2.6 SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Systems of Linear 1st Order ODEs

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2.6 SYSTEMS OF ODEs

Systems of Linear 1st order ODEs

In this section we will study the theory of the ***systems of linear 1st order ODEs***. It can be shown that such systems are equivalent to a single linear differential equation of a higher order; and for both of them the most theoretical results have a similar description. Although, we will try to avoid duplication of the theoretical justification, in a practical approach, the methods of solution for a single equation and for the systems are different. In many cases, the description of the physical model is more natural to perform with the systems of ODEs, and investigation of the physical models such as dynamic, stability etc. is more visual when it is made with the help of systems.

1. DEFINITIONS AND NOTATIONS:

Normal form

Consider a system of n linear first order ODE written in the normal form (solved for the derivatives of unknown functions):

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned} \tag{1}$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are unknown functions to be determined from the system (1), and coefficients $a_{ij}(t)$, $i, j = 1,2,...$ and functions $f_i(t)$ are continuous functions in $D \subset R$.

Matrix form

Introduce the following column vectors and a matrix with entries which are the elements of the system (1):

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \mathbf{x}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

Then the system (1) can be written in the compact matrix form:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{2}$$

This is a *non-homogeneous* system.

Without a free term $\mathbf{f}$, the system is *homogeneous:*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{3}$$

Solution vector

The solution vector (particular solution) is any column vector $\mathbf{x}(t)$, $t \in D$ which satisfies the system (2) or (3).

Initial value problem

Find the solution of the system of ODEs subject to initial conditions:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f} \quad \text{subject to } \mathbf{x}(t_0) = \mathbf{x}_0 \qquad \text{where } \mathbf{x}_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}, \; x_i^0 \in \mathbb{R} \tag{4}$$

The initial value problem (4) for normal system has exactly one solution (unique).

Existence Theorem

Theorem 1 *(existence theorem)*

Let the system of linear 1st order ODEs

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

be *normal* for $t \in D \subset \mathbb{R}$ (means that the functions $a_{ij}(t)$ and $f_i(t)$ are continuous on $D \subset \mathbb{R}$) and let $x_i^0 \in \mathbb{R}$.

Then there exist exactly one solution $\mathbf{x}(t)$ such that

$$\mathbf{x}(t_0) = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix} \qquad t_0 \in D$$

The particular solution $\mathbf{x}(t)$ of the system (2) or (3) is a vector valued function

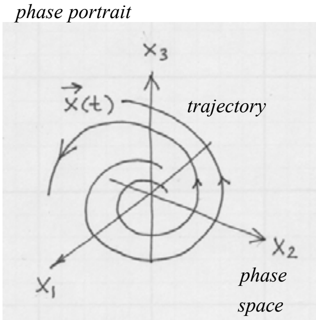
$$\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^n$$

which can be treated as a parametric definition of the curve in the space $\mathbb{R}^n$:

$$\mathbf{x}(t) = \begin{cases} x_1 = x_1(t) \\ x_2 = x_2(t) \\ \vdots \\ x_n = x_n(t) \end{cases} \qquad t \in D \tag{5}$$

with the coordinates $x_1, x_2, ..., x_n$.

phase portrait



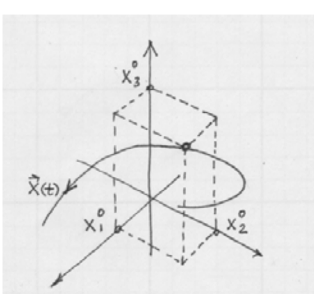
This space is called a ***phase space***, and the solution curve defined parametrically by the equation (5) is called the ***trajectory*** or the ***orbit*** in the phase space. For *2-D* and *3-D* cases, the traditional notations for the coordinate system and, correspondingly, for unknown functions in the system are used:

$$\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y + a_{13}(t)z + f_1(t) \\ y' &= a_{21}(t)x + a_{22}(t)y + a_{23}(t)z + f_2(t) \\ z' &= a_{31}(t)x + a_{32}(t)y + a_{33}(t)z + f_3(t) \end{aligned} \tag{1b}$$

with the particular solutions written as

$$\mathbf{x}(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in D \tag{5b}$$

The graph of the equation (5b) defines a trajectory in the phase space (phase plane, for *2-D* case). The independent variable $t \in D$ can be treated as the time (can also be negative), and for any moment of time equation (5b) defines the position of a point on the trajectory – therefore, the parametric equation (5b) can be interpreted as a motion of material point along the trajectories defined by the linear system of ODEs. The arrows on the trajectories indicate the direction of motion with the increase of time *t*. The family of all trajectories of the linear system is called the ***phase portrait***.



