$\boxed{y_2 = y_1 + h_2 f(x_1, y_1)}$$

Continuing this process, we get for point k, that

$$y_k \quad = y_{k-1} + h_k f(x_{k-1}, y_{k-1})$$

Starting from the point specified by the initial condition (x_0, y_0), we proceed following the direction field of the differential equation to get an approximate solution curve which is a piece-wise linear curve connecting points (x_k, y_k).

The algorithm for Euler's Method can be summarized as follows:

Euler's Method

$$\begin{aligned} & x_k \in \mathbb{R} \\ & h_k = x_k - x_{k-1} \\ & y_k = y_{k-1} + h_k f(x_{k-1}, y_{k-1}) \\ & k = 1,2,\ldots \end{aligned}$$

The accuracy of Euler's Method depends on the character of variation of the solution curve and the size of steps h_k. It can be shown that when step size h_k goes to zero, Euler's approximation approaches the exact solution. But it can easily deviate from the exact solution for coarrse steps. If we want an accurate solution, then step-size should be very small. It makes the Euler method a time consuming one. Some improvement can be made, in increasing the efficiency of approximation. In the modified Euler's Method the average slope of the tangent line between steps is taken into account:

Modified Euler's Method

$$\begin{aligned} & x_k \in \mathbb{R} \\ & h_k = x_k - x_{k-1} \\ & \tilde{y}_k = y_{k-1} + h_k f(x_{k-1}, y_{k-1}) \\ & y_k = y_{k-1} + \frac{h_k}{2}\left[f(x_{k-1}, y_{k-1}) + f(x_k, \tilde{y}_k)\right] \\ & k = 1,2,\ldots \end{aligned}$$

Further improvement can be obtained by taking into account the slope of the tangent line to the solution at the intermidiate points. Depending on the number of intemidiate steps these methods are called Runge-Kutta methods of different orders. The most popular is the Fourth Order Runge-Kutta Method. Its algorithm for regular step-size h, is traditionally written in the following form:

4th Order Runge-Kutta Method

$$\begin{aligned} & x_n \in \mathbb{R} \qquad h = x_n - x_{n-1} \quad \forall n \\ & k_1 = hf(x_{n-1}, y_{n-1}) \\ & k_2 = hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{k_1}{2}\right) \\ & k_3 = hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{k_2}{2}\right) \\ & k_4 = hf(x_{n-1} + h, y_{n-1} + k_3) \\ & y_n = y_{n-1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ & n = 1,2,\ldots \end{aligned}$$

3. Picard's Method of Successive Approximations

$$y' = f(x, y) \qquad y(x_0) = y_0$$

$$y_k = y_0 + \int_{x_0}^{x} f(x_{k-1}, y_{k-1})dx \qquad k = 1,2,\ldots$$

4. Newton's Method (Taylor series solution)

We assume that the solution of the IVP for the first order differential equation in normal form

$$y' = f(x, y) \qquad y(x_0) = y_0$$

can be obtained in the form of a Taylor's series

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots$$

For this expansion we need to determine the values of the unknown function and its derivatives at the point x_0:

From initial condition $\qquad y(x_0) = y_0$

And by substitution $x = x_0$ and $y = y_0$
into the equation and determining the derivative

$$y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$$

to obtained values of the higher derivatives.
Differentiate consecutively the equation as
an implicit function and substitute

$x = x_0$ and $y = y_0$

$$y''(x_0) = \frac{d}{dx} f(x_0, y_0)$$

$$y'''(x_0) = \frac{d^2}{dx^2} f(x_0, y_0)$$

$$\vdots$$

In the obtained approximate solution, sometimes a Taylor series expansion of a known function can be identified. Newton's method can be applied also and for higher order equations.

Example 1 Use Newton's Method to solve the following IVP

$$y'' + y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

The value of the function and first derivative are already known.
From the differential equation:

$$y'' = -y \qquad y''(x_0) = -y(x_0) = -y_0$$

Differentiate the ODE and substitute $x = x_0$ and $y = y_0$:

$$y''' = -y' \qquad y'''(x_0) = -y'(x_0) = -y_1$$
$$y^{iv} = -y'' \qquad y^{iv}(x_0) = -y''(x_0) = y_0$$
$$\vdots$$

$$y^{2j} = -y^{2j-2} \qquad y^{2j}(x_0) = (-1)^j y_0$$
$$y^{2j+1} = -y^{2j-1} \qquad y^{2j+1}(x_0) = (-1)^j y_1$$

Then the Taylor's series can be constructed as

$$y(x) = y_0 + y_1(x - x_0) - \frac{y_1}{2!}(x - x_0)^2 + \frac{y_0}{3!}(x - x_0) + \cdots$$
$$= \sum_{j=0}^{\infty}\left[y_0(-1)^j \frac{(x - x_0)^{2j}}{(2j)!} + y_1(-1)^j \frac{(x - x_0)^{2j+1}}{(2j+1)!} \right]$$

Where the Taylor series expansion of trigonometric functions can be recognized:

$$= y_0 \cos(x - x_0) + y_1 \sin(x - x_0)$$

2.2.9 Equations of reducible order

1. The unknown function does not appear in an equation explicitly

The general form of these equations which are solved for the second derivative is:

$$y'' = f(x, y') \tag{1}$$

This equation can be reduced to a 1st order ODE by the change of dependent variable

$$v = y' \tag{2}$$

then
$$v' = y'' \tag{3}$$

and substitution into the equation yields a 1st order ODE for the new function v.

Example 1 Solve the following 2nd order differential equation

$$y'' + y' = x$$

The dependant variable v is missing in this equation. Then substitutions (2-3) yield

$$v' + v = x$$

which is a first order linear ODE with constant coefficients.
The general solution can be obtained by variation of parameter (with an integrating factor $\mu(x) = e^{-x}$):

$$\begin{aligned} v &= ce^{-x} + e^{-x}\int e^{x} x\,dx \\ &= ce^{-x} + e^{-x}\left(xe^{x} - e^{x}\right) \\ &= ce^{-x} + x - 1 \end{aligned}$$

Then substitution into equation (2) yields the first order ODE for the unknown function y

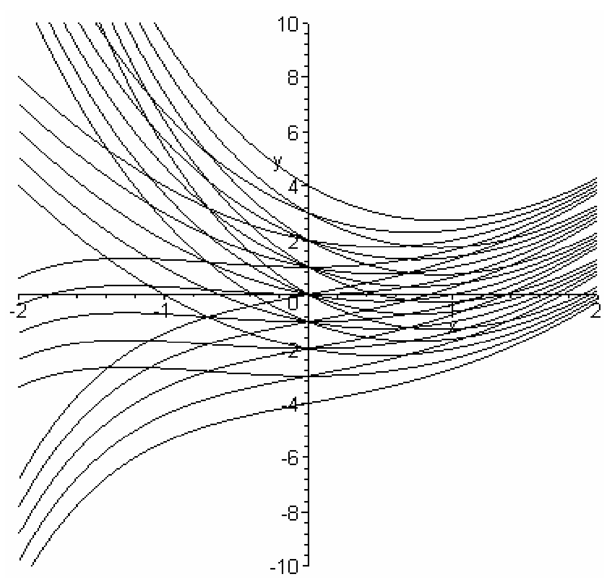
$$y' = ce^{-x} + x - 1$$

which can be solved by direct integration

$$y = c_1 e^{-x} + \frac{x^2}{2} - x + c_2$$

Solution curves can be sketched with the help of Maple:

```
> f:={seq(seq(i*exp(-x)+x^2-x+j,i=-2..2),j=-2..2)}:
> plot(f,x=-2..2,y=-10..10,color=black);
```



This approach can be applied for reduction of order of more general equations. Thus, an ODE of order n

$$F\left(x, y^{(n)}, y^{(n-1)}, \dots, y^{(k)}\right) = 0 \tag{4}$$

in which the unknown function y and its first $k-1$ derivatives are missing, by the change of variable

$$\begin{aligned} v &= y^{(k)} \\ v' &= y^{(k+1)} \\ &\vdots \\ v^{(n-k)} &= y^{(n)} \end{aligned} \tag{5}$$

is reduced to an ODE of order $n-k$:

$$F\left(x, v^{(n-k)}, y^{(n-k-1)}, \dots, v\right) = 0 \tag{6}$$

2. The independent variable does not appear in the equation explicitly (autonomous equation)

The normal form of these equations is:

$$y'' = f(y, y') \tag{7}$$

Such equations in which the independent variable does not appear explicitly, are called autonomous equations.
These equations can be transformed to 1st order ODE's by the change of the dependent variable to

$$v = y' \tag{8}$$

and then in the resulting equation consider y to be the independent variable and v to be the dependent variable. These transformations of the given ODE works as follows:

1) express derivatives of y in terms of a new function v:

$$y' = v$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}v = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v = v'v$$

2) substitution into equation (7) yields is 1st order ODE

$$vv' = f(y, v) \tag{9}$$

3) find (if possible) a general solution of equation (9) and write it in the form where it is solved for the function v (the general solution should include one parameter c_1):

$$v = F(y, c_1) \tag{10}$$

4) using back-substitution (8), set up the equation for the unknown function y

$$y' = F(y, c_1)$$

which formally can be solved by separation of variables

$$\frac{dy}{F(y, c_1)} = dx$$

$$\int \frac{dy}{F(y, c_1)} = x + c_2 \tag{11}$$

Equation (11) is an implicit form of the general solution of equation (7). It also can be written as an explicit function $x(y)$ with y as the independent variable:

$$x = \int \frac{dy}{F(y, c_1)} + c_2$$

Example 2 Solve the following 2nd order ODE

$$y'' - 2yy' = 0$$

Substitution (8) yields

$$vv' - 2vy = 0$$
$$v(v' - 2y) = 0$$

from which we have two equations:

$$v = 0$$
$$v' - 2y = 0$$

The first equation immediately leads to the solution

$$y' = 0 \quad \Rightarrow \quad y = c \qquad c \in R$$

The second equation is a 1st order ODE with the general solution

$$v = y^2 + c_1$$

Back-substitution gives the equation for y

$$y' = y^2 + c_1$$

which is a separable equation

$$\frac{dy}{y^2 + c_1} = dx$$

Depending on the sign of the constant c_1, integration yields the following solutions:

a) for $c_1 = c^2 > 0$ $\qquad \frac{1}{c} tan^{-1}\left(\frac{y}{c}\right) = x + c_2 \qquad c, c_2 \in R$

b) for $c_1 = -c^2 < 0$ $\qquad \frac{1}{2c} ln\left|\frac{y-c}{y+c}\right| = x + c_2 \qquad c, c_2 \in R$

c) for $c_1 = 0$ $\qquad \frac{1}{y} = x + c_2 \qquad c, c_2 \in R$

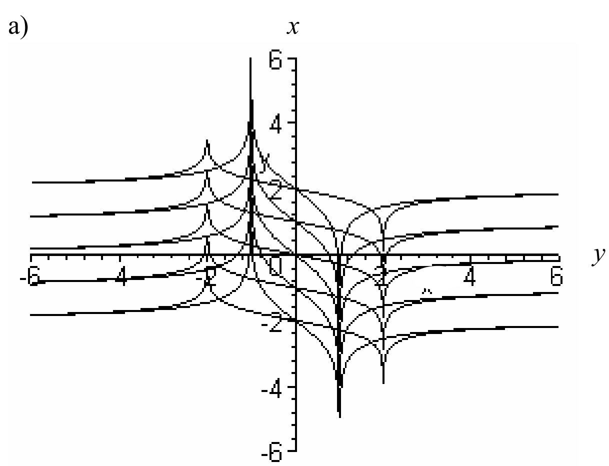
recall also the solution

d) $\qquad y = c \qquad c \in R$

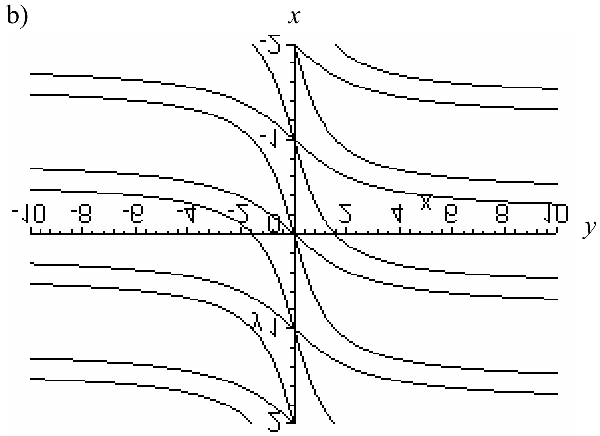
It is simpler to sketch the solution curves as explicit functions x of y; for each family of solutions they have the following form:

Note: x and y coordinates are interchanged

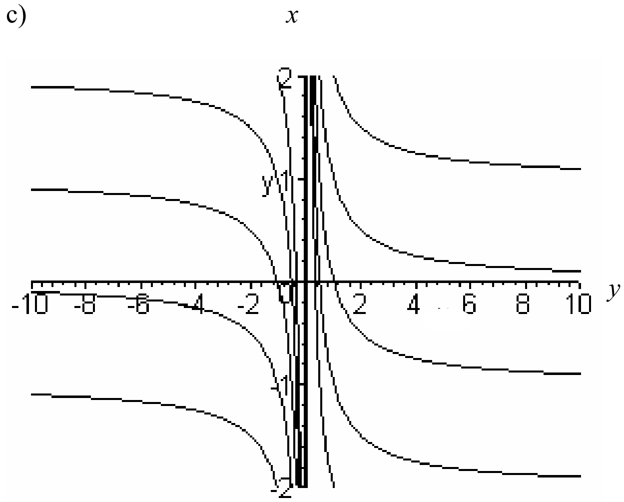
a)



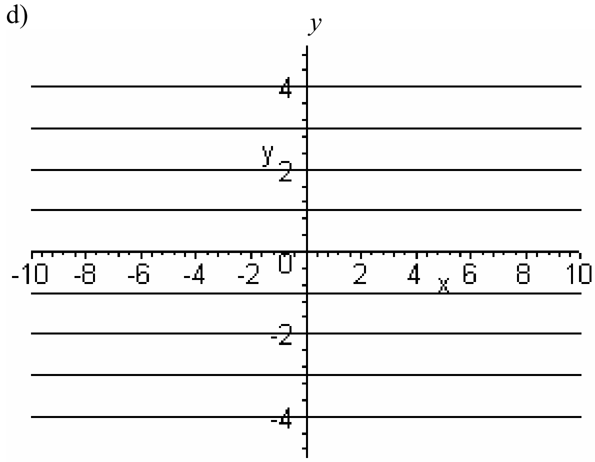
b)



c)



d)

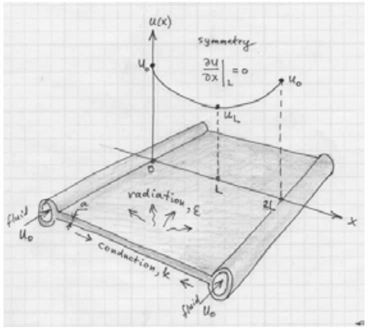


Example 3 *(outer-space radiator, example from [Siegel&Howell, ThermalRadiation Heat Transfer] with different solution)*

Excessive heat from space ships can be released only by radiating it from the surfaces exposed to outer-space which is assumed to be at zero absolute temperature. The schematic of one section of a radiator is shown in the Figure. Fluid heated inside of the ship to the temperature u_0 enters pipes connected by fins of thickness a and width $2L$. Fins are from material with thermal conductivity k and total surface emissivity ε.

Determine the steady state temperature distribution in the fin.

Assumptions for the physical and mathematical model describing heat transfer in the fins:



temperature varies only in the x-direction $u = u(x)$;
the ends of the fin attached to the pipes are at temperature u_0;
the fin surface is not exposed to direct sun radiation;
because of the symmetry, there is no heat flux at the middle of the plate:

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

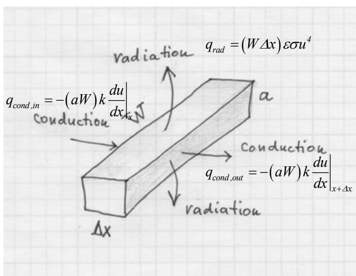
Energy balance for the control volume ($\Delta x \times a \times W$):

$$(aW)k\left[\left.\frac{du}{dx}\right|_{x+\Delta x} - \left.\frac{du}{dx}\right|_{x}\right] = (W\Delta x)\varepsilon\sigma u^4$$

at the limit $\Delta x \to 0$ yields a governing equation for temperature distribution

$$\frac{d^2u}{dx^2} = bu^4 \qquad b = \frac{2\varepsilon\sigma}{ka} \qquad x \in (0, L)$$

with boundary conditions:



$$u(0) = u_0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

The equation is a non-linear 2nd order ODE. This is an autonomous equation which can be reduced to the 1st order equation by the change of variable

$$u' = v$$
$$u'' = vv'$$

Then the equation becomes

$$vv = bu^4 \qquad \text{where } v' = \frac{dv}{du}$$

Separate variables

$$vdv = bu^4 du$$

and integrate to get a general solution

$$\frac{v^2}{2} = \frac{b}{5}u^5 + c_1$$

Apply the second boundary condition $v[u(L)] = \left.\frac{du}{dx}\right|_{x=L} = 0$ and notation

$$u_L = u(L)$$

for the fin's midpoint temperature to determine the constant of integration

$$0 = \frac{b}{5}u_L^{\ 5} + c_1$$

Then

$$v^2 = \frac{2b}{5}(u - u_L)^5$$

Because for the interval $(0, L)$ temperature of the fin is decreasing and v is in a direction of the temperature gradient, then the previous equation yields

$$v = -\sqrt{\frac{2b}{5}(u - u_L)^5}$$

which is followed with the back-substitution to

$$\frac{du}{dx} = -\sqrt{\frac{2b}{5}(u - u_L)^5}$$

and after separation of variables

$$\frac{-du}{\sqrt{\frac{2b}{5}(u - u_L)^5}} = dx$$

Definite integration of this equation for the change of temperature from u_0 to $u(x_0)$ when the space variable changes from 0 to x, yields

$$x = -\int_{u_0}^{u(x)} \frac{du}{\sqrt{\frac{2b}{5}\left(u-u_L\right)^5}} \qquad (☼)$$

This is an implicit equation for the value of the temperature at x. The value of the midpoint temperature u_L can be determined from the solution of the equation

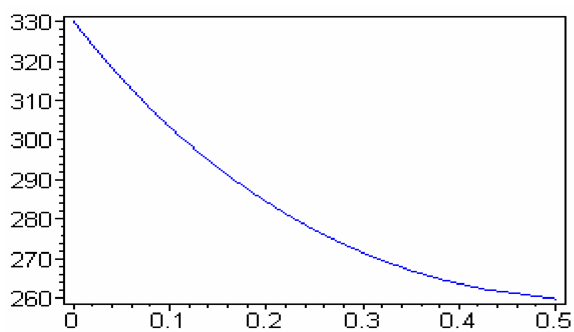
$$L = -\int_{u_0}^{u_L} \frac{du}{\sqrt{\frac{2b}{5}\left(u-u_L\right)^5}}$$

which can be solved numerically.
Then for fixed values of the coordinate x temperature values $u(x)$ can be found from the numerical solution of equation (☼).

Consider the particular case with the following values of parameters:

$$a = 0.01m\ ,\ \varepsilon = 0.8\ ,\ k = 100\frac{W}{m\cdot K}\ ,\ \sigma = 5.67\cdot 10^{-8}\frac{W}{m^2\cdot K^4}\ ,\ L = 0.5m\ ,\ u_0 = 330K$$

Then from equation (☼), the following temperature distribution follows with the midpoint temperature $u_L = 259.9K$ (Maple file: *fin3.mws*)



3. Reduction of the order of a linear equation if one solution is known

a) If any non-trivial solution $y_1(x)$ of a linear n^{th} order homogeneous differential equation is known

$$a_0(x)y^{(n)} + \cdots + a_{n-1}(x)y + a_n(x) = 0$$

then the order of the **homogeneous equation** can be reduced by one order by the change of dependent variable with
$y = y_1 v$ followed by the change of variable $v' = u$.
These two substitutions can be combined in one change of variable by

$$y = y_1\int u dx$$

which preserves linearity and homogeneity of the equation.

The order of the **non-homogeneous equation**

$$a_0(x)y^{(n)} + \cdots + a_{n-1}(x)y + a_n(x) = f(x)$$

can be reduced by one order by the change of dependent variable with the same substitution $y = y_1 v$, but the resulting equation will be non-homogeneous.
This method was used by Euler for solution of linear ODE's by systematic reduction of order.
b) Reduction formula for a 2nd order linear ODE:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

Let $y_1(x)$ be a non-trivial solution, then it satisfies

$$a_0(x)y_1'' + a_1(x)y_1' + a_n(x)y_1 = 0$$

Let $\quad y_2 = y_1 u \quad$ then

$$y_2' = y_1'u + y_1 u'$$

$$y_2'' = y_1''u + 2y_1'u' + y_1 u''$$

Substitute into the equation and collect terms in the following way

$$a_0(x)y_1 u'' + \left[2a_0(x)y_1' + a_1 y_1\right]u' + \left[a_0(x)y_1'' + a_1(x)y_1' + a_n(x)y_1\right]u = 0$$

The last term is equal to zero because $y_1(x)$ is a solution of the homogeneous equation

$$a_0(x)y_1 u'' + \left[2a_0(x)y_1' + a_1 y_1\right]u' = 0$$

Now this equation does not include the unknown function *u* explicitly, therefore, by substitution $u' = v$ it can be reduced to a 1st order equation

$$u' = v$$

$$u'' = v'$$

$$a_0(x)y_1 v' + \left[2a_0(x)y_1' + a_1 y_1\right]v = 0$$

$$v' + \frac{\left[2a_0(x)y_1' + a_1 y_1\right]}{a_0(x)y_1}v = 0$$

The integrating factor for this equation is

$$\mu = e^{\int \frac{\left[2a_0(x)y_1' + a_1 y_1\right]}{a_0(x)y_1}dx} = e^{\int \left[2\frac{y_1'}{y_1} + \frac{a_1}{a_0}\right]dx} = e^{2\int \frac{y_1'}{y_1}dx} e^{\int \frac{a_1}{a_0}dx}$$

$$= e^{2\ln y_1} e^{\int \frac{a_1}{a_0}dx} = e^{\ln y_1^2} e^{\int \frac{a_1}{a_0}dx} = y_1^2 e^{\int \frac{a_1}{a_0}dx}$$

then the general solution for *v* is

$$v = c_1 \frac{e^{-\int \frac{a_1}{a_0}dx}}{y_1^2}$$

Then the formal solution for the function *u* is

$$u = \int v dx + c_2 = c_1 \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{y_1^2} dx + c_2$$

then the second solution can be written as

$$y_2 = y_1 u = c_1 y_1 \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{y_1^2} dx + c_2 y_1$$

Choose arbitrary constants as $c_1 = 1, c_2 = 0$ then

$$\boxed{y_2 = y_1 \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{y_1^2} dx}$$

which is called the reduction formula.

Check if the solutions y_1, y_2 are linearly independent:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_1 u \\ y_1' & y_1'u + y_1 u' \end{vmatrix}$$

$$= y_1^2 u' = e^{-\int \frac{a_1}{a_0}dx} > 0$$

Therefore, the solutions are linearly independent and constitute the fundamental set for a 2nd order linear ODE.

2.3 Theory of Linear ODE

2.3.1. Linear ODE

The general form of linear ODE of the n^{th} order is given by equation

$$L_n y \equiv a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \qquad (1)$$

defined in the domain $x \in D \subset \mathbb{R}$, where coefficients $a_i(x)$ and $f(x)$ are continuous functions in D: $a_i(x), f(x) \in C[D]$.
If in addition, the leading coefficient $a_0(x) \neq 0$ for all $x \in D$, then equation (1) is said to be **normal**.

If $f(x) \equiv 0$, then equation $L_n y = 0$ is **homogeneous** or an equation without a right hand side; otherwise, the equation $L_n y = f(x)$ is **non-homogeneous** or an equation with a right hand side.

A **solution** of equation (1) is n times continuously differentiable in a D function $y(x) \in C^n[D]$ which after substitution into equation (1), turns it into an identity (in other words, $y(x)$ satisfies the differential equation).

A differential operator of n^{th} order L_n is linear in the sense that if we have two n times differentiable functions $y_1(x), y_2(x) \in C^n[D]$, then application of the operator L_n to their linear combination yields a linear combination:

$$L_n[\alpha y_1(x) + \beta y_2(x)] = \alpha L_n y_1(x) + \beta L_n y_2(x) \qquad (2)$$

This property for the operator L_n follows from the fact that the operation of differentiation is linear.

We should note, that if $y_1(x), y_2(x) \in C^n[D]$ are solutions of the non-homogeneous equation (1), then it does not necessarily yield that their linear combination is also a solution of equation (1):

$$L_n y_1(x) = f(x),\ L_n y_2(x) = f(x) \qquad \not\Rightarrow L_n[\alpha y_1(x) + \beta y_2(x)] = f(x)$$

superposition principle

Instead, we use a *superposition principle:* if functions $y_1(x), y_2(x) \in C^n[D]$ are solutions of equations $L_n y = f_1(x)$ and $L_n y = f_2(x)$ correspondingly, then their linear combination is a solution of the differential equation $L_n y = \alpha f_1(x) + \beta f_2(x)$ (see Theorem 10 for a more general form):

$$L_n y_1(x) = f_1(x),\ L_n y_2(x) = f_2(x) \ \Rightarrow L_n[\alpha y_1(x) + \beta y_2(x)] = \alpha f_1(x) + \beta f_2(x) \ (3)$$

For a homogeneous linear ODE $L_n y = 0$, the superposition principle reflects in full the linearity of the ODE:

$$L_n y_1(x) = 0,\ L_n y_2(x) = 0 \ \Rightarrow L_n[\alpha y_1(x) + \beta y_2(x)] = 0 \qquad (4)$$

therefore, any linear combination of the solutions of the homogeneous equation is also a solution of this equation. The last property is important for understanding the structure of the solution set for the homogeneous equation: if some functions are solutions of linear homogeneous ODE's then their span consists completely of solutions of this equation.

Initial Value Problem

The initial value problem (IVP) for an n^{th} order ODE is given by:

$$
\begin{aligned}
&\text{Solve} && L_n y = f(x) \quad \text{in } x \in D \subset \mathbb{R} \\
&\text{Subject to} && y(x_0) = k_1 \\
& && y'(x_0) = k_2 \\
& && \vdots \\
& && y^{(n-1)}(x_0) = k_n \qquad x_0 \in D,\ k_i \in \mathbb{R}
\end{aligned} \tag{5}
$$

The setting of the IVP for an ODE is important for the proper modeling of physical processes. Thus, the solution of the IVP should exist, and the development of the solution from the initial state should be unique. The other property of the solution should include the continuous dependence of the solution on their initial conditions. If it holds, then the IVP is said to be well-set (otherwise, it is said to be an ill-set problem). The following theorem (given here without proof) gives the sufficient condition for existence and uniqueness of the solution of the IVP.

Existence and uniqueness

Theorem1 If a linear ODE $L_n y = f(x)$ is normal in *D*, then the IVP (5) has a unique solution in *D*

Corollary If $y(x)$ is a solution of the IVP (5) for the homogeneous equation

$$L_n y = 0$$

with $k_1 = 0, k_2 = 0, ..., k_n = 0$, then $y(x) \equiv 0$.

Obviously, the trivial solution satisfies these initial conditions and because the solution of the IVP is unique, $y(x)$ is a zero function.

2.3.2. Homogeneous linear ODE

Further, if there is no special reason otherwise, equations are assumed to be normal. Because only a homogeneous linear ODE $L_n y = 0$ possesses the property that a linear combination of solutions is also a solution, we will study first the solutions of homogeneous equations.

But first we have to recall some concepts of linear algebra concerning vector spaces. Consider a vector space of all *n* times continuously differentiable functions in *D*:

$$f(x) \in C^n[D]$$

and formulate definitions for linear independence of the set of their elements:

Linear independent sets

Definition 1 Functions $y_1(x), y_2(x), ..., y_n(x)$ are **linearly independent** in *D,* if equation

$$c_1 y_1(x) + ... + c_n y_n(x) = 0 \quad c_i \in \mathbb{R} \tag{6}$$

is valid for all $x \in D$ only if all coefficients

$$c_i = 0$$

Definition 2 Functions $y_1(x), y_2(x), ..., y_n(x)$ are **linearly dependent** in *D,* if there exist coefficients $c_i \in \mathbb{R}$ not all equal to zero such that

$$c_1 y_1(x) + ... + c_n y_n(x) = 0$$

for all $x \in D$.

Obviously, if functions $y_1(x), y_2(x),..., y_n(x)$ are not linearly dependent, then they are linearly independent.

Definition 1 provides a unique representation of a zero function by a linear combination of linearly independent functions with all coefficients equal to zero. Now, if we consider a subspace spanned by a set of linearly independent functions:

$$V \equiv \langle y_1(x), y_2(x), ..., y_n(x) \rangle = \{c_1 y_1(x) + ... + c_n y_n(x), c_i \in \mathbb{R}\},$$

then representation of any function from V by a linear combination of $y_1(x), y_2(x),..., y_n(x)$ is also unique:

$$f(x) = c_1 y_1(x) + ... + c_n y_n(x) \tag{7}$$

Indeed, suppose that we have a linear combination with other coefficients that is valid for all $x \in D$:

$$f(x) = d_1 y_1(x) + ... + d_n y_n(x)$$

then subtracting it from equation (7) we get

$$0 = (c_1 - d_1) y_1(x) + ... + (c_1 - d_n) y_n(x)$$

this, according to definition 1, yields immediately

$$c_1 - d_1 = 0, ..., c_n - d_n = 0$$

or

$$c_1 = d_1, ..., c_n = d_n$$

therefore, functions $y_1(x), y_2(x),..., y_n(x)$ can serve as a basis for the vector space V.

How is it determined that a set of functions $y_1(x), y_2(x),..., y_n(x)$ is linearly independent in D? For this purpose we will use the properties of the determinant of a special matrix (called the Wronskian) which is constructed from functions $y_1(x), y_2(x),..., y_n(x)$ and their derivatives.

Wronskian

Definition 3 The Wronskian of $y_1(x), y_2(x),..., y_n(x) \in C^n[D]$ is a determinant

$$W[y_1, ..., y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \tag{8}$$

Important properties of the Wronskian are formulated in the following theorems:

Theorem 2 The set of functions $y_1(x), y_2(x),..., y_n(x) \in C^n[D]$ is linearly independent in D, if and only if

$$W[y_1, ..., y_n](x) \neq 0 \quad \text{for all } x \in D$$

Theorem 3 The set of functions $y_1(x), y_2(x),..., y_n(x) \in C^n[D]$ is linearly dependent in D, if and only if

$$W[y_1, ..., y_n](x) = 0 \quad \text{for all } x \in D$$

Theorem 4 If at some point $x_0 \in D$, $W[y_1, ..., y_n](x_0) \neq 0$, then

$$W[y_1, ..., y_n](x) \neq 0 \quad \text{for all } x \in D$$

Theorem 5 If at some point $x_0 \in D$, $W[y_1, ..., y_n](x_0) = 0$, then functions $y_1(x), y_2(x),..., y_n(x)$ are linearly dependent in D.

Proof of Theorem 2:

($\Leftarrow$ sufficient condition)

Let $\quad y_1(x), y_2(x), \ldots, y_n(x) \in C^n[D]$ and

$W[y_1, \ldots, y_n](x) \neq 0 \quad$ for all $x \in D$.

Construct a homogeneous system of linear algebraic equations:

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

For any $x \in D$, this system has only a trivial solution because the determinant of the matrix of coefficients $W[y_1, \ldots, y_n](x) \neq 0$. Therefore, the linear combination

$$c_1 y_1(x) + \ldots + c_n y_n(x) = 0$$

only if all coefficients c_i are equal to zero for all $x \in D$. Then, according to Definition 1, the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent in *D*. ■

($\Rightarrow$ necessary condition)

Let the set of functions $y_1(x), y_2(x), \ldots, y_n(x) \in C^n[D]$ be linearly independent in *D*. Then for all $x \in D$

$$c_1 y_1(x) + \ldots + c_n y_n(x) = 0 \text{ only if all } c_i = 0$$

Differentiate this equation *n-1* times and construct a linear system of algebraic equations

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If for some $x_0 \in D$, the determinant $W[y_1, \ldots, y_n](x_0) = 0$, then the linear system at this point has a non-trivial solution, and

$$c_1 y_1(x_0) + \ldots + c_n y_n(x_0) = 0 \quad \text{where some } c_i \neq 0$$

but that contradicts the theorem's assumption, therefore

$W[y_1, \ldots, y_n](x) \neq 0 \quad$ for all $x \in D$. ■

Solution space of $L_n y = 0$

The next two theorems determine the dimension of the solution space of a homogeneous linear ODE. In the first theorem, we constitute that there always exists *n* linear independent solutions of $L_n y = 0$, and in the second theorem, that there can not be more than *n* linearly independent solutions.

Theorem 6 There exist *n* linearly independent solutions of $L_n y = 0$

Proof: The existence will be shown by construction of such a set of *n* linearly independent solutions.

Let $x_0 \in D$ be an arbitrary point. The initial value problem for the normal equation $L_n y = 0$ subject to conditions $y(x_0) = k_1$, $y'(x_0) = k_2$,..., $y^{(n-1)}(x_0) = k_n$ $x_0 \in D$, $k_i \in R$ has a unique solution $y(x) \in D$ (Theorem 1). Denoted by vectors

$\mathbf{k}=\begin{bmatrix}k_1\\k_2\\\vdots\\k_n\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}y(x)\\y'(x)\\\vdots\\y^{(n-1)}(x)\end{bmatrix}$, then the initial condition is written in vector form as:

$$\mathbf{y}(x_0)=\mathbf{k}$$

Choose $\mathbf{k}_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, then IVP $\begin{matrix}L_n y=0\\ \mathbf{y}(x_0)=\mathbf{k_1}\end{matrix}$ has a unique solution $y_1(x)$

Choose $\mathbf{k}_2=\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}$, then IVP $\begin{matrix}L_n y=0\\ \mathbf{y}(x_0)=\mathbf{k}_2\end{matrix}$ has a unique solution $y_2(x)$

$\vdots$

Choose $\mathbf{k}_n=\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$, then IVP $\begin{matrix}L_n y=0\\ \mathbf{y}(x_0)=\mathbf{k}_n\end{matrix}$ has a unique solution $y_n(x)$

Show that when constructed this way the set of solutions $y_1(x), y_2(x),..., y_n(x)$ is linearly independent in *D*. Calculate the Wronskian of these functions at $x_0 \in D$:

$$W[y_1,...,y_n](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0)\\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0)\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_{n-1}^{(n-1)}(x_0)\end{vmatrix}$$

$$=\left|\mathbf{k_1} \quad \mathbf{k_2} \quad \cdots \quad \mathbf{k_n}\right| = \begin{vmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{vmatrix} = 1$$

Then according to theorem 4, the Wronskian is not equal to zero at all $x \in \mathbb{R}$; therefore, by theorem 2, functions $y_1(x), y_2(x),..., y_n(x)$ are linearly independent in *D*. ■

Theorem 7 Let $y_1(x), y_2(x),..., y_n(x)$ be a set of *n* linearly independent solutions of the normal homogeneous linear ODE

$$L_n y = 0 \qquad x \in D \subset \mathbb{R} \tag{9}$$

If the function $y(x)$, $x \in D$ is any solution of equation (9), then there exists coefficients $c_i \in R$ such that

$$y(x) = c_1 y_1(x) + ... + c_n y_n(x) \tag{10}$$

for all $x \in D$

Proof: Let $y(x)$, $x \in D$ be an arbitrary solution of equation (9) in *D*. Show that we can find coefficients $c_i \in \mathbb{R}$ such that

$$y(x) = c_1 y_1(x) + ... + c_n y_n(x)$$

Differentiate this equation consequently *n-1* times (as the solutions of equation (9), $y(x), y_i(x) \in C^n[D]$):

$$\begin{aligned} &y(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \\ &y'(x) = c_1 y_1'(x) + \ldots + c_n y_n'(x) \\ &\vdots \\ &y^{(n-1)}(x) = c_1 y_1^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) \end{aligned} \tag{11}$$

And applying this equations at some point $x_0 \in D$, write a system of algebraic equations:

$$\begin{aligned} &y(x_0) = c_1 y_1(x_0) + \ldots + c_n y_n(x_0) \\ &y'(x_0) = c_1 y_1'(x_0) + \ldots + c_n y_n'(x_0) \\ &\vdots \\ &y^{(n-1)}(x_0) = c_1 y_1^{(n-1)}(x_0) + \ldots + c_n y_n^{(n-1)}(x_0) \end{aligned} \tag{12}$$

which in the matrix form can be rewritten as

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

or in the short form as

$$\mathbf{y}(x_0) = \mathbf{Ac} \tag{13}$$

The determinant of matrix **A** is a Wronskian $W[y_1, \ldots, y_n](x_0)$ of functions $y_1(x), y_2(x), \ldots, y_n(x)$ calculated at $x_0 \in D$. Since functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent in *D*, $W[y_1, \ldots, y_n](x_0) \neq 0$. Therefore, the matrix **A** is invertable, and the system of algebraic equations (13) has a unique solution

$$\mathbf{c} = \mathbf{A}^{-1}\mathbf{y}(x_0) \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \neq 0 \tag{14}$$

Using determined coefficients c_i (14), construct a function in *D*:

$$z(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \qquad x \in D \tag{15}$$

As a linear combination of solutions, according to the superposition principle for the homogeneous equation, $z(x)$ is also a solution of $L_n y = 0$, and, therefore, $z(x) \in C^n[D]$. Then differentiate equation (15) consequently *n-1* times:

$$\begin{aligned} &z(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \\ &z'(x) = c_1 y_1'(x) + \ldots + c_n y_n'(x) \\ &\vdots \\ &z^{(n-1)}(x) = c_1 y_1^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) \end{aligned} \tag{16}$$

Consider $y(x) - z(x)$, the function $y(x) - z(x)$ also is a solution of $L_n y = 0$. Calculate $y(x_0) - z(x_0)$ and its derivatives comparing equations (12) and (16):

$$\begin{aligned} &y(x_0) - z(x_0) = 0 \\ &y'(x_0) - z'(x_0) = 0 \\ &\vdots \\ &y^{(n-1)}(x_0) - z^{(n-1)}(x_0) = 0 \end{aligned}$$

Then, according to the Corollary for Theorem 1, the function $y(x) - z(x)$ which is a solution of $L_n y = 0$ satisfying zero initial conditions, is identically equal to zero:

$$y(x) - z(x) \equiv 0 \qquad \text{for all } x \in D$$

therefore,

$$y(x) \equiv z(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \quad \text{for all } x \in D$$

where the coefficients c_i are not all equal to zero, are uniquely determined by equation (14). ■

Equation $L_n y = 0$ always has *n* linearly independent solutions, but not more. If we have *n* linearly independent solutions $y_1(x), y_2(x), ..., y_n(x)$ of $L_n y = 0$, then any other solution is represented by their linear combination. So, the complete solution set of $L_n y = 0$ is a vector space spanned by any set of its *n* linearly independent solutions:

$$S = span\{y_1(x), y_2(x), ..., y_n(x)\} \tag{17}$$

solution space

The set $\{y_1(x), y_2(x), ..., y_n(x)\}$ is a basis for the solution space of dimension n.