An ***initial value problem*** defines the trajectory which goes through the prescribed point. According to the Existence Theorem, the solution of any initial value problem of the linear system is unique – it means that there is only one trajectory which goes through any point of the phase space, and that the trajectories of the linear system do not intersect.

2. THEORY OF LINEAR SYSTEMS OF ODEs

It can be shown that a linear n^{th} order ODE can be transformed to a system of n linear 1^{st} order ODEs, and a system of n linear 1^{st} order ODEs can be reduced to a linear n^{th} order ODE. Therefore, the description and the properties of the general solution of the system will be similar to those of the general solution of a linear n^{th} order ODE. Here, we will list the major results of the theory of the systems of linear 1^{st} order ODEs.

I) Homogeneous System:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{3}$$

linear independent solutions

Vectors (vector functions) $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ are *linearly independent* if their linear combination equals zero for all $t \in D$

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + ... + c_n\mathbf{x}_n(t) = 0 \tag{6}$$

only if all coefficients are zero $c_1 = c_2 = ... = c_n = 0$.

If vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ are not linearly independent, then they are *linearly dependent.* It means that in the linear combination at least one coefficient $c_k \neq 0$ can be non-zero.

Wronskian

The Wronskian of the set of solution vectors of the homogeneous system $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ is defined as a determinant of the matrix which columns are the vectors $\mathbf{x}_k(t)$:

$$W(t) = det\left[\mathbf{x}_1(t) \;\; \mathbf{x}_2(t) \;\; ... \;\; \mathbf{x}_n(t)\right] \tag{7}$$

There is a relationship of the Wronskian (7) to the Wronskian defined in 5.3.

If $W(t) \neq 0$ at least at one point $t \in D$, then the solution vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ are linearly independent.

There always exist n linear independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ of the homogeneous system (3). But any $n+1$ solutions of the homogeneous system (3) are linearly dependent.

Fundamental set

Any set of n linearly independent solutions of the system (3) $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)$ is called a ***fundamental set***.

It is obvious that a homogeneous systems always possesses a zero solution $\mathbf{x}(t) \equiv \mathbf{0}$ (***trivial solution***). But any set which includes the zero vector is linearly dependent. Therefore, the fundamental set cannot include the trivial solution.

Fundamental matrix

A matrix with the columns which are constructed from the fundamental set is said to be the ***fundamental matrix***:

$$\mathbf{X}(t) = \left[\mathbf{x}_1(t) \;\; \mathbf{x}_2(t) \;\; ... \;\; \mathbf{x}_n(t)\right] \tag{8}$$

General Solution

Any solution of the homogeneous system (3) can be written as a linear combination of the vectors from the fundamental set. Therefore, the ***general solution (complete solution, complimentary solution)*** of the homogeneous system is a set of all its solutions and it is given by all linear combinations of the vectors from the fundamental set (span of the fundamental set) and it can be defined as:

$$\mathbf{x}_c(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) = \mathbf{Xc} \tag{9}$$

where $\mathbf{c}$ is a vector of arbitrary constants. Equation (9) defines a family of curves in the phase space which represents the solutions of the homogeneous system.