Definition4 Any set of *n* linearly independent solutions of $L_n y = 0$

$$\{y_1(x), y_2(x), ..., y_n(x)\}$$

is called a **fundamental set.**

Definition5 If $\{y_1(x), y_2(x), ..., y_n(x)\}$ is a fundamental set of $L_n y = 0$, then its solution space

$$\begin{aligned} y_c(x) &= span\{y_1(x), y_2(x), ..., y_n(x)\} \\ &= c_1 y_1(x) + ... + c_n y_n(x), \ x \in D, \ c_i \in \mathbb{R} \end{aligned}$$

is called a **complimentary solution.**

Theorem 8 Let $\{y_1(x), y_2(x), ..., y_n(x)\}$ be a fundamental set of $L_n y = 0$, then any solution of $L_n y = 0$ belongs to its complementary solution $y_c(x) = span\{y_1(x), y_2(x), ..., y_n(x)\}$.

That means that any solution of the homogeneous equation $L_n y = 0$ is represented by a linear combination $y = c_1 y_1(x) + ... + c_n y_n(x)$.

2.3.3 Non-Homogeneous linear ODE

Consider the structure of the solution set of the non-homogeneous linear ODE

$$L_n y = f(x) \quad x \in D \tag{18}$$

General solution of $L_n y = f(x)$

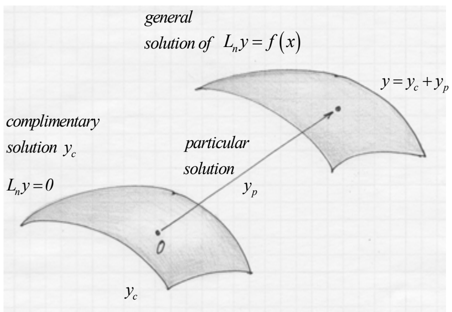
Theorem 9 Let $\{y_1(x), y_2(x), ..., y_n(x)\}$ be a fundamental set of $L_n y = 0$ on *D,* and let y_p be any particular solution of the non-homogeneous equation $L_n y = f(x)$ on *D*, then the complete solution of the non-homogeneous linear ODE

$$L_n y = f(x)$$

is described by a complimentary solution of the homogeneous equation translated by a particular solution of the non-homogeneous equation

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= c_1 y_1(x) + \cdots + c_n y_n(x) + y_p(x) \qquad c_i \in R \end{aligned} \tag{19}$$

Proof: Show that any solution $u(x)$ of the non-homogeneous equation $L_n u \equiv f(x)$ is represented by the sum $u(x) = y_c(x) + y_p(x)$



Let $\quad L_n u = f(x)$

$\quad\quad L_n y_p = f(x)$

and subtract the second equation from the first equation

$$L_n(u - y_p) = 0$$

Therefore, $u - y_p$ is a solution of the homogeneous equation, and, according to Theorem 8, is represented by a linear combination of the fundamental set

$$u - y_p = c_1 y_1(x) + \cdots + c_n y_n(x)$$

The claimed result can be obtained by moving the particular solution to the right hand side of the equation

$$u = c_1 y_1(x) + \cdots + c_n y_n(x) + y_p \qquad \blacksquare$$

Superposition principle ***Theorem 10*** Let $y_i(x)$ be the particular solutions of $L_n y = f_i(x)$ $i = 1, \ldots, k$

$$L_n y_i \equiv f_i(x) \quad x \in D$$

Then $\sum_{i=1}^{k} y_i(x)$ is a solution of the equation $L_n y = \sum_{i=1}^{k} f_i(x)$

$$L_n \sum_{i=1}^{1} y_i \equiv \sum_{i=1}^{k} f_i(x)$$

This theorem is a simple corollary of the linearity of the differential operator L_n.

It follows that if the fundamental set and any particular solution are known then the general solution of the non-homogeneous linear ODE can be constructed. In the next section we will study how they can be obtained for the particular form of linear equations.

2.3.4. Fundamental set of linear ODE with constant coefficients

Consider first differential equations of the second order because they are the most popular in engineering modeling (for example, dynamical systems with forces) :

2nd order

$$\boxed{a_0 y'' + a_1 y' + a_2 y = 0} \qquad x \in D \tag{20}$$

Ansatz: look for the solution of equation (20) in the form of an exponential function

$$y = e^{mx} \tag{21}$$

with some undetermined parameter *m*. Differentiate (21) twice

$$y' = me^{mx} \qquad y'' = m^2 e^{mx}$$

and substitute into equation (20)

$$a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$\left(a_0 m^2 + a_1 m + a_2\right) e^{mx} = 0$ factor e^{mx}

because $e^{mx} > 0$ for all x, the necessary condition for the trial function $y = e^{mx}$ to be a solution is

$a_0 m^2 + a_1 m + a_2 = 0$

This equation is called an auxiliary equation. From the auxiliary equation we can find the values of the parameter m satisfying this equation

$$m_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0} \tag{22}$$

Consider the following cases:

a) $\boxed{a_1^2 - 4a_0 a_2 > 0}$ There are two distinct roots $m_1 \neq m_2$, then corresponding solutions of the differential equation (20) are

$$y_1 = e^{m_1 x} \text{ and } y_2 = e^{m_2 x} \tag{23}$$

Use the Wronskian to show that these functions are linearly independent

$$W\left(e^{m_1 x}, e^{m_2 x}\right) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = m_2 e^{(m_1+m_2)x} - m_1 e^{(m_1+m_2)x} = \left(m_2 - m_1\right) e^{(m_1+m_2)x}$$

because $m_1 \neq m_2$ and $e^{(m_1+m_2)x} > 0$ for all x, $W\left(e^{m_1 x}, e^{m_2 x}\right) \neq 0$ for all x, therefore, according to Theorem 2, solutions (23) are linearly independent and constitute the fundamental set for the homogeneous equation (20):

$$\left\{e^{m_1 x}, e^{m_2 x}\right\} \tag{24}$$

b) $\boxed{a_1^2 - 4a_0 a_2 = 0}$ There is only one repeated root of the auxiliary equation of multiplicity two: $m_1 = m_2 = m = \dfrac{-a_1}{2a_0}$, then there is only one corresponding solution of the trial form

$y_1 = e^{mx}$

To construct a second solution, use a reduction formula ( ):

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{y_1^2} dx && \text{substitute } \frac{a_1}{a_0} = -2m \text{ and } y_1 = e^{m} \\ &= e^{mx} \int \frac{e^{2\int m dx}}{e^{2mx}} dx && \text{perform integration} \\ &= e^{mx} \int \frac{e^{2mx}}{e^{2mx}} dx \\ &= e^{mx} \int dx \\ &= x e^{mx} \end{aligned}$$

Use the Wronskian to check if the obtained solutions are linearly independent:

$$W\left(e^{mx}, x e^{mx}\right) = \begin{vmatrix} e^{mx} & x e^{mx} \\ m e^{mx} & e^{mx} + m x e^{mx} \end{vmatrix} = e^{2mx} + m x e^{2mx} - m x e^{2mx} = e^{2mx} > 0$$

for all x. Therefore, the solutions are linearly independent and the fundamental set is

$$\left\{e^{mx}, x e^{mx}\right\} \tag{25}$$

c) $\boxed{a_1^2 - 4a_0 a_2 < 0}$ There is a conjugate pair of two complex roots

$$m_1 = a + ib \quad a = \frac{-a_1}{2a_0}$$
$$m_2 = a - ib \quad b = \frac{\sqrt{4a_0a_2 - a_1^2}}{2a_0}$$

and corresponding solutions are

$$y_1 = e^{(a+ib)x}$$
$$y_2 = e^{(a-ib)x}$$

Use the Euler formula (1.xx) for conversion of the solution to trigonometric form

$$y_1 = e^{ax}\left(\cos bx + i\sin bx\right)$$
$$y_2 = e^{ax}\left(\cos bx - i\sin bx\right)$$

Any linear combination of solutions of a linear homogeneous equation is also a solution, therefore, by adding and subtracting these solutions and neglecting constant coefficients, two real valued solutions can be obtained:

$$y_1 = e^{ax}\cos bx$$
$$y_2 = e^{ax}\sin bx$$

More formally, the following fact can be proved:
If a linear homogeneous differential equation (in general, of order n) has a complex solution

$$y(x) = u(x) + iv(x)$$

then each real-valued function $u(x)$ and $v(x)$ are also the solutions of the same homogeneous equation. It can be shown by direct substitution into the differential equation (in exercise).
Use the Wronskian to check if the obtained solutions are linearly independent:

$$W\left(e^{ax}\cos bx, e^{ax}\sin bx\right) = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ ae^{ax}\cos bx - be^{ax}\sin bx & ae^{ax}\sin bx + be^{ax}\cos bx \end{vmatrix}$$

$$= ae^{2ax}\sin bx\cos bx + be^{2ax}\cos^2 bx - ae^{2ax}\sin bx\cos bx + be^{2ax}\sin^2 bx$$
$$= be^{2ax}\left(\cos^2 bx + \sin^2 bx\right)$$
$$= be^{2ax}$$
$$\neq 0 \qquad \text{for all } x \text{ because } b = \frac{\sqrt{4a_0a_2 - a_1^2}}{2a_0} \neq 0$$

then the fundamental set of the homogeneous differential equation is

$$\left\{e^{ax}\cos bx, e^{ax}\sin bx\right\} \tag{26}$$

n^th order

$$\boxed{a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0} \qquad x \in D \tag{27}$$

The same form of the trial solution $y = e^{mx}$ is used for the n^{th} order homogeneous equation. Its application yields an auxiliary equation

$$a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0 \tag{28}$$

which is an n^{th} order algebraic equation for parameter m. According to the fundamental theorem of algebra, it has exactly n roots which can be real distinct or repeated or complex roots appearing in conjugate pairs (also they can be repeated). Consider the typical cases of the roots of the auxiliary equation:

a) If the auxiliary equation has a real root m of multiplicity k. Then there are k linearly independent solutions in the fundamental set corresponding to root m:

$$\left\{e^{mx}, xe^{mx}, x^2e^{mx}, ..., x^{k-1}e^{mx}\right\} \tag{29}$$

b) If the auxiliary equation has a conjugate pair of complex roots $m_{1,2} = a \pm ib$. Then the fundamental set includes two solutions corresponding to these roots

$$\left\{e^{ax}\cos bx, e^{ax}\sin bx\right\} \tag{30}$$

If the complex roots $m_{1,2} = a \pm ib$ are of multiplicity *k*, the corresponding solutions in the fundamental set are

$$\left\{e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, ..., x^{k-1}e^{ax}\cos bx, x^{k-1}e^{ax}\sin bx\right\} \tag{31}$$

2.3.5. Particular solution of linear ODE

In general, the particular solution of a non-homogeneous equation is not easy to find. There are two main methods for finding particular solutions: the variation of parameters (called Lagrange's method, however, published first by L.Euler and D.Bernulli , 1740) and the method of undetermined coefficients.

variation of parameter

Let the linear differential equation of n^{th} order

$$L_n y \equiv a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \tag{32}$$

be normal and let

$$\left\{y_1(x), y_2(x), ..., y_n(x)\right\}$$

be its fundamental set (linearly independent solutions of the homogeneous equation)

We will look for a particular solution of the non-uniform equation (32) in the form of a combination of fundamental solutions

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + ... + u_n(x)y_n(x) = \sum_{i=1}^{n} u_i(x)y_i(x) \tag{33}$$

where coefficients $u_i(x)$ are, in general, functions of *x* (variation of coefficients), and we will try to determine them in such a way that equation (32) is satisfied. We will feel free to impose any additional condition on these coefficients if it makes our task simpler, because we need just any particular solution (the number of such additional conditions can be at least *n-1* because initially for *n* unknown coefficients $u_i(x)$ we required satisfaction only of equation (32)).

Differentiate equation (33) using the product rule

$$y_p'(x) = \sum_{i=1}^{n} u_i(x)y_i'(x) + \sum_{i=1}^{n} u_i'(x)y_i(x)$$

For simplification, require the second sum to be equal to zero

$$\sum_{i=1}^{n} u_i'(x)y_i(x) = 0$$

then

$$y_p'(x) = \sum_{i=1}^{n} u_i(x)y_i'(x)$$

Differentiate the particular solution the second time

$$y_p''(x) = \sum_{i=1}^{n} u_i(x)y_i''(x) + \sum_{i=1}^{n} u_i'(x)y_i'(x)$$

and again simplify it by the condition

$$\sum_{i=1}^{n} u_i'(x)y_i'(x) = 0$$

then

$$y_p''(x) = \sum_{i=1}^{n} u_i(x)y_i''(x)$$

Continue this process until for the $(n-1)^{th}$ derivative we obtain

$$\sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) = 0$$

then

$$y_p^{(n-1)}(x) = \sum_{i=1}^{n} u_i(x) y_i^{(n-1)}(x)$$

One more differentiation of particular solution yields

$$y_p^{(n)}(x) = \sum_{i=1}^{n} u_i(x) y_i^{(n)}(x) + \sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x)$$

and now we can substitute all derivatives into the non-homogeneous equation (32):

$$\begin{aligned} a_0(x)\left[\sum_{i=1}^{n} u_i(x) y_i^{(n)}(x) + \sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x)\right] \quad & + a_1(x) \sum_{i=1}^{n} u_i(x) y_i^{(n-1)}(x) \\ & + \ldots \\ & + a_{n-2}(x) \sum_{i=1}^{n} u_i(x) y_i''(x) \\ & + a_{n-1}(x) \sum_{i=1}^{n} u_i(x) y_i'(x) \\ & + a_n(x) \sum_{i=1}^{n} u_i(x) y_i(x) \quad = f(x) \end{aligned}$$

Collect terms including coefficients $u_i(x)$

$$\begin{aligned} a_0(x) \sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) \quad & + a_0(x) \sum_{i=1}^{n} u_i(x) y_i^{(n)}(x) \\ & + a_1(x) \sum_{i=1}^{n} u_i(x) y_i^{(n-1)}(x) \\ & + \ldots \\ & + a_{n-2}(x) \sum_{i=1}^{n} u_i(x) y_i''(x) \\ & + a_{n-1}(x) \sum_{i=1}^{n} u_i(x) y_i'(x) \\ & + a_n(x) \sum_{i=1}^{n} u_i(x) y_i(x) \quad = f(x) \end{aligned}$$

and combine them in a single summation

$$\begin{aligned} & a_0(x) \sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) \\ & + \sum_{i=1}^{n} u_i(x)\left[a_0(x) y_i^{(n)}(x) + a_1(x) y_i^{(n-1)}(x) + \cdots + a_{n-1}(x) y_i'(x) + a_n(x) y_i(x)\right] \\ & \qquad = f(x) \end{aligned}$$

Since all functions $y_i(x)$ are solutions of the homogeneous equation the second term disappears and we have

$$a_0(x) \sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) \quad = f(x)$$

Because the differential equation is normal, we can divide it by the leading coefficient $a_0(x)$:

$$\sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) \quad = \frac{f(x)}{a_0(x)}$$

Combine now this equation with other additional conditions for $u_i'(x)$ into a system:

$$\sum_{i=1}^{n} u_i'(x) y_i(x) = 0$$
$$\sum_{i=1}^{n} u_i'(x) y_i'(x) = 0$$
$$\vdots$$
$$\sum_{i=1}^{n} u_i'(x) y_i^{(n-2)}(x) = 0$$
$$\sum_{i=1}^{n} u_i'(x) y_i^{(n-1)}(x) = \frac{f(x)}{a_0(x)}$$

where unknowns are derivatives of coefficients $u_i(x)$. This system can be written in the following matrix form

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \\ \vdots \\ u_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dfrac{f(x)}{a_0(x)} \end{bmatrix}$$

Since the determinant of the matrix is a Wronskian of the fundamental set of solutions, it never equals zero, therefore, the system of algebraic equations for the coefficients $u_i(x)$ has a unique solution, which using Cramer's rule can be written as

$$u_m'(x) = \frac{\det \begin{bmatrix} y_1(x) & y_2(x) & \cdots\ 0 \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots\ 0 \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots \dfrac{f(x)}{a_0(x)} \cdots & y_n^{(n-1)}(x) \end{bmatrix}}{\det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}} \qquad m = 1,\ldots,n$$

Expand the determinant in the numerator over the m^{th} column, and using the definition of the Wronskian, the solution can be written in the following way

$$u_m'(x) = (-1)^m \frac{W(y_1(x),\ldots,y_{m-1}(x),y_{m+1}(x),\ldots,y_n(x))}{W(y_1(x),\ldots,y_n(x))} \frac{f(x)}{a_0(x)} \qquad m = 1,\ldots,n$$

Then by integration, the unknown functions can be determined as

$$\boxed{u_m(x) = (-1)^m \int \frac{W(y_1(x),\ldots,y_{m-1}(x),y_{m+1}(x),\ldots,y_n(x))}{W(y_1(x),\ldots,y_n(x))} \frac{f(x)}{a_0(x)} dx} \qquad m = 1,\ldots,n \tag{34}$$

2nd order linear ODE

For the important case of the second order linear equation, variation of parameters yields the following particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$u_1(x) = -\int \frac{y_2}{y_1y_2' - y_2y_1'}\frac{f(x)}{a_0(x)}dx \qquad (35)$$

$$u_2(x) = \int \frac{y_1}{y_1y_2' - y_2y_1'}\frac{f(x)}{a_0(x)}dx$$

This method was derived without any restrictions and should lead to the calculation of the particular solution of the non-homogeneous equation provided that integration can be performed.

Undetermined Coefficients

In many cases of the special form of the function $f(x)$ in the right hand side of the non-homogeneous equation, the method of undetermined coefficients is simpler and more convenient. In this method, the particular solution is found in the form determined by the function $f(x)$. It can be generalized to the following description:

If the function $f(x)$ is a combination of the exponential, trigonometric functions and polynomial in the form

$$f(x) = e^{ax}\left[p_i(x)\cos bx + q_j(x)\sin bx\right] \qquad (36)$$

where $p_i(x)$ and $q_j(x)$ are polynomials of orders *i* and *j*, respectively

1) and $a \pm ib$ is not a root of the auxiliary equation (28) then look for the particular
solution in the form

$$y_p(x) = e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right] \qquad k = max\{i, j\} \qquad (37)$$

2) and if $a \pm ib$ is a root of the auxiliary equation (28) of multiplicity s then look for the particular solution in the form

$$y_p(x) = x^s e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right] \qquad k = max\{i, j\} \qquad (38)$$

where
$$P_k(x) = A_0x^k + A_1x^{k-1} + \ldots + A_{k-1}x + A_k$$
$$Q_k(x) = B_0x^k + B_1x^{k-1} + \ldots + B_{k-1}x + B_k$$

are polynomials with unknown coefficients which are determined by substitution of the trial solution into the non-homogeneous equation (1)

Example 1 Find the particular solution of the linear equation

$$y'' - 3y' = e^{3x}$$

The auxiliary equation $m^2 - 3m = 0$ has two roots $m_1 = 0$ and $m_2 = 3$. Because the coefficient *a* in the function in the right hand side is equal to one of the roots of the auxiliary equation, the particular solution should be determined in the form

$$y_p = Axe^{3x},$$ then differentiation yields

$$y'_p = Ae^{3x} + 3Axe^{3x}$$

$$y''_p = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x}$$

Substitute it into the equation

$$6Ae^{3x} + 9Axe^{3x} - 3\left(Ae^{3x} + 3Axe^{3x}\right) = e^{3x}$$

$$3Ae^{3x} = e^{3x}$$

$$A = \frac{1}{3}$$

Therefore, the particular solution of the given equation is

$$y_p = \frac{1}{3}xe^{3x}$$

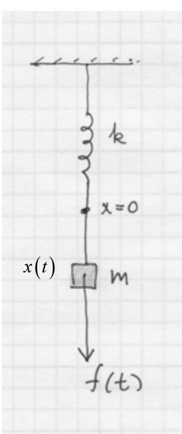
Example 2 *(application of Maple for the solution of 2^nd order linear ODE's)*

Find the general solution of a linear ODE with constant coefficients

$$y'' + 3y' + 2y = e^{2x}$$

Use Maple to follow the required steps for construction of the solution: the fundamental set and the particular solution of the non-homogeneous equation.

Example 3 *(Forced motion of a mass-spring system with damping)*



Consider a spring suspended point mass m vibrating under the external force $f(t)$. The equation of motion for the position of mass $x(t)$ is

$$\frac{d^2x(t)}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

where $\lambda = \dfrac{\beta}{2m}$, β is a damping coefficient, $\omega = \sqrt{\dfrac{k}{m}}$ is a circular frequency of vibration, k is a spring constant, $F(t) = \dfrac{f(t)}{m}$.

The equation of motion is a 2nd order linear ODE with constant coefficients. It describes deflection of the point mass from an equilibrium position. Therefore any 2nd order linear ODE can be treated as a model of such a dynamical system.

Consider, for example, damping of the system under a periodic sinusoidal force:

$$\frac{d^2x(t)}{dt^2} + 2\frac{dx}{dt} + 2x = sin(t)$$

which initially ($t = 0$) was at the point $x(0) = 2$ at rest $\left.\dfrac{dx}{dt}\right|_{t=0} = 0$.

The solution with undetermined coefficients yields:

Auxiliary equation $\quad m^2 + 2m + 2 = 0$
Roots $\quad m_{1,2} = -1 \pm i$

Complimentary solution $y_c = c_1 e^{-t}\sin t + c_2 e^{-t}\cos t$

Ansatz for the particular solution

$$y_p = A\sin t + B\cos t$$
$$y_p' = A\cos t - B\sin t$$
$$y_p'' = -A\sin t - B\cos t$$

Substitute into equation

$(A-2B)\sin t + (2A+B)\cos t = \sin t$

$$A - 2B = 1$$
$$2A + B = 0$$

Find coefficients $A = \frac{1}{5} \quad B = \frac{-2}{5}$

Particular solution $x_p = \frac{1}{5}\sin t - \frac{2}{5}\cos t$

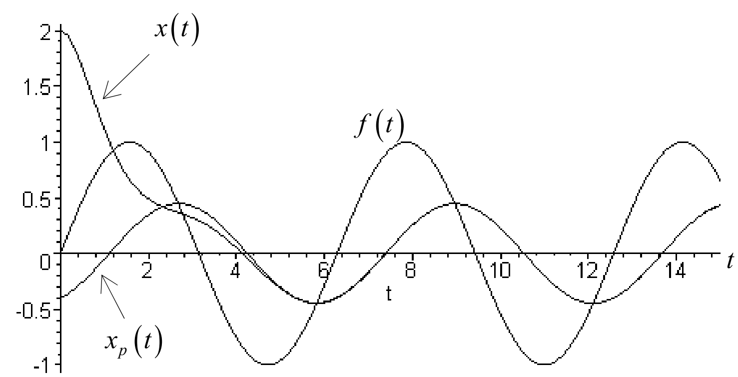
General solution

$$x(t) = c_1 e^{-t}\sin t + c_2 e^{-t}\cos t + \frac{1}{5}\sin t - \frac{2}{5}\cos t$$

From initial conditions $c_1 = \frac{11}{5} \quad c_2 = \frac{12}{5}$

Solution of initial value problem:

$$x(t) = \frac{11}{5}e^{-t}\sin t + \frac{12}{5}e^{-t}\cos t + \frac{1}{5}\sin t - \frac{2}{5}\cos t$$



The graph of solution of the initial value problem together with the graphs of the particular solution and the external force are shown in this Figure. We can see that for this periodic case, the solution curve due to damping approaches the particular solution (which plays the role of steady state for this system). Steady state follows the trend of the external force with some delay.

2.3.6. Euler-Cauchy Equation A linear differential equation with variable coefficients is called an Euler-Cauchy (or just Euler equation) if the coefficients have the form

$$a_k(x) = a_k x^{n-k} \quad \text{where } a_k \in R \text{ are constants, } k = 0,1,\ldots,n$$

therefore, an Euler equation has the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x) \tag{39}$$

It can be noticed that the scaling of the x variable in the Euler equation to αx by any non-zero constant α does not change the form of the equation – that is why this equation is also called the **equidimensional equation**.
An equation of the form

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = f(x) \tag{40}$$

is also called an Euler equation, it can be reduced to the form (39) by the change of variable $\xi = ax + b$.

This equation appears in a solution of partial differential equations in cylindrical and spherical coordinates by separation of variables.

These type of equations with variable coefficients can be reduced to linear differential equations with constant coefficients with the help of a change of independent variable

$$x = e^z \qquad \text{with inverse substitution} \qquad z = ln|x|$$

Derivatives of the unknown function y are replaced according to the following representation (use for convenience the differential operator $D = \frac{d}{dz}$):

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \Rightarrow \qquad y' = \frac{1}{x}Dy$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{1}{x^2}\left(-\frac{dy}{dz} + \frac{d^2 y}{dz^2}\right) \qquad \Rightarrow \qquad y'' = \frac{1}{x^2}D(D-1)y$$

$$\frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^3 y}{dx^3}\right) = \frac{1}{x^3}\left(2\frac{dy}{dz} - 3\frac{d^2 y}{dz^2} + \frac{d^3 y}{dz^3}\right) \qquad \Rightarrow \qquad y''' = \frac{1}{x^3}D(D-1)(D-2)y$$

…

then, noticing the pattern, assume that for the n^{th} derivative

$$\frac{d^n y}{dx^n} = \frac{1}{x^n}\left(\ldots + \frac{d^n y}{dz^n}\right) \qquad \Rightarrow \qquad y^{(n)} = \frac{1}{x^n}D(D-1)(D-2)\cdots[D-(n-1)]y$$

For mathematical induction, show the validity of this equation for $n+1$:

$$\begin{aligned}\frac{d^{n+1} y}{dx^{n+1}} &= \frac{d}{dx}\left(\frac{d^n y}{dx^n}\right)\\ &= \frac{d}{dz}\left(\frac{d^n y}{dx^n}\right)\frac{dz}{dx}\\ &= \frac{d}{dz}\left(\frac{d^n y}{dx^n}\right)\frac{1}{x}\\ &= \frac{d}{dz}\left\{\frac{1}{x^n}D(D-1)(D-2)\cdots[D-(n-1)]y\right\}\frac{1}{x}\end{aligned}$$

Use the product rule and $\frac{d}{dz}\left(\frac{1}{x^n}\right) = -nx^{-n-1}\frac{dx}{dz} = -nx^{-n-1}e^z = -nx^{-n-1}e^{\ln x} - nx^{-n-1}x = \frac{-n}{x^n}$

$$= \left\{\frac{-n}{x^n}D(D-1)(D-2)\cdots\left[D-(n-1)\right]y + \frac{1}{x^n}DD(D-1)(D-2)\cdots\left[D-(n-1)\right]y\right\}\frac{1}{x}$$

$$= \frac{-1}{x^n}(-n+D)D(D-1)(D-2)\cdots\left[D-(n-1)\right]y\frac{1}{x}$$

$$= \frac{1}{x^{n+1}}D(D-1)(D-2)\cdots\left[D-(n-1)\right](D-n)y$$

From these formulas it is obvious that their substitution into equation (39) yields a differential equation with constant coefficients. Thus, for 2nd order and 3rd order ODE's the results are

2nd order

$$a_0\frac{d^2y}{dz^2} + (a_1 - a_0)\frac{dy}{dz} + a_2 y = f\left(e^z\right)$$

3rd order

$$a_0\frac{d^3y}{dz^3} + (a_1 - 3a_0)\frac{d^2y}{dz^2} + (2a_0 - a_1 + a_2)\frac{dy}{dz} + a_3 y = f\left(e^z\right)$$

Example 4 (ODE/Euler.mws)

Find the solution of the linear equation

$$x^2y'' + 3xy' + y = \ln x$$

This is a 2nd order non-homogeneous Euler equation. Solve it with the substitution $x = e^z$ which yields a linear ODE with constant coefficients

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = z$$

The general solution of this linear ODE is

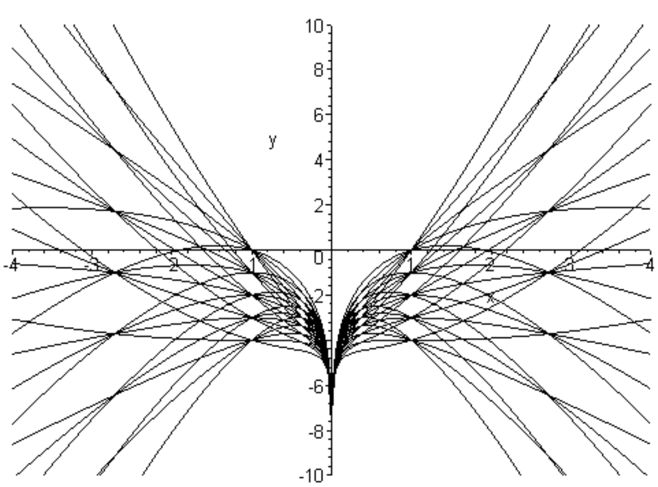
$$y(z) = c_1e^{-z} + c_2ze^{-z} + z - 2$$

then replacing $z = \ln|x|$, we obtain a general solution of the original problem

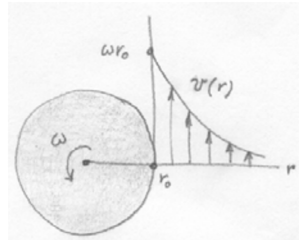
$$y(z) = c_1e^{-\ln|x|} + c_2\ln|x|e^{-\ln|x|} + \ln|x| - 2$$

$$y(x) = \frac{c_1}{x} + c_2x\ln|x| + \ln|x| - 2$$

Solution curves have the following form



Example 5 Consider a viscous flow around a long circular cylinder of radius r_0 rotating around its axis with angular velocity ω.



The fluid angular velocity component $v(r)$ is defined by the differential equation

$r^2v'' + rv' - v = 0$

with the boundary condition:

$v(r_0) = \omega r_0$ (non-slip condition).

It is reasonable to assume that the steady state velocity profile is developed in the initially stagnant fluid, therefore, the velocity of the fluid when radial coordinates approach infinity is zero

$v(r)\big|_{r\to\infty} = 0$ (is still at rest).

Find the steady radial dependence of angular velocity $v(r)$.

Solution: The differential equation is the 2nd order homogeneous Euler equation. The auxiliary equation is

$m^2 + (+1-1)m - 1 = 0$

$m^2 = 1$ the two real roots are $m_{1,2} = \pm 1$.

Then the general solution of Euler equation is

$v(r) = c_1 r + c_2\dfrac{1}{r}$

To satisfy the condition $v(r)\big|_{r\to\infty} = 0$, we have to demand that the constant of integration $c_1 = 0$.

Then the boundary condition at $r = r_0$ yields

$v(0) = c_2\dfrac{1}{r_0} = \omega r_0 \quad \Rightarrow \quad c_2 = \omega r_o^2$

and the solution of the problem is

$v(r) = \dfrac{\omega r_0^2}{r}$

This field is exactly like the potential vortex driven by the viscous no-slip condition.

Example 6 Consider steady state conduction in the long cylindrical region between two isothermal surfaces $r = r_1$ at temperature u_1 and $r = r_2$ at temperature u_2. The radial temperature distribution under assumption of angular symmetry in the absence of volumetric heat generation is described by the differential equation

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0$$

Find the temperature distribution $u(r)$, $r \in (r_1, r_2)$.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2.4 Power Series Solutions

2.4.1 Introduction

In the previous chapter, we have developed and studied solution techniques for special kinds of ordinary linear differential equations, namely those which had constant coefficients or those which were of "Cauchy-Euler type", so-called "equi-dimensional equations". We will now consider a more general class of linear differential equations. Again we will confine ourselves to equations of order 2, although the methods can also be applied to higher order equations.

We will consider differential equations of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \tag{2.1}$$

$$y(x_0) = y_0 \tag{2.2}$$

$$y'(x_0) = y_1 \tag{2.3}$$

where the coeffient functions $a_j : I_0 \to \mathbb{R}$ are "analytic" on the open interval I_0, $x_0 \in I_0$ and $a_0(x_0) \neq 0$. Equation (2.1) is called the *general form* of the linear ordinary differential equation (of order 2).

We will see, that under the specified assumptions on the coefficient functions, every solution of the differential equation (2.1) is analytic and has a "power series" representation

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

at x_0, whose "radius of convergence" R is at least as great as the distance from x_0 to the nearest (not necessarily real!) "singular" point of (2.1). The coefficients c_n of the solution series can be easily computed using a procedure called the *power-series method* or the *method of undetermined coefficients* for power series.

Before we present this solution technique, we will introduce some terminology and review some well-known(!) facts on power series etc.

2.4.2 Basic Definitions and Results

Ordinary Points

The equation (2.1) can be rewritten as

$$y'' + p(x)y' + q(x)y = 0, \tag{2.4}$$

which is called the *normal form* or the *standard form* of the general linear differential equation. Here, clearly

$$p(x) = \frac{a_1(x)}{a_0(x)} \tag{2.5}$$

and

$$q(x) = \frac{a_2(x)}{a_0(x)} \tag{2.6}$$

for $x \in I_0$.

In this chapter, we will consider the special and pleasant case, were the solutions of (2.1) are centered at a point $x_0 \in I_0$ for which $a_0(x_0) \neq 0$. Those points of I_0 are called *ordinary points* of the differential equation (2.1). In the next chapter, we will consider the more general situation, were we are looking for solutions around points $x_0 \in I_0$, for which $a_0(x_0) = 0$, so-called *singular points* of the equation (2.1). Note that in this case, the coefficient functions p and q of the normal form (2.4) of our differential equation have "singularities" (poles) at the point x_0.

Binomial Coefficients, etc.

Recall that the factorial of a number $n \in \mathbb{N}^0 := \{0, 1, 2, 3, \ldots\}$, can be defined recursively by

$$0! := 1 \quad \text{and} \quad n! := n \cdot \big((n-1)!\big) \tag{2.7}$$

The number $n! = 1 \cdot 2 \cdot \ldots \cdot n$, is called n-*factorial* and counts the number of permutations of n objects along a line. For numbers $n \in \mathbb{N}$, we define the numbers

$$\binom{n}{k} := \prod_{j=1}^{k} \frac{n-j+1}{j} = \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{k!},$$

which are called *binomial coefficients*, since they occur in the binomial theorem. Recall that $\binom{n}{k}$ counts the number of ways in which we can choose k objects out of a set of n objects. Then we can prove the

Theorem 2.1
Suppose $n, k \in \mathbb{N}$, *then*

1. $\binom{n}{k} = 0 \quad$ *if* $k > n$.

2. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$ *for* $0 \leq k \leq n$.

3. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *for* $1 \leq k \leq n$.

4. *For* $x, y \in \mathbb{R}$ *and* $n \in \mathbb{N}^0$, *we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

   which is called the "Binomial Theorem".

If we set $x = y = 1$ and $x = 1$ and $y = -1$, respectively, in the Binomial Theorem, we obtain from Theorem 2.1 equations (1) and (2) of the following corollary. Equation (3), follows by subtracting equation (2) from equation (1) and dividing the difference by 2.

Corollary 2.1
For all $n \in \mathbb{N}$ *we have*

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

2. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

3. $\displaystyle\sum_{j=0}^{m} \binom{2m+1}{2j+1} = 2^{2m}$

Some Basic Facts on Power Series

Recall that a series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called a *power series* in $x - x_0$. The point x_0 is called the *expansion point* of the power series, a_k its *coefficients.*

Example 2.1
The well-known *geometric series*

$$\sum_{k=0}^{\infty} x^k$$

is a power series with expansion point $x_0 = 0$ and coefficients $a_j = 1$ for all $k \in \mathbb{N}^0 := \{0, 1, 2, \ldots, \}$. It converges, as we have seen in calculus, for all $x \in \mathbb{R}$ with $|x| < 1$, i.e. for all $x \in (-1, 1)$ and represents there the function

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for all } x \in (-1, 1).$$

It diverges for all $x \geq 1$ although the function $1/(1-x)$ is defined for all $x \in \mathbb{R} \setminus \{1\}$.

The following "maple worksheet" gives some visualizations:

```
> restart:n:=4;m:=51;
```

$$n := 4$$
$$m := 51$$

the n-th partial sum

```
> p[n](x):=sum(x^k,k=0..n);p(x):=Sum(x^k,k=0..infinity)
> ;p[m](x):=sum(x^k,k=0..m):
```

$$p_4(x) := 1 + x + x^2 + x^3 + x^4$$

$$\mathrm{p}(x) := \sum_{k=0}^{\infty} x^k$$

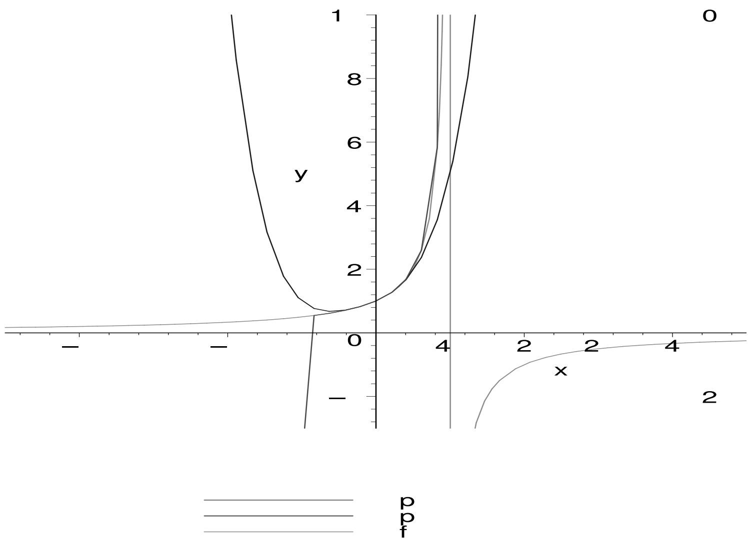
the limit function

```
> f(x):=1/(1-x);
```

$$\mathrm{f}(x) := \frac{1}{1-x}$$

The graphs of p[n], p[m] and f, note that the interval of the power series p is (-1,1) and that p has a pole at 1!

```
> plot({p[n](x),p[m](x),f(x)
> },x=-5..5,y=-3..10,color=[red,blue,gold]);
```



Let C denote the set of all numbers $x \in \mathbb{R}$ for which the power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges. Since $\sum_{k=0}^{\infty} a_k(x_0-x_0)^k = \sum_{k=0}^{\infty} a_k \cdot 0 = 0$ clearly converges, we know that $C \neq \emptyset$ is not empty. The number

$$R := \sup_{x \in C} |x - x_0| \qquad (0 \leq R \leq \infty)$$

is called the *radius of convergence* of the power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$, the open interval

$$(x_0 - R, x_0 + R),$$

the *interval* of convergence of the power series (which becomes $(-\infty, \infty)$ if $R = \infty$). The association

$$x \mapsto \sum_{k=0}^{\infty} a_k(x-x_0)^k$$

defines a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ which is called the *limit function* of the power series, or the function represented by the power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$. In the previous example, the radius of convergence R is 1, i.e. $R = 1$, its interval of convergence is $(-1, 1)$ and the limit function is $1/(1-x)$. The following theorem describes the "quality" of the convergence of a power series.

Theorem 2.2 (Convergence)
Suppose the power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ has radius of convergence $R > 0$. Then, $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges absolutely on its interval of convergence $I := (x_0 - R, x_0 + R)$ and uniformly on each compact subinterval of I.

Recall that a series $\sum_{k=0}^{\infty} b_k$ converges *absolutely* if the associated series $\sum_{k=0}^{\infty} |b_k|$ converges. A sequence $(f_n : I \to \mathbb{R})_{n\in\mathbb{N}}$ converges *uniformly* towards the (limit) function $f : I \to \mathbb{R}$ if for any $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \varepsilon \quad \text{for all } n \geq n_0 \text{ and all } x \in I.$$

Intuitively, this condition means that the sequence $(f_n(x))$ (note the free parameter x !) converges towards $f(x)$ at the same rate for all points $x \in I$. In the case of Theorem 2.2, the functions f_n are the partial sums $f_n(x) := \sum_{k=0}^{n} a_k(x - x_0)^k$ or (nth Taylor polynomials) of the infinite series $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ at x_0. In Example 2.1, the functions f_n are given by

$$f_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \in I := (-1, 1).$$

Recall that an absolutely convergent series can be rearranged in any order or regrouped in any way without affecting the convergence and the sum of the series. In contrast a conditionally convergent series (convergent but not absolutely convergent series) can be rearranged so that it converges to any given limit S between $-\infty$ and ∞ (Riemann Rearrangement Theorem). Since power series converge absolutely on their interval of convergence, they can be arbitrarily rearranged.