Solution of IVP

The solution of the initial value problem for a homogeneous system:

$$\mathbf{x}' = \mathbf{Ax} \qquad \text{subject to } \mathbf{x}(t_0) = \mathbf{x}_0 \tag{10}$$

is given by

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 \tag{11}$$

where $\mathbf{X}(t)$ is the fundamental matrix and

$\mathbf{X}^{-1}(t_0)$ is the inverse of the fundamental matrix at $t = t_0$

<u>II) Non-Homogeneous System:</u>

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{f} \tag{2}$$

Denote by $\mathbf{x}_p(t)$ any particular solution of the system (2). A particular solution can be found by the method of undetermined coefficients (similar to the case of linear ODE) or by the method of variation of parameter:

$$\mathbf{x}_p(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\,dt \tag{12}$$

The general solution of the non-homogeneous system (2) is given by a sum of the general solution of the homogeneous system (complementary solution) and a particular solution:

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$$

General Solution

Using equations (9) and (12), the general solution of the non-homogeneous system can be written as:

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\,dt \tag{13a}$$

Solution of IVP

The solution of the IVP for the non-homogeneous equation (4) can be given by:

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds \tag{13b}$$

In a conclusion, the general solution and the solution of IVP for homogeneous and non-homogeneous systems (9), (11), (12) and (13) can be determined if the fundamental matrix of the homogeneous system $\mathbf{X}(t)$ is known. In the next section we will consider the case of linear systems with constant coefficients (autonomous systems) for which there exist the developed methods of finding the fundamental matrix.

3. FUNDAMENTAL SET OF LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Consider the homogeneous system of linear 1st order ODEs

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{14}$$

where all coefficients $a_{ij} \in \mathbb{R}$, $i, j = 1,2,...$ are constants.

In the matrix form:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{15}$$

The Eigenvalue Problem, matrix diagonalization, matrix exponential are among the techniques used for construction of the fundamental set for the system with constant coefficients.

Eigenvalue Problem:

Because the linear system is of the first order, we look for the non-trivial solution of the exponential form

$$\mathbf{x}(t) = \mathbf{k}e^{\lambda t}, \qquad \mathbf{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \neq \mathbf{0} \tag{16}$$

$\mathbf{k}$ is a non-zero (non trivial) vector of constants, where k_i and λ can be the real or the complex numbers which have to be found from satisfying the equation (15). Substitute the trial form (16) into matrix equation (15):

$$\begin{aligned} &\left(\mathbf{k}e^{\lambda t}\right)' = \mathbf{A}\left(\mathbf{k}e^{\lambda t}\right) \\ &\lambda\mathbf{k}e^{\lambda t} = \mathbf{A}\mathbf{k}e^{\lambda t} \\ &\lambda\mathbf{k}e^{\lambda t} - \mathbf{A}\mathbf{k}e^{\lambda t} = \mathbf{0} \\ &\lambda\mathbf{I}\mathbf{k}e^{\lambda t} - \mathbf{A}\mathbf{k}e^{\lambda t} = \mathbf{0} && \mathbf{I} \text{ is the identity matrix} \\ &\left(\lambda\mathbf{I} - \mathbf{A}\right)\mathbf{k}e^{\lambda t} = \mathbf{0} && \text{can be divided by } e^{\lambda t} > 0 \\ &\left(\lambda\mathbf{I} - \mathbf{A}\right)\mathbf{k} = \mathbf{0} \\ &\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{k} = \mathbf{0} \end{aligned} \tag{17}$$

This is the homogeneous system of algebraic equations, which according to the Theorem has the non-trivial solution if the determinant of the matrix of coefficients is equal to zero. Therefore, the following condition should be satisfied:

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0 \tag{18}$$

Equation (18) is the n^{th} order algebraic equation for constant λ :

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right)$$

$$= det\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$= c_n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$

Expansion of the determinant yields an algebraic equation with real coefficients which is called the ***characteristic equation***. According to the Fundamental Theorem of Algebra it has n roots $\lambda_1, \lambda_2, ..., \lambda_n$ which can be real or complex, distinct or repeated. These roots are called the ***eigenvalues***. After the eigenvalues are determined, they can be substituted into equation (17) and the corresponding non-zero solutions $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n$ of the vector equation can be found. These solutions $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n$ are called the ***eigenvectors*** corresponding to eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. The solution of the eigenvalue problem (17) is not unique; they can be chosen in such a way that the desired solutions (16) have only the real-valued components. Then the constructed fundamental matrix also will have only real-valued entries. Let us show how it can be made:

Fundamental Matrix:

Case 1: All eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ are real and distinct. Then the corresponding eigenvectors vectors $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n$ are also real-valued and linearly independent. Therefore, the fundamental matrix can be defined as

$$\mathbf{X}(t) = \left[\mathbf{k}_1 e^{\lambda_1 t} \quad \mathbf{k}_2 e^{\lambda_2 t} \ \ldots \ \mathbf{k}_n e^{\lambda_n t} \right] \tag{19}$$

Exercise: show that the Wronskian is not equal to zero for any $t \in \mathbb{R}$.
In general, for real distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ the corresponding entries of the fundamental matrix are

$$\mathbf{k}_1 e^{\lambda_1 t} \quad \mathbf{k}_2 e^{\lambda_2 t} \ \ldots \ \mathbf{k}_m e^{\lambda_m t} \tag{20}$$

Case 2: Let eigenvalue λ be the repeated real root of the characteristic equation (18) of multiplicity m. Then if:

a) there are m linearly independent eigenvectors $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_m$ corresponding to the eigenvalue λ. Then the fundamental matrix includes

$$\mathbf{k}_1 e^{\lambda_1 t} \quad \mathbf{k}_2 e^{\lambda_2 t} \ \ldots \ \mathbf{k}_m e^{\lambda_m t}$$

b) there is only one linearly independent eigenvector $\mathbf{k}$ corresponding to the eigenvalue λ. Then the other linearly independednt solutions can be constructed in the following way: find the vectors $\mathbf{k}, \mathbf{p}, \mathbf{q}, ...$ which are solution of the vector equations

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{k} = \mathbf{0}$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{p} = \mathbf{k}$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{q} = \mathbf{p}$$
$$\ldots$$

Then the fundamental matrix will include the vectors

$$\mathbf{k} e^{\lambda t}, \ \mathbf{k} t e^{\lambda t} + \mathbf{p} e^{\lambda t}, \ \mathbf{k} t^2 e^{\lambda t} + \mathbf{p} t e^{\lambda t} + \mathbf{q} e^{\lambda t}, \ \ldots \tag{21}$$

It can be verified with the help of the Wronskian that these vectors are linearly indenedent.

Case 3: The eigenvalue λ is complex. We know that the complex roots of the algebraic equation with real coefficients appear in conjugate pairs:

$\lambda_{1,2} = \alpha \pm \beta i$

The corresponding eigenvectors are also a complex conjugates $\mathbf{k}_1$ and $\overline{\mathbf{k}}_1$. Therefore for distinct eigenvalues $\lambda_{1,2} = \alpha \pm \beta i$ there are two linearly independent solutions $\mathbf{k}_1 e^{(\alpha + \beta i)t}$ and $\overline{\mathbf{k}}_1 e^{(\alpha - \beta i)t}$. But they are complex-valued functions which is not convenient for representation of the physical problems. It can be shown that the linear combinations of these two solutions and application of the Euler formula ($e^{a+bi} = (\cos b + i \sin b) e^{a}$) yield the two independent real-valued solutions which can be included in the fundamental matrix:

$$\mathbf{x}_1 = (\mathbf{b}_1 \cos \beta t - \mathbf{b}_2 \sin \beta t) e^{\alpha t}$$
$$\mathbf{x}_2 = (\mathbf{b}_2 \cos \beta t + \mathbf{b}_1 \sin \beta t) e^{\alpha t} \qquad (22)$$

where vectors are $\mathbf{b}_1 = Re(\mathbf{k}_1)$ and $\mathbf{b}_2 = Im(\mathbf{k}_1)$.

Conclusion: The solution of the eigenvalue problem for the homogeneous linear system of 1st order ODEs with constant coefficients yields the fundamental matrix $\mathbf{X}$.

Matrix exponential:

The system of 1st order ODEs in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$ resembles a 1st order ODE for which it is very tempting to write the solution in the traditional exponential form $e^{\mathbf{A}t}$. But how can the exponential function with the matrix be calculated? Again, we can use the analogy with the calculus of functions of a single variable and define the matrix-valued exponential function in the form of the Taylor series:

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(t\mathbf{A})^k}{k!} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + ... \qquad (23)$$

in which we know how to calculate the powers of the matrix $\mathbf{A}^k = \overbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}^{k\ times}$ (it can be shown that the infinite series (23) is always convergent for any t).