The following theorem gives some efficient formulae to determine the radius of convergence of a given power series:

Theorem 2.3 (Radius of Convergence)
Given a power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$, its radius of convergence can be computed using the

1. *formula of Cauchy–Hadamard*

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}$$

2. *or, if $a_k \neq 0$ for all $k \in \mathbb{N}_0$*

$$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}.$$

Example 2.2
We would like to revisit the geometric series

$$\sum_{k=0}^{\infty} x^k,$$

and recompute its radius of convergence R. Note that $a_k = 1$ for all $k \in \mathbb{N}_0$. We first try the root test:

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{1}} = \frac{1}{1} = 1.$$

We will now use the quotient test and again obtain

$$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} \frac{1}{1} = 1,$$

as claimed above.

Example 2.3
Find the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k}(x-1)^k.$$

Clearly, $a_k = k^2/2^k$. We will use the root test and obtain

$$R = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{\dfrac{k^2}{2^k}}} = \frac{1}{\lim\limits_{k\to\infty} \dfrac{\sqrt[k]{k^2}}{\sqrt[k]{2^k}}} = \frac{1}{\lim\limits_{k\to\infty} \dfrac{\left(\sqrt[k]{k}\right)^2}{2}}$$

and by the limit theorem of calculus

$$= \frac{1}{\frac{1}{2}\left(\lim\limits_{k\to\infty} \sqrt[k]{k}\right)^2} = \frac{2}{1^2} = 2,$$

since $\lim_{k\to\infty} \sqrt[k]{k} = 1$.

Power series behave very much like polynomials:

Theorem 2.4 (Differentiation and Integration)
Suppose $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series with interval of convergence $I := (x_0 - R, x_0 + R)$, $R > 0$, representing the function $f : I \to \mathbb{R}$ as limit function, i.e. $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ for all $x \in I$. Then

1. *f is differentiable on I and its derivative f' has a power series representation with the same radius of convergence, which can be obtained from the power series for f by termwise differentiation:*

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1} \qquad \text{for all } x \in I.$$

2. *f is integrable on I and its integral $\int f(x)\,dx$ has a power series representation with the same radius of convergence, which can be obtained from the power series for f by termwise integration:*

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-x_0)^{k+1} + C \qquad \text{for all } x \in I.$$

Through repeated application of Theorem 2.4, we obtain the following corollary

Corollary 2.2
Suppose $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ *for* $x \in (x_0 - R, x_0 + R)$, $R > 0$, *then* f *possesses derivatives of any order, which can be obtained by repeated termwise differentiation of the power series. In particular, we have*

$$a_k = \frac{f^{(k)}(x_0)}{k!} \qquad \text{for all } n \in \mathbb{N}_0.$$

Theorem 2.4 is a very useful tool for computing new power series expansion from old ones. We demonstrate this technique by some examples and add a list of fundamental expansions, which can be obtained by Taylor's theorem.

Example 2.4
Expand the function f given by

$$f(x) := \frac{1}{1+x^2}$$

as a power series in terms of x. Then find a power series representation of $\arctan$.

The given function f has the form of the limit function of the geometric series studied above. Therefore, we obtain

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} \left(-x^2\right)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

Since the geometric series has radius of convergence $R = 1$, we conclude that $|-x^2| < 1 = R$, which is equivalent to $|x| < 1$. Hence, the power series $\sum_{k=0}^{\infty} (-1)^k x^{2k}$ representing the function f has radius of convergence $R = 1$, too.

We will now determine the power series expansion for $\arctan$. Recall that

$$\arctan'(x) = \frac{1}{1+x^2}.$$

Therefore, we can conclude that

$$\arctan'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}. \tag{2.8}$$

for all $|x| < 1$ and apply Theorem 2.4(2) and integrate equation 2.8 in order to obtain a series representation for $\arctan(x) + C = \int \arctan'(x)\, dx$, which will have the same radius of convergence, namely $R = 1$. Hence

$$\begin{aligned}
\arctan(x) &= \int \arctan'(x), dx - C && (2.9)\\
&= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx - C && (2.10)\\
&= \sum_{k=0}^{\infty} \int \left((-1)^k x^{2k} \right) dx - C && (2.11)\\
&= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} - C. && (2.12)
\end{aligned}$$

By evaluating both $\arctan(x)$ and the power series 2.12 at $x = 0$, we find that $C = 0$. Thus

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1}.$$

Theorem 2.5 (Fundamental Power Series Expansions)
Here are some fundamental power series expansions and their radius of convergence:

1. $\displaystyle \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1$ *(Geometric Series).*

2. $\displaystyle (1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad |x| < 1$ *(Binomial Series).*

3. $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad |x| < \infty.$

4. $\displaystyle \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad |x| < \infty.$

5. $\displaystyle \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad |x| < \infty$

6. $\displaystyle \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad |x| < \infty.$

7. $\displaystyle \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad |x| < \infty$

8. $\displaystyle \ln(x+1) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \quad |x| < 1.$

9. $\arctan(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}, \quad |x| < 1.$

A power series is uniquely determined by its coefficients. In other word, if two power series with common expansion point x_0 have the same values on a neighborhood of x_0 then they are identical. In fact, this result follows even under weaker assumptions:

Theorem 2.6 (Identity Theorem)
Suppose x_0 is limit point of the set S which is contained in the interval of convergence of both of the power series below. If

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty} b_k(x-x_0)^k$$

for all $x \in S$ (note that S could be a tiny neighborhood of the point x_0) then $a_k = b_k$ for all $k \in \mathbb{N}_0$.

Recall that a point $x_0 \in \mathbb{R}$ is called a *limit point* of the set $S \subset \mathbb{R}$ if any neighborhood of x_0 contains at least one point of S distinct from x_0.

The assertion of Theorem 2.6 is fundamental for the "power series method" for solving linear differential equations, which we will study below. Also crucial for this solution method is a working knowledge of the algebraic operations which can be performed with power series. Given two functions f and g which are representated at x_0 by the power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ and $\sum_{k=0}^{\infty} b_k(x-x_0)^k$ on some open subinterval $(x_0 - r, x_0 + r)$ $r > 0$ of their intervals of convergence, respectively, the following theorem indicates, that the functions $f + g$, αf and fg can also be represented by a power series at x_0 and how their power series can be obtained from the power series of f and g:

Theorem 2.7 (Algebraic Operations)
Given two power series $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ and $g(x) = \sum_{k=0}^{\infty} b_k(x-x_0)^k$ with common expansion point x_0 and positive radius of convergence. Let $r > 0$ be chosen such that both power series converge for all $|x - x_0| < r$ (for example, could r be chosen to be the smaller of the two radii of convergence). Then

1. $f(x) + g(x) = \sum_{k=0}^{\infty} \big(a_k + b_k\big)(x-x_0)^k.$

2. $(\alpha f)(x) = \sum_{k=0}^{\infty} (\alpha a_k)(x-x_0)^k.$

3. $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right)(x-x_0)^k.$ *(Cauchy Product)*

Keep in mind that polynomials are special power series, namely power series, for which "almost all" coefficients are zero. "Almost all" is just short for "all but finitely many".

Example 2.5
Suppose $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$ are two polynomials. Since f and g coincide with the polynomials representing them everywhere, they are just called polynomials. We will compute their product and see that the Cauchy product is just a natural extension of the common polynomial multiplication to absolutely convergent series.

$$\begin{aligned} f(x) \cdot g(x) &= (a_0 + a_1x + a_2x^2 + a_3x^3) \cdot (b_0 + b_1x + b_2x^2) \\ &= (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\ &\quad + (a_0b_3 + a_1b_2 + a_2b_0)x^3 + (a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1)x^4. \end{aligned}$$

Example 2.6
Supose $f(x) = x^3$ and the function $g(x) = 1/(1-x)$, which can be represented by the power series $\sum_{k=0}^{\infty} x^k$ on the open interval $(-1, 1)$. Then, with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

denoting the Kronecker-delta, we obtain

$$\begin{aligned} f(x) \cdot g(x) &= x^3 \cdot \frac{1}{1-x} = x^3 \cdot \sum_{k=0}^{\infty} x^k \\ &= \left(0 + 0x + 0x^2 + 1x^3 + 0x^4 + 0x^5 + \cdots\right) \cdot \sum_{k=0}^{\infty} x^k \\ &= \left(\sum_{k=0}^{\infty} \delta_{3,k} x^k\right) \cdot \left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \delta_{3,j} \cdot 1\right) x^k = \sum_{k=3}^{\infty} x^k \end{aligned}$$

which is the same as

$$= \sum_{k=0}^{\infty} x^{k+3}.$$

Example 2.7
Suppose $f(x) = x^{n_0}$ and $g(x) = \sum_{k=0}^{\infty} a_k x^k$, then, as above, we obtain

$$\begin{aligned} f(x) \cdot g(x) &= \left(0 + 0x + \cdots 0x^{n_0 - 1} + 1x^{n_0} + 0x^{n_0+1} + \cdots\right) \cdot \sum_{k=0}^{\infty} a_k x^k \\ &= \left(\sum_{k=0}^{\infty} \delta_{n_0,k} x^k\right) \cdot \left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \delta_{n_0,j} \cdot a_{k-j}\right) x^k \end{aligned}$$

and, since $\delta_{n_0,j}$ is only different from 0 if $j = n_0$ (in which case it assumes the value 1), this expression is

$$= \sum_{k=n_0}^{\infty} a_{k-n_0} x^k = \sum_{k=0}^{\infty} a_k x^{k+n_0}.$$

We can therefore conclude that multiplying a power series in $(x - x_0)$ by the binom $(x - x_0)^{n_0}$ yields

$$(x - x_0)^{n_0} \cdot \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{k=0}^{\infty} a_k (x - x_0)^{n_0+k}.$$

Example 2.8

$$\begin{aligned} e^x \cdot e^x &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \cdot \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{1}{j!} \cdot \frac{1}{(k-j)!} \right) x^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!k!} \right) x^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \right) \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} \right) \frac{x^k}{k!} = \sum_{k=0}^{\infty} 2^k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = e^{2x}. \end{aligned}$$

Definition 2.1 (Real-Analytic Functions)
Suppose I is an open interval and $x_0 \in I$. Then the function $f : I \to \mathbb{R}$ is called *analytic* at x_0, if there exists a neighborhood of x_0, in which it can be represented by a power series. The function is called *analytic* on I if it is analytic at each point $x_0 \in I$.

Example 2.9
Clearly, every polynomial is analytic on $\mathbb{R}$.

Example 2.10
The function f given by the expression

$$f(x) := \frac{1}{x} \qquad \text{for all } x \in (0, \infty),$$

is analytic at $x_0 = 1$ (actually it is analytic on $(0, \infty)$. The following computation can be easily extended to a general proof:

$$f(x) = \frac{1}{x} = \frac{1}{1 + (x - 1)}$$

and, using the geometric series

$$= \sum_{k=0}^{\infty} \Big(-(1-x)\Big)^k \qquad \text{for all } |-(x-1)| < 1.$$

Hence,

$$f(x) = \sum_{k=0}^{\infty} (-1)^k (1-x)^k \qquad \text{for all } |x-1| < 1,$$

or in other words, for all $x \in (0, 2)$. In this example, the neighborhood mentioned in Definition 2.1, is the open interval $(0, 2)$ containing the point $x_0 = 1$. Note that the power series $\sum_{k=0}^{\infty} (-1)^k (1-x)^k$ represents f only on this neighborhood, outside it does not even converge.

Theorem 2.8
Suppose the functions f and g are analytic on the open interval I. Then $f + g$ and fg are analytic on I. If $g(x) \neq 0$ for all $x \in I$, then also $1/g$ is analytic on I.

2.4.3 The Power-Series Method

Existence and Uniqueness of Solutions

Recall the following theorem, which guarantees the existence of n linear independent solutions of a "normal" linear differential equation of order n.

Theorem 2.9 (Existence)
Suppose the functions a_j $(j = 0, 1, \ldots, n)$ are continuous on the open interval I_0 and $a_0(x) \neq 0$ for all $x \in I_0$. Moreover, suppose $k_j \in \mathbb{R}$ for $j = 0, \ldots, n-1$ and $x_0 \in I_0$. Then there exists exactly one solution $y : I_0 \to \mathbb{R}$ of the linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

satisfying the initial conditions $y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$.

We confine ourselves again to the special but for applications most important case of linear differential equations of order 2. Here, it can be shown that under the stronger assumptions requiring that the coefficient functions a_j are not just continuous on I_0 but analytic on I_0 then the solutions are analytic at x_0.

Theorem 2.10 (Analyticity of the Solutions)
Suppose the functions a_2, a_1 and a_2 are analytic on the open interval I_0 and suppose that $x_0 \in I_0$ is an ordinary point of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{2.13}$$

Then every solution of (2.13) has a power series representation whose radius of convergence R is a least as great as the distance from x_0 to the nearest singular point of (2.13).

Determining Solutions

We will now demonstrate by an example, how the solutions of a linear differential equation can be obtained using the "power series method".

Example 2.11
We would like to solve the linear differential equation

$$y'' - x^2 y = 0$$

whose coefficients $a_0(x) = 1$, $a_1(x) = 0$ and $a_2(x) = -x^2$ are polynomials and as such analytic on $I_0 = \mathbb{R}$. We are looking for solutions around the origin, i.e. $x_0 = 0$, which is an ordinary point of the equation.

By Theorem 2.9 and Theorem 2.10, the differential equation possesses two linearly independent solutions and every solution is analytic at x_0. Therefore, we can assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We determine its first two derivatives using Theorem 2.4 and obtain

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1},$$
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substituting these expressions into the original differential equation yields

$$0 = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x^2 \sum_{n=0}^{\infty} c_n x^n$$

which by Example 2.7 equals

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

and applying the index substitution $m = n - 2$ to the first and the index substitution $m = n + 2$ to the second power series, we obtain

$$= \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{m=2}^{\infty} c_{m-2} x^m$$

rewriting the sums such that they have the same starting index yields

$$\begin{aligned} &= 2 \cdot 1 \cdot c_2 + 3 \cdot 2 \cdot c_3 \cdot x + \\ &\quad + \sum_{m=2}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{m=2}^{\infty} c_{m-2} x^m \end{aligned}$$

using Theorem 2.7 we can combine the two power series and obtain the power series

$$= 2c_2 + 6c_3 x + \sum_{m=2}^{\infty} \Big((m+2)(m+1)c_{m+2} - c_{m-2} \Big) x^m.$$

Thus, summarizing, we have obtained the following identity

$$0 = 2c_2 + 6c_3 x + \sum_{m=2}^{\infty} \Big((m+2)(m+1)c_{m+2} - c_{m-2} \Big) x^m$$

for all $x \in (-R, R)$, where R denotes the radius of convergence of the power series. Thus, by Theorem 2.6, we can conclude that

$$2c_2 = 0, \quad 6c_3 = 0$$

and

$$(m+2)(m+1)c_{m+2} - c_{m-2} = 0 \quad \text{for all } m \geq 2.$$

Hence, the coefficients c_m satisfy the following recursion formula:

$$\begin{aligned} c_0 &= \text{arbitrary} \\ c_1 &= \text{arbitrary} \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

and for all $m \geq 2$

$$c_{m+2} = \frac{1}{(m+1)(m+2)} \cdot c_{m-2}.$$

Thus, we can compute the first values of c_m:

$$
\begin{aligned}
c_4 &= \frac{1}{3\cdot 4}\cdot c_0 \\
c_5 &= \frac{1}{4\cdot 5}\cdot c_1 \\
c_6 &= \frac{1}{5\cdot 6}\cdot c_2 = 0 \\
c_7 &= \frac{1}{6\cdot 7}\cdot c_3 = 0 \\
c_8 &= \frac{1}{7\cdot 8}\cdot c_4 = \frac{1}{3\cdot 4}\cdot\frac{1}{7\cdot 8}\cdot c_0 \\
c_9 &= \frac{1}{8\cdot 9}\cdot c_5 = \frac{1}{4\cdot 5}\cdot\frac{1}{8\cdot 9}\cdot c_1 \\
c_{10} &= \frac{1}{9\cdot 10}\cdot c_6 = 0 \\
c_{11} &= \frac{1}{10\cdot 11}\cdot c_7 = 0 \\
c_{12} &= \frac{1}{11\cdot 12}\cdot c_8 = \frac{1}{3\cdot 4}\cdot\frac{1}{7\cdot 8}\cdot\frac{1}{11\cdot 12}\cdot c_0 \\
c_{13} &= \frac{1}{12\cdot 13}\cdot c_9 = \frac{1}{4\cdot 5}\cdot\frac{1}{8\cdot 9}\cdot\frac{1}{12\cdot 13}\cdot c_1.
\end{aligned}
$$

By induction, we obtain

$$
\begin{aligned}
c_0 &= \text{arbitrary} \\
c_1 &= \text{arbitrary} \\
c_2 &= 0 \\
c_3 &= 0 \\
c_{4k} &= \frac{c_0}{\prod_{j=1}^{k}(4j-1)4j} = \frac{c_0}{4^k k!\prod_{j=1}^{n}(4j-1)} \\
c_{4k+1} &= \frac{c_1}{\prod_{j=1}^{k}4j(4j+1)} = \frac{c_1}{4^k k!\prod_{j=1}^{k}(4j+1)} \\
c_{4k+2} &= 0 \\
c_{4k+3} &= 0
\end{aligned}
$$

for all $k \geq 1$. Hence, we obtain the following solution, which still contains the free parameters c_0 and c_1

$$
y(x) = \sum_{k=1}^{\infty}\Big(c_{4k}x^{4k} + c_{4k+1}x^{4k+1}\Big)
$$

which can be rearranged, since power series converge absolutely on their interval of convergence

$$= c_0 \left(\sum_{k=0}^{\infty} \frac{1}{4^k k! \prod_{j=1}^{k}(4j-1)} x^{4k} \right) + c_1 \left(\sum_{k=0}^{\infty} \frac{1}{4^k k! \prod_{j=1}^{k}(4j+1)} x^{4k+1} \right)$$

since the empty products $\prod_{k=1}^{0}(4j-1)$ and $\prod_{k=1}^{0}(4k+1)$ (by definition) equal 1, also recall that $4^0 = 1$ and $0! = 1$; hence $y(x) = c_0 y_0(x) + c_1 y_1(x)$, where $y_0(x)$ and $y_1(x)$ stand for the first and second power series, respectively, which are clearly linearly independent. Since the equation $y'' - x^2 y = 0$ does not have any singular points, by Theorem 2.10, the radius of convergence of the solutions y_0 and y_1 clearly is $R = \infty$.

HOMEWORK SET 2.1

1. Prove that for $1 \le k \le n$

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

2. Evaluate

$$\sum_{k=0}^{n} \binom{2n}{2k} - \sum_{k=0}^{n-1} \binom{2n}{2k+1}$$

3. Using power series and Cauchy products to verify the "First Pythagorean Identity" $\cos^2 x + \sin^2 x = 1$. HINT: Use the identity established in the previous exercise!

4. Give the power series expansions of $x/(x-1)$ and $1/(x-1)$ in x and determine their radius of convergence.

5. Show that the Cauchy product of the two power series in x representing $1/(1+x)$ and $1/(1-x)$ reduces to the power series expansion of $1/(1-x^2)$.

6. Find the radius of convergence of the power series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+3)^2}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x-1)^n$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$

7. Using the power series method, find complete solutions around the origin for the following differential equations

(a) $y' - \lambda y = 0$ (b) $y'' - \lambda^2 y = 0$ (c) $y'' + xy' + 2y = 0$

2.5 The Method of Frobenius

2.5.1 Introduction

Recall that any solution of the ordinary linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{2.14}$$

whose coefficient functions a_0, a_1, a_2 are analytic at x_0 and for which $a_0(x_0) \neq 0$, is analytic at x_0 and therefore has the form

$$y(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n \tag{2.15}$$

The radius of convergence of y is at least as large as the distance from x_0 to the nearest singular point of (2.14) (including complex zeroes of a_0 !). As we have seen in the previous chapter, the power series method allowed us to find a complete solution of (2.14).

However, in applications, for instance the "separation of variable method for the classical partial differential equation, we are often lead to linear differential equations, which need to be solved around so-called "regular" singular points. We just mention the Bessel and Legendre differential equations as examples.

To address this need, we will study an extension to the power series method, called "Frobenious method", which will allow us to also develop series representations of solutions of (2.14) around regular singular points.

2.5.2 Singular Points

We will rewrite equation (2.14) in "normal form"

$$y'' + p(x)y' + q(x)y = 0 \tag{2.16}$$

where

$$p(x) := \frac{a_1(x)}{a_0(x)}, \quad q(x) := \frac{a_2(x)}{a_0(x)}, \tag{2.17}$$

$x_0 \in I_0$ and $I_0 \subseteq \mathbb{R}$ is an open interval containing the point x_0.

Definition 2.2 ((Ir)regular singular point)
Suppose $x_0 \in I_0$ with $a_0(x_0) = 0$. Then the point x_0 is called a

1. *regular singular point* of the equation (2.16) if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at x_0, i.e. if there exists power series $\sum_{j=0}^{\infty} p_j(x - x_0)^j$ and $\sum_{j=0}^{\infty} q_j(x - x_0)^j$ with positive radii of convergence $R_p, R_q > 0$, respectively, such that

$$(x - x_0)p(x) = \sum_{j=0}^{\infty} p_j(x - x_0)^j \tag{2.18}$$

and

$$(x-x_0)^2 q(x)=\sum_{j=0}^{\infty} q_j (x-x_0)^j \tag{2.19}$$

for all x of their common domain $(x_0-R_p, x_0+R_p)\cap(x_0-R_q, x_0+R_q)$.

2. Otherwise the point x_0 is called an *irregular singular point* of (2.16).

Example 2.12
The linear differential equation

$$x(x-1)^2 y'' - 3y' + 5y = 0$$

has normal form

$$y'' + \underbrace{\frac{-3}{x(x-1)^2}}_{p(x)} y' + \underbrace{\frac{5}{x(x-1)^2}}_{q(x)} y = 0.$$

We will classify the singular points $x_0=0$ and $x_1=1$ of this equation by applying Definition 2.2:

1. We will first consider $x_0=0$:

$$\begin{aligned}(x-x_0)p(x) &= (x-0)\frac{-3}{x(x-1)^2} = \frac{-3}{(x-1)^2}\\ (x-x_0)^2 q(x) &= (x-0)^2\frac{5}{x(x-1)^2} = \frac{5x}{(x-1)^2}\end{aligned}$$

which are both analytic at $x_0=0$. Hence, $x_0=0$ is a regular singular point of the given equation.

2. We will next check $x_1=1$:

$$\begin{aligned}(x-x_1)p(x) &= (x-1)\frac{-3}{x(x-1)^2} = \frac{-3}{x(x-1)}\\ (x-x_1)^2 q(x) &= (x-1)^2\frac{5}{x(x-1)^2} = \frac{5}{x}\end{aligned}$$

where the first quotient is *not* analtyic at $x_1=1$. Hence, $x_1=1$ is an irregular singular point of the given equation.

Remark 2.1 (Necessary Condition for Analyticity)
The following list contains two useful necessary—but not *sufficient—conditions for a function to be analytic at a point x_0 derived from its associated power series representation:*

1. $\lim_{x\to x_0} f(x)$ *exists (it equals the constant term of the power series representation of* f *at* x_0 *!)*

2. f *possess derivatives of any order at* x_0 *(since a power series possess derivative of any order !)*

Example 2.13
In the equation

$$y'' + \sqrt{x}\, y = 0,$$

which is already in normal form, the functions $p(x) = 0$ and $q(x) = \sqrt{x}$ for all $x \in [0, \infty)$. Clearly, both p and q are analytic for all $x_0 > 0$. However, q is *not* analytic at $x_0 = 0$, since $(x - x_0)^2 q(x) = (x - 0)^2 \sqrt{x} = x^{5/2}$ is not analytic at $x_0 = 0$ because its third derivative does not exist at $x_0 = 0$ as can be easily verified.

HOMEWORK SET 2.2

1. Find the singular points of the equations

   (a) $y'' + 2xy' + y = 0$

   (b) $2x^2y'' - 3\lambda^2 y = 0$

   (c) $(1 - x^2)y'' + y' - y = 0$

   (d) $x^2y'' - y' + xy = 0$

2. Show that $(x - 1)p(x) = 2(x - 1)/x$ is analytic at $x_0 = 1$ by computing its Taylor series expansion in $x - 1$.

3. Show that $x^2q(x) = 3x/(x - 1)^3$ is analytic at $x_0 = 0$ by computing its Taylor series expansion in x.

2.5.3 The Solution Method

Suppose $x_0 \in I_0$ is a regular singular point of the equation

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I_0. \tag{2.20}$$

For convenience, we assume that $x_0 = 0$ (otherwise shift using the substitution $\xi := x - x_0$!) and $I_0 := (0, a)$ for some $a > 0$. We multiply the equation by x^2 and obtain

$$x^2y'' + x\big(xp(x)\big)y' + \big(x^2q(x)\big)y = 0. \tag{2.21}$$

Since, by assumption, $x_0 = 0$ is a regular singular point of (2.20), there exist two power series $\sum_{j=0}^{\infty} p_j x^j$ and $\sum_{j=0}^{\infty} q_j x^j$ such that

$$xp(x) = \sum_{j=0}^{\infty} p_j x^j, \quad |x| < R_p \tag{2.22}$$

$$x^2q(x) = \sum_{j=0}^{\infty} q_j x^j, \quad |x| < R_q. \tag{2.23}$$

Remark 2.2
In a small neighborhood of the point $x_0 = 0$, equation (2.21) can be "approximated"—using just the 0th term of the power series (2.22, 2.23)—by the equation

$$x^2 y'' + p_0 x y' + q_0 y = 0, \tag{2.24}$$

which is a Cauchy-Euler equation. As we have seen, the solutions of (2.24) have the form

$$y(x) = x^r f(x) \tag{2.25}$$

for some $r \in \mathbb{R}$ and $f(x) = 1$ or $f(x) = \ln x$.

In view of Remark 2.2, it is natural to expect that the solutions of equation (2.20) have the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n. \tag{2.26}$$

We will first demonstrate the solution technique in general to make transparent steps where the method has the potential of failing and provide remedies. Afterwards, we will study examples.

Suppose

$$y(x) = x^r \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} c_j x^{j+r}, \quad c_0 \neq 0, r \in \mathbb{R} \tag{2.27}$$

is a solution of the the equation

$$x^2 y'' + x \left(\sum_{j=0}^{\infty} p_j x^j \right) y' + \left(\sum_{j=0}^{\infty} q_j x^j \right) y = 0, \quad 0 < x < \min(R_p, R_q). \tag{2.28}$$

Then since,

$$x^2 y'' = x^2 \sum_{j=0}^{\infty} (j+r)(j+r-1) c_j x^{j+r-2} \tag{2.29}$$

$$= \sum_{j=0}^{\infty} (j+r)(j+r-1) c_j x^{j+r} \tag{2.30}$$

and

$$x y' = x \sum_{j=0}^{\infty} (j+r) c_j x^{j+r-1} \tag{2.31}$$

$$= \sum_{j=0}^{\infty} (j+r) c_j x^{j+r}, \tag{2.32}$$

we obtain

$$0 = \sum_{j=0}^{\infty}(j+r)(j+r-1)c_j x^{j+r} \tag{2.33}$$

$$+ \left(\sum_{j=0}^{\infty} p_j x^j\right)\left(\sum_{j=0}^{\infty}(j+r)c_j x^{j+r}\right) + \left(\sum_{j=0}^{\infty} q_j x^j\right)\left(\sum_{j=0}^{\infty} c_j x^{j+r}\right) \tag{2.34}$$

$$= \sum_{j=0}^{\infty}(j+r)(j+r-1)c_j x^{j+r} \tag{2.35}$$

$$+ \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}(k+r)c_k\right)x^{n+r} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}c_k\right)x^{n+r} \tag{2.36}$$

$$= x^r\left[\sum_{n=0}^{\infty}\left((n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big(p_{n-k}(k+r)+q_{n-k}\Big)c_k\right)x^n\right]. \tag{2.37}$$

Division by x^r (note $x > 0$) yields

$$0 = \sum_{n=0}^{\infty}\left((n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big(p_{n-k}(k+r)+q_{n-k}\Big)c_k\right)x^n, \tag{2.38}$$

which implies by the Identity Theorem that

$$(n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big(p_{n-k}(k+r)+q_{n-k}\Big)c_k = 0 \tag{2.39}$$

for all $n \in \mathbb{N}^0$. For $n = 0$, we obtain

$$\big(r(r-1)+p_0 r+q_0\big)c_0 = 0, \tag{2.40}$$

which implies, since we chose $c_0 \neq 0$ that

$$r(r-1)+p_0 r+q_0 = 0 \tag{2.41}$$

and therefore

$$h(r) := r^2 + (p_0-1)r + q_0 = 0. \tag{2.42}$$

Equation (2.42) is called the *indicial equation* of (2.20).

Remark 2.3
We mention that the coefficients p_0 and q_0 occuring in the indicial equation (2.42) and which are the constant terms of the power series expansions of $xp(x)$ and $x^2q(x)$, respectively, can be easily obtained by computing the limits

$$p_0 = \lim_{x\to 0} xp(x), \quad q_0 = \lim_{x\to 0} x^2 q(x). \tag{2.43}$$

We summerize: If (2.27) is a solution of the differential equation (2.20), then the parameter r has to satisfy the indicial equation (2.42).

We will now study the conditions which the coefficients c_n have to satisfy if (2.27) is a solution of the differential equation (2.20). To this we first rewrite equation (2.39) collecting the terms containing c_n on the left side:

$$\underbrace{\big[\underbrace{(n+r)(n+r-1)}_{(n+r)^2-(n+r)}+\big(p_0(n+r)+q_0\big)\big]}_{h(n+r)}\,c_n+\sum_{k=0}^{n-1}\Big(p_{n-k}(k+r)+q_{n-k}\Big)c_k=0, \tag{2.44}$$

noting that the coefficient of c_n coincides with $h(n+r)$, i.e.:

$$h(n+r)c_n=-\sum_{k=0}^{n-1}\Big(p_{n-k}(k+r)+q_{n-k}\Big)c_k \tag{2.45}$$

for all $n\in\mathbb{N}$.

We will now use the recurrence relation (2.45) to determine the coefficients c_n of the first solution y_1 of (2.20) . In order to facilitate the computation, we introduce the convention that the solutions $r_1,r_2\in\mathbb{C}$ of the indicial equation (2.42) are indexed such that $\mathrm{Re}(r_1)\geq\mathrm{Re}(r_2)$. Then, if we start our computation with the “larger” zero r_1 of the indicial equation, then First solution y_1

$$h(n+r_1)\neq 0\quad\text{for all } n\in\mathbb{N}, \tag{2.46}$$

(otherwise $n+r_1$ would be a larger zero of h, since $n>0$!) and we can divide the recurrence (2.45) by $h(n+r_1)$ solving for c_n for all $n\in\mathbb{N}$ and obtain

$$c_n=-\frac{1}{h(n+r_1)}\sum_{k=0}^{n-1}\Big(p_{n-k}(k+r_1)+q_{n-k}\Big)c_k\quad\text{for all } n\in\mathbb{N}, \tag{2.47}$$

which uniquely determines the sequence of coefficients c_n once c_0 has been chosen. Thus we obtain the first solution

$$y_1(x):=|x|^{r_1}\sum_{n=0}^{\infty}c_nx^n\quad\text{for all } 0<|x|<R_1, \tag{2.48}$$

with free parameter c_0.

We will now compute the second solution y_2 of (2.20) and consider the second zero r_2 of the indicial equation (2.42). Renaming the coefficients c_n in the recurrence (2.45) to d_n and replacing r by r_2 we obtain the recursion Second solution y_2

$$h(n+r_2)d_n=-\sum_{k=0}^{n-1}\Big(p_{n-k}(k+r_2)+q_{n-k}\Big)d_k \tag{2.49}$$

for all $n\in\mathbb{N}$. However, since by assumption $\mathrm{Re}(r_1)\geq\mathrm{Re}(r_2)$, it might happen that $r_2+n=r_1$ for some $n\in\mathbb{N}$. Therefore, we have to distinguish the following three settings:

1. $r_1 - r_2 \notin \mathbb{Z}$: Then, since $\text{Re}(r_1) \geq \text{Re}(r_2)$, $n + r_2 \neq r_1$ for all $n \in \mathbb{N}$, i.e. $h(n + r_2) \neq 0$ for all $n \in \mathbb{N}$, we can solve for d_n by diving equation (2.49) by $h(n + r_2)$ and obtain the recurrence

$$d_n = -\frac{1}{h(n + r_2)} \sum_{k=0}^{n-1} \Big(p_{n-k}(k + r_2) + q_{n-k}\Big) d_k \tag{2.50}$$

for all $n \in \mathbb{N}$, which yields the second solution

$$y_2(x) := |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n \quad \text{for all } 0 < |x| < R_2 \tag{2.51}$$

with free parameter d_0.

2. $r_1 - r_2 \in \mathbb{N}$: Then $r_2 + n_0 = r_1$ for some $n_0 \in \mathbb{N}$. In this case, we can compute the coefficients $d_1, d_2, \ldots, d_{n_0 - 1}$. However, if $n = n_0$ then the recurrence (refeq:recurrencet) yields the identity

$$\underbrace{h(n_0 + r_2)}_{=0 \text{ !!!}} d_{n_0} = -\sum_{k=0}^{n_0 - 1} \Big(p_{n_0 - k}(k + r_2) + q_{n_0 - k}\Big) d_k. \tag{2.52}$$

If the right side of equation (2.52) is also zero for $n = n_0$, then d_{n_0} can be chosen arbitrarily and we obtain the second solution

$$y_2(x) := |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n \quad \text{for all } 0 < |x| < R_2 \tag{2.53}$$

with the free parameters d_0 and d_{n_0}. If the right side of equation (2.52) is not zero for $n = n_0$, then the solution method has to be modified or the method of "Reduction of Order" can be applied to obtain a second solution.

3. $r_1 - r_2 = 0$: In this case this method only yields one solution. To obtain a second solution this method has to be modified or "Reduction of Order" has to be used.

The general result is formulated in the following

Theorem 2.11
If the equation $y'' + p(x)y' + q(x)y = 0$ *has a regular singular point at* $x_0 = 0$ *and if* $xp(x) = \sum_{j=0}^{\infty} p_j x^j$ *and* $x^2 q(x) = \sum_{j=0}^{\infty} q_j x^j$ *for* $|x| < R_p$ *and* $|x| < R_q$, *respectively, and if* $r_1, r_2 \in \mathbb{C}$ *are the solutions of the indicial equation* $r^2 + (p_0 - 1)r + q_0 = 0$ *indexed such that* $\text{Re}(r_1) \geq \text{Re}(r_2)$, *then the differential equation has a complete solution*

$$y(x) = a y_1(x) + b y_2(x), \quad a, b \in \mathbb{R} \tag{2.54}$$

on a "deleted neighborhood" $0 < |x| < R$ *where* $R \geq \min\{R_p, R_q\}$ *and the functions* y_1 *and* y_2 *have the form*

1. *If* $r_1 - r_2 \notin \mathbb{Z}$*, then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0 \tag{2.55}$$

$$y_2(x) = |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n, \quad d_0 \neq 0. \tag{2.56}$$

2. *If* $r_1 - r_2 \in \mathbb{N}$*, then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0 \tag{2.57}$$

$$y_2(x) = |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n + c y_1(x) \ln|x|, \quad d_0 \neq 0 \tag{2.58}$$

   where $c \in \mathbb{R}$ *(can be zero!).*

3. *If* $r_1 - r_2 = 0, \quad$ *then*

$$y_1(x) = |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0 \tag{2.59}$$

$$y_2(x) = |x|^{r_1} \sum_{n=1}^{\infty} d_n x^n + y_1(x) \ln|x|. \tag{2.60}$$

Example 2.14 (Case 1: $r_1 - r_2 \notin \mathbb{Z}$**)**
We consider the differential equation

$$2x^2 y'' + (x^2 - x) y' + y = 0 \tag{2.61}$$

which has normal form

$$y'' + \frac{x^2 - x}{2x^2} y' + \frac{1}{2x^2} y = 0. \tag{2.62}$$

We are seeking a solution near the point $x_0 = 0$, which is a regular singular point of equation (2.61), since

$$xp(x) = \tfrac{1}{2}x \underbrace{-\tfrac{1}{2}}_{p_0} \quad \text{and} \quad x^2 q(x) = \underbrace{\tfrac{1}{2}}_{q_0}. \tag{2.63}$$

1. The indicial equation of (2.61) is given by

$$h(r) = r^2 + (p_0 - 1)r + q_0 = r^2 - \tfrac{3}{2}r + \tfrac{1}{2} = 0, \tag{2.64}$$

   which is equivalent to

$$2h(r) = 2r^2 - 3r + 1 = (2r-1)(r-1) = 0 \tag{2.65}$$

   and implies that $r_1 = 1$ and $r_2 = \frac{1}{2}$. Since $r_1 - r_2 = 1 - \frac{1}{2} = \frac{1}{2} \notin \mathbb{Z}$, we are in case 1.

2. We will now determine the recurrence relations for computing the solutions. We are seeking solutions of the form

$$y(x) = x^r \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} c_j x^{j+r} \tag{2.66}$$

where we assume that $c_0 \neq 0$ and $x > 0$. Its derivatives are

$$y'(x) = \sum_{j=0}^{\infty} (j+r) c_j x^{j+r-1} \tag{2.67}$$

$$y''(x) = \sum_{j=0}^{\infty} (j+r-1)(j+r) c_j x^{j+r-2}. \tag{2.68}$$

Substituting these functions into the equation (2.61) yields

$$\begin{aligned}
0 &= \sum_{j=0}^{\infty} 2(j+r-1)(j+r) c_j x^{j+r} \\
&\quad + \underbrace{\sum_{j=0}^{\infty} (j+r) c_j x^{j+r+1}}_{\text{sub: } \ell := j+1} - \sum_{j=0}^{\infty} (j+r) c_j x^{j+r} + \sum_{j=0}^{\infty} c_j x^{j+r} \\
&= \sum_{j=0}^{\infty} \Big[2(j+r-1)(j+r) - (j+r) + 1 \Big] c_j x^{j+r} \\
&\qquad\qquad\qquad + \sum_{\ell=1}^{\infty} (\ell - 1 + r) c_{\ell-1} x^{\ell+r} \\
&= \underbrace{\big[2(r-1)r - r + 1 \big]}_{2h(r) = 2r^2 - 3r + 1} c_0 x^r \\
&\quad + \sum_{j=1}^{\infty} \Big[\underbrace{2(j+r-1)(j+r) - (j+r) + 1}_{2h(j+r) = 2(j+r)^2 - 3(j+r) + 1} \Big] c_j x^{j+r} \\
&\qquad\qquad\qquad + \sum_{\ell=1}^{\infty} (\ell - 1 + r) c_{\ell-1} x^{\ell+r}.
\end{aligned}$$

Division by x^r yields

$$\begin{aligned}
0 &= \big[(r-1)(2r-1) \big] c_0 \\
&\quad + \sum_{j=1}^{\infty} \Big\{ \big[(j+r-1)\big(2(j+r) - 1\big) \big] c_j + (j+r-1) c_{j-1} \Big\} x^j,
\end{aligned}$$

from which, by the "Identity Theorem", follows that

$$\big[(r-1)(2r-1) \big] c_0 = 0$$

and

$$\big[(j+r-1)\big(2(j+r)-1\big)\big]c_j + (j+r-1)c_{j-1} = 0 \quad \text{for all } j \in \mathbb{N}.$$

Since, by assumption, $c_0 \neq 0$, it follows that $(r-1)(2r-1) = 0$ which is the "indicial equation" of (2.61). We thus obtain the recurrence relation

$$\underbrace{\big[(j+r-1)\big(2(j+r)-1\big)\big]}_{2h(j+r)} c_j = -(j+r-1)c_{j-1} \quad \text{for all } j \in \mathbb{N}. \tag{2.69}$$

3. We will now use the recurrence (2.69) to compute the two linearly independent solutions of equation (2.61). We start with the larger solution $r_1 = 1$ and substitute it into the recursion (2.69) and obtain (note that $h(j+1) \neq 0$ for all $j \in \mathbb{N}$, since $r_1 = 1$ is the smallest root of the equation $h(r) = 0$.)

$$c_j = \frac{-c_{j-1}}{2j+1} \quad \text{for all } j \in \mathbb{N}. \tag{2.70}$$

With $c_0 \in \mathbb{R}$ arbitrary, we then compute

$$c_1 = \frac{-c_0}{3} \tag{2.71}$$

$$c_2 = \frac{-c_1}{5} = \frac{(-1)^2 c_0}{3\cdot 5} \tag{2.72}$$

$$c_3 = \frac{-c_2}{7} = \frac{(-1)^3 c_0}{3\cdot 5\cdot 7} \tag{2.73}$$

and thus

$$c_j = \frac{(-1)^j c_0}{\prod_{k=1}^{j}(2k+1)} = \frac{(-1)^j j! 2^j c_0}{(2j+1)!}. \tag{2.74}$$

for all $j \in \mathbb{N}$. For convenience, we set $c_0 := 1$. Then

$$y_1(x) = x^1 \sum_{j=0}^{\infty} \frac{(-1)^j j!\, 2^j}{(2j+1)!} x^j = \sum_{j=0}^{\infty} \frac{(-1)^j j!\, 2^j}{(2j+1)!} x^{j+1} \tag{2.75}$$

is the first solution of the equation (2.61).

The following maple worksheet computes the 10th partial sum of y_1 and plots this function:

```
> restart:c[0]:=1;
```

$$c_0 := 1$$

```
> for j from 1 to 10 do c[j]:=(-1)^j*j!/((2*j+1)!)  od;
```

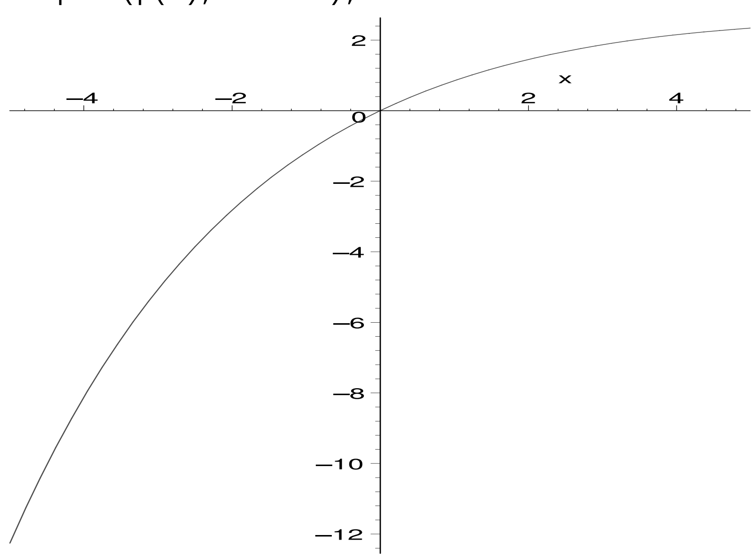
$$c_1 := \frac{-1}{6}$$

$$c_2 := \frac{1}{60}$$
$$c_3 := \frac{-1}{840}$$
$$c_4 := \frac{1}{15120}$$
$$c_5 := \frac{-1}{332640}$$
$$c_6 := \frac{1}{8648640}$$
$$c_7 := \frac{-1}{259459200}$$
$$c_8 := \frac{1}{8821612800}$$
$$c_9 := \frac{-1}{335221286400}$$
$$c_{10} := \frac{1}{14079294028800}$$

```
> p(x):=sum(c[k]*x^(k+1),k=0..10);
```

$$\mathrm{p}(x) := x - \frac{1}{6}x^2 + \frac{1}{60}x^3 - \frac{1}{840}x^4 + \frac{1}{15120}x^5 - \frac{1}{332640}x^6 + \frac{1}{8648640}x^7 - \frac{1}{259459200}x^8 + \frac{1}{8821612800}x^9 - \frac{1}{335221286400}x^{10} + \frac{1}{14079294028800}x^{11}$$

```
> plot(p(x),x=-5..5);
```



HOMEWORK SET 2.3

1. Determine the second solution for Example 2.14.

2. Find two linearly independent solutions near the origin for the differential equation $2x^2y'' + (2x^2 + x)y' - y = 0$.

Example 2.15 (Case 2: $r_1 - r_2 \in \mathbb{N}$)
We will now consider the differential equation

$$9x^2y'' + 3xy' + 2(x-4)y = 0 \tag{2.76}$$

or in normal form

$$y'' + \frac{1}{3x}\,y' + \frac{2(x-4)}{9x^2}\,y = 0. \tag{2.77}$$

We are seeking a solution near the point $x_0 = 0$, which is a regular singular point of equation (2.76), since

$$xp(x) = \underbrace{-\tfrac{1}{3}}_{p_0} \quad \text{and} \quad x^2q(x) = \tfrac{2}{9}x \underbrace{-\tfrac{8}{9}}_{q_0}. \tag{2.78}$$

1. The indicial equation of (2.76) is given by

$$h(x) = r^2 + (p_0 - 1)r + q_0 = r^2 - \tfrac{2}{3}r - \tfrac{8}{9} = 0, \tag{2.79}$$

   which is equivalent to

$$9r^2 - 6r - 8 = (3r-4)(3r+2) = 0 \tag{2.80}$$

   and implies that $r_1 = \frac{4}{3}$ and $r_2 = -\frac{2}{3}$. Since $r_1 - r_2 = \frac{4}{3} + \frac{2}{3} = 2 \in \mathbb{N}$, we are in case 2.

2. We will now determine the recurrence relations for computing the solutions. We are seeking solutions of the form

$$y(x) = x^r \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} c_j x^{j+r} \tag{2.81}$$

   where we assume that $c_0 \neq 0$ and $x > 0$. Its derivatives are

$$y'(x) = \sum_{j=0}^{\infty} (j+r)c_j x^{j+r-1} \tag{2.82}$$

$$y''(x) = \sum_{j=0}^{\infty} (j+r-1)(j+r)c_j x^{j+r-2}. \tag{2.83}$$

Substituting these functions into the equation (2.76) yields

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty} 9(j+r-1)(j+r)c_j x^{j+r} \\ &\quad + \sum_{j=0}^{\infty} 3(j+r)c_j x^{j+r} + \underbrace{\underbrace{\sum_{j=0}^{\infty} 2c_j x^{j+r+1}}_{\text{sub: } \ell := n+1}}_{\sum_{\ell=1}^{\infty} 2c_{\ell-1}x^{\ell+r}} - \sum_{j=0}^{\infty} 8c_j x^{j+r} \\ &= \Big[9r(r-1) + 3r - 8\Big] c_0 x^r \\ &\quad + \sum_{j=1}^{\infty} \Big(\big[9(j+r)(j+r-1) + 3(j+r) - 8\big] c_j + 2c_{j-1} \Big) x^{j+r} \end{aligned}$$

Division by x^r yields

$$\begin{aligned} 0 &= \big[(r-1)(2r-1)\big] c_0 \\ &\quad + \sum_{j=1}^{\infty} \Big\{ \big[2(j+r-1)\big(2(j+r)-1\big)\big] c_j + (j+r-1)c_{j-1} \Big\} x^j, \end{aligned}$$

from which, by the “Identity Theorem”, follows that

$$\big[(r-1)(2r-1)\big] c_0 = 0$$

and

$$\big[2(j+r-1)\big(2(j+r)-1\big)\big] c_j + (j+r-1)c_{j-1} = 0 \quad \text{for all } j \in \mathbb{N}.$$

2.5.4 The Bessel Functions

Introduction

Before we construct the Bessel functions, we would like to recall that several important functions can be defined through differential equations. We illustrate this by some simple and well-known examples.

Example 2.16 (The Exponential Function)
Consider the homogenius first order linear differential equation

$$y' - y = 0. \tag{2.84}$$

To illustrate the similarity to the case of the Bessel functions, we will use the power series method and seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Substituting the series and its derivative into (2.84), we obtain

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

and with the index substitutions $k = n$ in the first and $k = n + 1$ in the second sum

$$= \sum_{k=1}^{\infty} k c_n x^{k-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k-1} = \sum_{k=1}^{\infty} \Big(k c_k - c_{k-1} \Big) x^{k-1},$$

from which follows by the Identity Theorem that $k c_k - c_{k-1} = 0$ for all $k \in \mathbb{N}$ and thus

$$c_k = \frac{1}{k} c_{k-1} = \frac{1}{k} \cdot \frac{1}{k-1} \cdot c_{k-2} = \cdots = \frac{1}{k!} \cdot c_0.$$

Hence, if we set $c_0 := 1$, we obtain the particular solution

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

of the equation (2.84) which is called the (natural) exponential function and usually denoted denoted by $\exp(x)$ or simply by e^x. As the differential equation does not have any singular points, by the theorem on the analyticity of the solution from the previous lecture, we can conclude that the series converges for $x \in \mathbb{R}$. Thus the general solution of (2.84) is given by $y(x) = c \cdot e^x$ for $c \in \mathbb{R}$.

Example 2.17 (The Trigonmetric Functions Sine And Cosine)
Consider the homogenius linear second order constant coefficients equation

$$y'' + y = 0. \tag{2.85}$$

Again, for the purpose of illustration, we seek solutions of the form

$$y(x) = \sum_{k=0}^{\infty} c_n x^n.$$

Substituting the series and its secon derivative into the equation (2.85), we obtain

$$0 = \sum_{n=2}^{\infty} n(n-2) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n$$

from which follows using the index substitution $k = n$ in the first and $k = n + 2$ in the second sum

$$= \sum_{k=2}^{\infty} k(k-2) c_k x^{k-2} + \sum_{k=2} c_{k-2} x^{k-2}$$

and

$$= \sum_{k=2}^{\infty} \Big(k(k-2)c_k + c_{k-2} \Big) x^{k-2},$$

and by the Identity theorem finally the recurrence relation $k(k-2)c_k + c_{k-2} = 2$ or, solved for c_k,

$$c_k = \frac{-1}{k(k-2)} \cdot c_{k-2} \quad \text{for all } k \geq 2.$$

Therefore,

$$\begin{aligned} c_0 &= \text{arbitrary} \\ c_1 &= \text{arbitrary} \\ c_2 &= \frac{-1}{2 \cdot 1} c_0 \\ c_3 &= \frac{-1}{3 \cdot 2} c_1 \\ c_4 &= \frac{-1}{4 \cdot 3} c_2 = \frac{(-1)^2}{4 \cdot 3 \cdot 2 \cdot 1} c_0 \\ c_5 &= \frac{-1}{5 \cdot 4} c_1 = \frac{(-1)^2}{5 \cdot 4 \cdot 3 \cdot 2} c_1 \end{aligned}$$

and

$$\begin{aligned} c_{2k} &= (-1)^k \frac{1}{(2k)!} c_0 \\ c_{2k+1} &= (-1)^k \frac{1}{(2k+1)!} c_1 \end{aligned}$$

for all $k \in \mathbb{N}^0$. Hence,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} \left(\frac{(-1)^k c_0}{(2k)!} x^{2k} + \frac{(-1)^k c_1}{(2k+1)!} x^{2k+1} \right) \\ &= c_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

The series solution

$$y_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

is commonly abbreviated $\cos x$, while the solution

$$y_2(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

is usually denoted by $\sin x$.

The Bessel Equation of order ν

The second equation

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \tag{2.86}$$

is called the *Bessel equation* of order ν with parameter λ. This equation arises in a great variety of problems, including almost all application involving partial differential equations like the wave equation, heat equation in regions with circular symmetry.

The substitution $t := \lambda x$ reduces equation (2.86) using

$$\frac{dy}{dx} = \lambda\frac{dy}{dt} \quad \text{and} \quad \frac{dy^2}{dx^2} = \lambda^2\frac{d^2y}{dt^2}$$

to the form

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0, \tag{2.87}$$

which is called the *Bessel equation* of order ν. We will solve this form of the equation, but choose the simpler formulation

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0,$$

or in normal form

$$y'' + \frac{1}{t}y' + \frac{t^2 - \nu^2}{t^2}\,y = 0.$$

Note that $tp(t) = 1$ and $t^2q(t) = -\nu^2 + t^2$, which implies that $p_0 = 1$ and $q_0 = -\nu^2$. Therefore, $t_0 = 0$ is a regular singular point of equation (2.87) and its only singular point. We will systematically solve equation (2.87) using the Frobenius method.

(a) Indicial equation:

$$\begin{aligned} h(r) &= r^2 + (p_0 - 1)r + q_0 \\ &= r^2 + (1-1)r + (-\nu^2) \\ &= r^2 - \nu^2 = 0. \end{aligned}$$

Hence, $r_1 = \nu$ where we choose $\nu \geq 0$, and $r_2 = -\nu$.

(b) Recurrence relation:
We are seeking a solution of the form

$$y(t) = t^r\sum_{n=0}^{\infty}c_nt^n = \sum_{n=0}^{\infty}c_nt^{n+r}, \quad t > 0.$$

Substitution its derivatives

$$y'(y) = \sum_{n=0}^{\infty}(n+r)c_nt^{n+r-1}$$

and

$$y''(y) = \sum_{n=0}^{\infty} (n+r-1)(n+r)c_n t^{n+r-2}$$

into equation (2.87) yields

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (n+r-1)(n+r)c_n t^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n t^{n+r} + \\
&\qquad + \sum_{n=0}^{\infty} c_n t^{n+r+2} + \sum_{n=0}^{\infty} (-\nu^2)c_n t^{n+r} \\
&= \sum_{n=0}^{\infty} \Big((n+r-1)(n+r) + (n+r) - \nu^2 \Big) c_n t^{n+r} \\
&\qquad + \sum_{n=0}^{\infty} c_n t^{n+r+2}
\end{aligned}$$

and with the index substitution $k = n$ in the first and $k = n+2$ in the second sum and simplfying the coefficient of c_n we obtain

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \Big((k+r)^2 - \nu^2 \Big) c_k t^{k+r} + \sum_{k=2}^{\infty} c_{k-2} t^{k+r} \\
&= t^r \Bigg((r^2 - \nu^2)c_0 + \big((r+1)^2 - \nu^2 \big) c_1 t + \\
&\qquad + \sum_{k=2}^{\infty} \Big[\Big((k+r)^2 - \nu^2 \Big) c_k + c_{k-2} \Big] t^k \Bigg).
\end{aligned}$$

Using the Identity theorem, we conclude that

$$(r^2 - \nu^2)c_0 = 0, \quad \big((r+1)^2 - \nu^2 \big) c_1 = 0$$

and

$$\big((k+r)^2 - \nu^2 \big) c_k + c_{k-2} = 0 \quad \text{for all } k \geq 2.$$

Since by definition c_0, we obtain (a second time) the indicial equation

$$h(r) := r^2 - \nu^2 = 0$$

and the recurrence relation

$$h(k+r)c_k + c_{k-2} = 0 \quad \text{for all } k \geq 2.$$

Note that our assumption $\nu \geq 0$ implies that $c_1 = 0$. Thus we obtain the recurrence

$$c_0 \neq 0, \quad c_1 = 0$$

and

$$h(k+r)c_k + c_{k-2} = 0 \tag{2.88}$$

(c) Using the recurrence relation to determine the coefficients of the first solution y_1:

We set $r := r_1 = \nu \geq 0$ which is the “greater” of the two solutions of the indicial equation $h(r) = r^2 - \nu^2 = 0$. Therefore, $h(k+\nu) \neq 0$ for all $k > 0$, since $r_1 = \nu \geq -\nu = r_2$. Hence we can solve equation (2.88) for c_k by dividing the equation by $h(k+\nu) = k(k+2\nu)$ yielding

$$c_k = -\frac{1}{h(k+\nu)}\, c_{k-2} = -\frac{1}{k(k+2\nu)}\, c_{k-2} \quad \text{for } k = 2, 3, \ldots \tag{2.89}$$

Thus we obtain

$$\begin{aligned}
c_0 &= \text{arbitrary but} \neq 0 \\
c_1 &= 0 \\
c_2 &= -\frac{1}{2(2+2\nu)}\, c_0 \\
c_3 &= -\frac{1}{3(3+2\nu)}\, c_1 = 0 \\
c_4 &= -\frac{1}{4(4+2\nu)}\, c_2 = \frac{-1}{4(4+2\nu)} \cdot \frac{-1}{2(2+2\nu)}\, c_0 \\
c_5 &= -\frac{1}{5(5+2\nu)}\, c_3 = 0 \\
c_6 &= -\frac{1}{6(6+2\nu)}\, c_4 = \frac{-1}{6(6+2\nu)} \cdot \frac{-1}{4(4+2\nu)} \cdot \frac{-1}{2(2+2\nu)}\, c_0 \\
&= \frac{(-1)^3}{\big(3 \cdot 2 \cdot 2(3+\nu)\big)\big(2 \cdot 2 \cdot 2(2+\nu)\big)\big(1 \cdot 2 \cdot 2(1+\nu)\big)}\, c_0,
\end{aligned}$$

and finally

$$c_{2m} = \frac{(-1)^m\, c_0}{2^{2m} m! \prod_{j=1}^{m}(j+\nu)} \quad \text{for all } m \in \mathbb{N}.$$

To emphasize the fact that c_{2m} is the coefficient of $t^{2m+\nu}$ we rewrite the expression above

$$c_{2m} = \frac{(-1)^m\, (2^\nu c_0)}{2^{2m+\nu} m! \prod_{j=1}^{m}(j+\nu)}$$

which in turn can be rewitten using the Gamma function notation as

$$c_{2m} = \frac{(-1)^m\, (2^\nu c_0)\, \Gamma(\nu+1)}{2^{2m+\nu} m!\, \Gamma(\nu+1) \prod_{j=1}^{m}(j+\nu)}$$

which using the properties of the Gamma function simplifies to

$$c_{2m} = \frac{(-1)^m\, [2^\nu \Gamma(\nu+1)\, c_0]}{2^{2m+\nu}\, m!\, \Gamma(m+\nu+1)} \quad \text{for all } m \in \mathbb{N}.$$

If we chose c_0 to be

$$c_0 := \frac{1}{2^\nu \Gamma(\nu+1)},$$

the formula for the coefficients c_{2m} simplifies to

$$c_{2m} = \frac{(-1)^m}{2^{2m+\nu}\, m!\, \Gamma(m+1+\nu)} \quad \text{for all } m \in \mathbb{N}.$$

The solution

$$y_1(t) = J_\nu(t) := \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu}\, m!\, \Gamma(m+1+\nu)} t^{2m+\nu} \tag{2.90}$$

of the equation (2.87) is called a *Bessel function of the first kind and order* ν. Since $t_0 = 0$ is the only singular point of the Bessel equation, we can conclude that the radius of convergence R of this Frobenius series is $R = \infty$, i.e. it converges for any $x \in \mathbb{R}$. Recall that $R \geq \min\{R_p, R_q\}$ and $R_p = R_q = \infty$, since $tp(x) = 1$ and $t^2 q(t) = -\nu^2 + t^2$ are polynomials in t.

(d) Finding the second solution y_2:

We will now consider the case $r_2 = -\nu$, where we might encounter "overlapping" series. Clearly, $r_1 - r_2 = \nu - (-\nu) = 2\nu \in \mathbb{N}^0 := \{0, 1, 2, \ldots\}$ if and only if $\nu = k \cdot \frac{1}{2}$ for some $k \in \mathbb{N}^0$, i.e. if ν is a non-negative multiple of $\frac{1}{2}$. Summarizing, if ν is not an integer then, $h(k + (-\nu) = 0$ if and only if $0 = 0$ or

We begin with the special case $\nu \in \mathbb{R} \setminus \mathbb{Z}$, i.e. the case, where ν is not an integer. Note that this assumption includes the case $\nu = (2\ell + 1) \cdot \frac{1}{2}$ for some $\ell \in \mathbb{N}^0$, i.e. the case, where ν is an odd non-negative multiple of $\frac{1}{2}$ and where $r_1 - r_2$ differ by a positive integer (case 2 in the textbook). However, we will see that we are in the "fortunate" situation where in the recurrence equation (2.88) the term $h\big(k + (-\nu)\big) = 0$ at the same time when $c_{k-2} = 0$, namely for all odd subscripts.

We first rewrite the recurrence relation (2.88) exchanging c_k with d_k and c_{k-2} with d_{k-2} and obtain

$$d_0 \neq 0, \quad d_1 = 0$$

and

$$h(k + r)d_k + d_{k-2} = 0 \quad \text{for } k > 1. \tag{2.91}$$

Clearly, as mentioned above, $h\big(k+(-\nu)\big) = k^2 - 2k\nu + \nu^2 - \nu^2 = k(k-2\nu) = 0$ if and only if $k = 0$ or $k = 2\nu$, which implies that $2\nu \in \mathbb{N}^0$ i.e. that ν is a multiple of $\frac{1}{2}$. Since, by assumption, ν is not an integer, $\nu = (2\ell + 1) \cdot \frac{1}{2}$ for some $\ell \in \mathbb{N}^0$ and thus $k = 2\nu = 2(2\ell+1)\frac{1}{2} = 2\ell + 1$. Thus, under the assumption that ν is not an integer, $h\big(k + (-\nu)\big) \neq 0$ for all even integers $k > 1$. For odd integers $k = 2\ell + 1$, the recurrence (2.91) in conjunction with the initial value $d_1 = 0$ yields that

$$h\big(2\ell + 1) + (-\nu)\big)d_{2\ell+1} = (-1)d_{2\ell-1} = \cdots = (-1)^\ell d_1 = 0.$$

Hence, we can set $d_k = 0$ for all odd $k \in \mathbb{N}$. For even $k \in \mathbb{N}$, we can solve for d_k as in the case $r_1 = \nu$ and obtain

$$d_k = \begin{cases} -\dfrac{1}{k\big(k+2(-\nu)\big)}\, d_{k-2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

which coincides with (2.89) if we replace ν by $-\nu$. Hence, since we have the same start values as in the previous case, the second solution can be obtained from the first by replacing ν by $-\nu$ yielding

$$y_2(t) = J_{-\nu}(t) := \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m-\nu}\, m!\, \Gamma(m+1-\nu)} t^{2m-\nu}. \tag{2.92}$$

Thus

$$y(y) := c_1 J_\nu(t) + c_2 J_{-\nu}(t)$$

constitutes a complete solution of the Bessel equation of order ν.

(e) Modifications of the basis function of the complete solution of the Bessel equation:

(i) For any $\nu \in \mathbb{R} \setminus \mathbb{Z}$ (any real number that is not an integer), we can define the function

$$Y_\nu(t) := \frac{\cos(\nu\pi) J_\nu(t) - J_{-\nu}(t)}{\sin(\nu\pi)},$$

which is called the *Bessel function of the second kind of order* ν. As a linear combination of J_ν and $J_{-\nu}$, Y_ν is also a solution of the Bessel equation and

$$y(y) := c_1 J_\nu(t) + c_2 Y_\nu(t)$$

also constitutes a complete solution of the Bessel equation (2.87).

(ii) For $\nu \in \mathbb{R} \setminus \mathbb{Z}$, the complex valued functions

$$H_\nu^{(1)}(t) := J_\nu(t) + iY_\nu(t)$$

and

$$H_\nu^{(2)}(t) := J_\nu(t) - iY_\nu(t)$$

are called the *Hankel functions* or the *Bessel functions of the third kind of order* ν. Again as linear combination of solutions of the Bessel equation they are also solutions and

$$y(y) = c_1 H_\nu^{(1)}(t) + c_2 H_\nu^{(2)}(t)$$

constitutes a complete solution of the Bessel equation (2.87).

Some Basic Facts On The Gamma Function

The improper integral

$$\int_0^\infty e^{-\xi}\xi^{x-1}d\xi$$

converges for all $x > 0$ and defines a function

$$\Gamma(x) := \int_0^\infty e^{-\xi}\xi^{x-1}d\xi,$$

which is called *Euler's Gamma function* and which is positiv for all $x > 0$. Clearly,

$$\Gamma(1) = \int_0^\infty e^{-\xi}d\xi = [-e^{-\xi}]_0^\infty = 1.$$

Partial integration for $x > 0$ yields

$$\Gamma(x+1) := \int_0^\infty e^{-\xi}\xi^{x}d\xi = [-e^{-\xi}\xi^x]_0^\infty + x\int_0^\infty e^{-\xi}\xi^{x-1}d\xi,$$

which, considering that

$$[-e^{-\xi}\xi^x]_0^\infty = \lim_{\xi\to\infty}(e^{-\xi}\xi^x) - \lim_{\xi\to 0^+}(e^{-\xi}\xi^x) = 0 - 0,$$

implies that

$$\Gamma(x+1) = x\Gamma(x).$$

For natural numbers $n \in \mathbb{N}$, it can be shown using mathematical induction that

$$\Gamma(n+1) = n!$$

The Gamma function possesses derivatives of any order, which can be obtained by differentiation under the integral sign:

$$\Gamma^{(n)}(x) = \int_0^\infty \big(\ln\xi\big)^n e^{-\xi}\xi^{x-1}d\xi.$$

Thus, the Gamma function can be considered as a differential extension of the factorial.

Chapter 2 Ordinary Differential Equations Exercises:

2.1 *Basics*

1. Give an example of a differential equation (simple or sophisticated) used for modeling in engineering. Describe briefly what process is governed by this equation. What assumptions and conditions are applied?

2. Describe each of the following differential equations, giving its order and telling whether it is ordinary or partial, its order, whether it is linear or nonlinear, and homogeneous or non-homogeneous:

a) $x^2 \dfrac{dy}{dx} = \sin x$ (*example: ODE, 1st order, linear, non-homogeneous*)

b) $\ddot{x} + a\dot{x} + bx = c\sin t$

c) $(v')^2 + av = b$

d) $X'' + \lambda^2 X = 0$

e) $a^2 \dfrac{\partial^4 y}{\partial x^4} = -\dfrac{\partial^2 y}{\partial t^2}$

f) $y'' + xy = \sin y$

g) $y'' - 6y' - 4y = \sin^2 x$

h) $y''' + x^2 y = \sqrt{x}$

i) $y'' + yy' = 0$

j) $\dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial}{\partial x}\left(k(u)\dfrac{\partial u(x,t)}{\partial x}\right)$

k) $\dfrac{d^2u}{dx^2} + \left(\dfrac{du}{dx}\right)^{-1} = 0$

3. Find the values of m for which $y = x^m$ is a solution of the differential equation $3x^3 y''' - x^2 y'' + 2xy' - 2y = 0$

4. Solve the following differential equation
$y' = xe^{3x}$ subject to the initial condition $y(0) = 0$

5. Find a complete solution of the differential equation $y' = 4x^{\frac{1}{3}}$ and sketch the solution curve through the point $(2,3)$.

6. Find a complete solution of the differential equation $y' = xe^{\frac{-x^2}{2}}$ and sketch the solution curve through the point $\left(1,\dfrac{1}{\sqrt{e}}\right)$.

7. Find the integral of the differential equation $4x + 9yy' = 0$; sketch the integral curve through the point $(3,-4)$.

8. Sometimes it is necessary to find a differential equation which has a given general solution. Usually, it can be found by differentiation of the given solution and elimination of the parameters by algebraic manipulations.
Find a second order differential equation which has a general solution given by

$y = c_1 e^{-x} + c_2 \sin x$, where c_1 and c_2 are arbitrary constants.

9. Find a solution of the initial value problem $(x+1)y' = 2y$ subject to $y(-1) = 0$ and state whether or not it is unique.

10. The homogeneous spherical wall (inner radius r_1 and outer radius r_2) with thermal conductivity k is in a steady state with temperature of the inner surface T_1 and of the outer surface T_2, respectively. Assuming that due to conservation of energy, in steady state the same rate of heat transfer Q is passing through a spherical surface of any radius, find the temperature $T(r)$ at an arbitrary radius $r_1 \le r \le r_2$. Derive the differential equation using Fourier's Law and relation $Q = qA_s$, where A_s is the surface area. Sketch the graph of solution for $r_2 = 5r_1 = 2$ and $T_1 = 4T_2 = 80^o C$, $k = 0.5$, and determine the rate of heat transfer Q.

2.2.1-2 *1st order ODE*

1) Solve the following differential equations and sketch the solution curves:

a) $(x+y+1)dx + (x - y^2 + 3)dy = 0$

b) $(2xy + x^3)dx + (x^2 + y^2)dy = 0$

c) $\left(e^x + \dfrac{y}{x}\right)dx + \left(\ln x + \dfrac{1}{y}\right)dy = 0$

d) $(x + x^2 y + y^3)dx + (y + xy^2 + x^3)dy = 0$

e) $(3y - 2x)y' + 2y = 0$

f) $e^x \sin y dx - (y - 2e^x \cos y)dy = 0$

g) $xy' + y + y \ln xy = 0$

2) Prove Theorem 2.

3) a) Determine an equation for the multiplication factor if it is a function of x only.
b) Determine an equation for the multiplication factor if it is a function of y only.

2.2.3-4 *Separable*

1. Solve the differential equation and sketch the solution curves:

a) $x(1+y^2)dx - y(1+x^2)dy = 0$

b) $yy' = 2xy + 2x$

c) $2xyy' + 2xy' = y$

2. Solve the differential equation and sketch the solution curves:

a) $(2xy^3 - x^4)y' + 2x^3 y - y^4 = 0$

b) $(x+y)dx - (y-x)dy = 0$

c) $(x-y)dx + (y-x)dy = 0$

d) $2xy' = y - x$

e) $y' = \sec\left(\dfrac{y}{x}\right) + \dfrac{y}{x}$ subject to $y(2) = \pi$

3. Solve the differential equation by conversion to polar coordinates and sketch the solution curves:

a) $y' = \dfrac{x+y}{x-y}$

4. a) Show that the equation $y' = f(ax+by)$ where $a,b \in \mathbb{R}$ can be transformed to a separable differential equation with the help of a change of the dependent variable by the substitution $z = ax + by$.

b) Solve the differential equation and sketch the solution curves:

i) $y' = 2x + y$

ii) $y' = \dfrac{1}{x-y} + 1$

2.2.5 *Linear 1st order*

1. Solve the differential equation and sketch the solution curves:

a) $xy' + (1+x)y = e^{-x}$

b) $y' = y + 4e^{x}\sin^{2} x$ subject to $y(0) = 10$

c) $y' - \dfrac{y}{x} = x^{2}$

2.2.6 *Special*

1. Show that if $u(x)$ is a particular solution of the Ricatti equation

$y' = P(x)y^2 + Q(x)y + R(x)$

Then the substitution

$y = \dfrac{1}{z} + u$

reduces the Ricatti equation to a Bernulli equation

$u' = (Q + 2Pz)u + Pu^{2}$

2. Solve the differential equation and sketch the solution curves:

a) $y' + y = xy^{2}$

b) $y' = xy^{2} + (1-2x)y + x - 1$ (particular solution $u = 1$)

c) $y = xy' - e^{y'}$

2.2.7 *Applications*

1. Find a family of orthogonal curves defined by the equation:

$x^{2} + 3y^{2} = c$

and sketch the graph of curves

2.2.8 *Numerical*

A skier is skiing down a hill with a constant slope θ. The dependence of his speed v on time t is described by the equation:

$$m\frac{dv}{dt} = -\frac{\rho A C_D}{2}v^{2} + mg\sin\theta - \mu mg\cos\theta$$

with the initial condition $v(0) = v_0$.

Solve this IVP by:

a) exact methods,
b) as a special equation,
c) the Euler method,
d) the modified Euler,
e) the Runge-Kutta, and
f) Newton's method.

Use the following values of parameters:

$\rho = 1$

$C_D = 0.9$

$A = 0.4$

$m = 80$

$g = 9.82$

$\theta = 30^{o}$

$\mu = 0.1$

$v_0 = 2$ Make some observations.

2.2.9 *Reducible*

Solve by reduction of order
$xy'' - xy' + y = 0$ (hint: $y_1 = x$ is a solution)

2.3 Linear ODE

1) Solve the following equations and sketch solution curves:

a) $y'' + y = \sec x$
b) $y'' + y = 4x\cos x$
c) $y''' - y = \cos t$
d) $y'' - 5y' + 6y = 2e^{2x}$
e) $y^{(iv)} + y = 0$
f) $y^{(v)} + y = 0$
g) $y^{(vi)} + y = 0$
h) $y'' - 3y' = e^{3x}$
i) $y''' + 3y'' - 4y = e^{2x}$
k) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 20x$

2. Let y_1, y_2 be two linearly independent solutions of the normal linear ODE
$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$
Show that any solution of this equation can be written as a linear combination
$y(x) = c_1 y_1(x) + c_2 y_2(x)$

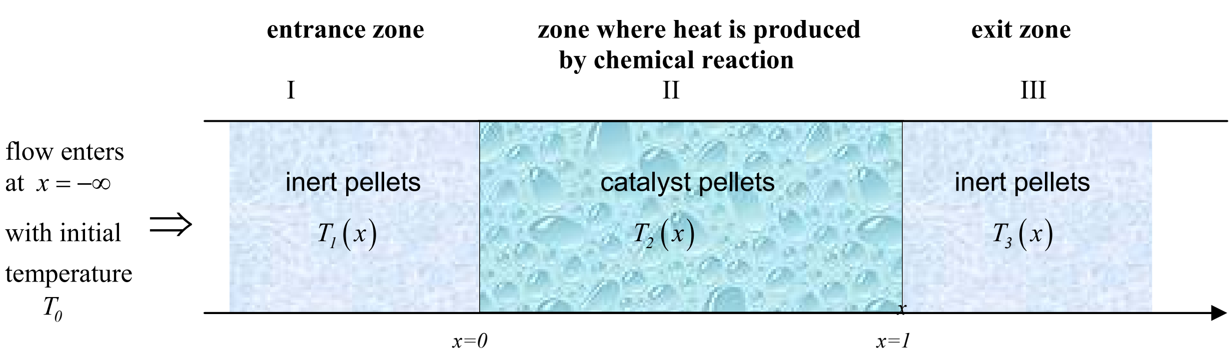
3. Show that any solution of $L_n y = f(x)$ is represented by $y = y_c + y_p$

4. Consider steady state conduction in the spherical region between two isothermal surfaces $r = r_1$ at temperature u_1 and $r = r_2$ at temperature u_2.
Radial temperature distribution under assumption of angular symmetry in the absence of volumetric heat generation is described by the differential equation
$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0$$
Find the temperature distribution $u(r)$, $r \in (r_1, r_2)$

5. *Conjugate problem for multilayer region.*
[modification from Transport Phenomena] Fixed-bed flow reactor

Assume that flow enters a three zone fixed-bed flow reactor at $x = -\infty$ at initial temperature T_0. Zones I $(-\infty < x < 0)$ and III $(1 < x < \infty)$ are filled with inert pellets, and zone II $[0 \le x \le 1]$ is filled with catalyst pellets, respectively.
Temperature distribution is described by the function $T_i(x)$ where the index i denotes the corresponding zone of reactor, $i = I, II, III$, and has to satisfy the following differential equations with corresponding boundary conditions:

Zone:	**Equation:**	**Boundary and Conjugate Condition:**
		$T_1\big\vert_{x\to-\infty} = T_0$
I $\left(-\infty < x < 0\right)$	$\dfrac{1}{b}\dfrac{d^2T_1(x)}{dx^2} - \dfrac{dT_1(x)}{dx} = 0$	
		$T_1\big\vert_{x=0} = T_2\big\vert_{x=0}$
		$\dfrac{dT_1}{dx}\bigg\vert_{x=0} = \dfrac{dT_2}{dx}\bigg\vert_{x=0}$
II $\left(0 \le x \le 1\right)$	$\dfrac{1}{b}\dfrac{d^2T_2(x)}{dx^2} - \dfrac{dT_2(x)}{dx} + aT_2(x) = 0$	
		$T_2\big\vert_{x=1} = T_3\big\vert_{x=1}$
		$\dfrac{dT_2}{dx}\bigg\vert_{x=1} = \dfrac{dT_3}{dx}\bigg\vert_{x=1}$
III $\left(1 < x < \infty\right)$	$\dfrac{1}{b}\dfrac{d^2T_3(x)}{dx^2} - \dfrac{dT_3(x)}{dx} = 0$	
		$T_3\big\vert_{x\to\infty}$ *bounded*



a and b are physical parameters of the reactor ($a > 0, b > 0$) and the fluid with the following properties:
a is bigger for a longer reactor, lower flow rate, and faster reaction;
b is bigger for a longer reactor, higher flow rate, and smaller conduction between zones.
Conditions between zones provide continuity of temperature and heat flux.

Find the temperature distribution $T_i(x)$ for all zones and investigate the influence of the parameters a and b .
Sketch the graph.

6. Solve problem in the Example 6 of section 2.3.6.

2.4-5 Power Series Solution

1. Evaluate $\sum_{k=0}^{n}\binom{2n}{2k}-\sum_{k=0}^{n-1}\binom{2n}{2k+1}$

2. Using power series and Cauchy product , verify the "First Pythagorean Identity" $sin^2 x+cos^2 x=1$ (Hint: use the identity established in the previous exercise).

3. Give the power series expansions of $\frac{x}{x-1}$ and $\frac{1}{x-1}$ in x and determine their radius of convergence.

4. Find the radius of convergence of the power series:

a) $\sum_{n=0}^{\infty}\frac{x^n}{(n+3)^2}$

b) $\sum_{n=0}^{\infty}\frac{n^2}{2^n}(x-1)^n$

c) $\sum_{n=0}^{\infty}\frac{n!\,x^n}{n^n}$

5. Find the singular points of the equations

a) $y''+2xy'+y=0$

b) $2x^2y''-3\lambda^2y=0$

c) $(1-x^2)y''+y'-y=0$

d) $x^2y''-y'+xy=0$

6. Show that $\frac{2(x-1)}{x}$ is analytic at $x_0=1$ by computing its Taylor series expansion in terms of $x-1$.

7. Show that $\frac{3x}{(x-1)^3}$ is analytic at $x_0=0$ by computing its Taylor series expansion in x.

8. Using the power series method, find complete solutions for the following differential equations:

a) $y'-\lambda y=0$

b) $y''-\lambda^2y=0$

c) $y''+xy'+2y=0$

d) $y''-x^3y=0$

e) $y''-xy=0 \quad y(0)=1; y'(0)=0$

f) $y''+\frac{3}{x}y'-2y=0$

g) $2x^2y''-x(x-1)y'-y=0$

h) $xy''+y'-xy=0$

i) $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$

k) $2x^2y''+(2x^2+x)y'-y=0$

9. Determine the second solution for Example 73 (p. 99).

10. Consider the differential equation $(x-1)y''-xy'+y=0$

a) find the general solution of the given ODE in the form of power series about the point $x_0=0$;

b) What is the radius of convergence of the obtained power series solution?

c) Sketch the solution curves.

d) Find the solution subject to the initial conditions: $y(0)=-2; y'(0)=6$.