Then the fundamental matrix of the linear system (15) can be written as

$$\mathbf{X} = e^{t\mathbf{A}} \qquad (23)$$

Then the general solution in matrix exponential form is

$$\mathbf{x} = e^{t\mathbf{A}}\mathbf{c} + e^{t\mathbf{A}} \int e^{-t\mathbf{A}} \mathbf{f}(t)\, dt \qquad (24)$$

The solution of the IVP can be defined by

$$\mathbf{x} = e^{(t-t_0)\mathbf{A}}\mathbf{x}_0 + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-s\mathbf{A}} \mathbf{f}(s)\, ds \qquad (25)$$

The matrix exponential form of the fundamental matrix is not used very often for the actual solution of the linear system of ODEs. But it is very convenient for derivation and proofs of the theoretical results such as existence theorems etc.

4. AUTONOMOUS SYSTEMS:

A system of 1st order ODEs is called ***autonomous*** if it can be written in the form:

$$\begin{aligned} x_1' &= f_1(x_1, x_2, \ldots x_n) \\ x_2' &= f_2(x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= f_n(x_1, x_2, \ldots x_n) \end{aligned} \quad (26a)$$

and in a matrix form:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \quad (26b$$

The unknown functions $x_1(t), x_2(t), \ldots x_n(t)$ are functions of *t*, but the independent variable *t* does not appear ***explicitly*** in the right hand side of the system (26). Autonomous systems are not necessarily linear.

plane linear autonomous systems

Here, we will consider only plane linear autonomous systems, which for simplicity can be written as

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad a,b,c,d \in \mathbb{R} \quad -\infty < t < \infty \quad (27a)$$

and in the matrix form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \quad (27b)$$

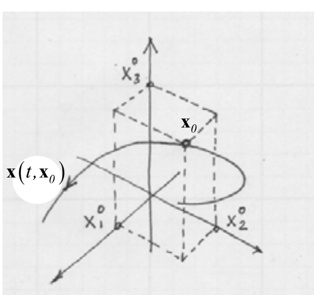
The particular solution of the plane system is a 2-dimensional vector which parametrically describes a trajectory (orbit) on the phase plane:

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \qquad -\infty < t < \infty \quad (28)$$

The general solution also includes an arbitrary constant vector

$$\mathbf{x}(t, \mathbf{c}) = \begin{bmatrix} x(t, c_1) \\ y(t, c_2) \end{bmatrix} \quad (29)$$

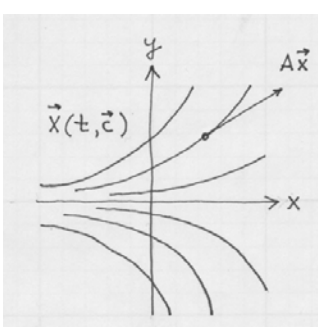
It defines the family of trajectories in the phase plane (phase portrait) and describes the motion of the points along the solution curves with the change of time t. The arrows on the trajectories indicate the direction of the motion of the point with the increase of time. This mapping defined by the equation (29) is called a ***dynamical system***.



The initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution

$$\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0) = \begin{bmatrix} x(t, x_0) \\ y(t, y_0) \end{bmatrix}$$

the trajectory which goes through the prescribed point $\mathbf{x}_0$.



The right hand side of the vector equation (27b) defines the vector field in the phase plane . At any point on the plane $\mathbf{x}$ we can draw a vector $\mathbf{A}\mathbf{x}$ and obtain a geometrical representation of the directional field. These directional vectors are tangent to the trajectories defined by the system (27b). The directional vector field can be drawn even without solving the system (27), but it provides a qualitative picture of the dynamical system.

Critical Points:

The important characteristics of the autonomous systems are the ***critical*** (equilibrium, stationary) points, which can be defined as the solutions not changing in time (constant solutions): if a point is placed at the equilibrium point it will remain there forever. The critical points can be defined as the solutions of the equation:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

For the plane system, critical points are the solutions of the system of equations:

$$ax + by = 0$$

$$cx + dy = 0$$

If the determinant

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0$$

there is only one critical point

$$\mathbf{x}_{cr} = (0,0)$$

(isolated critical point).
If the determinant

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 0 \quad \Rightarrow \quad ad = bc$$

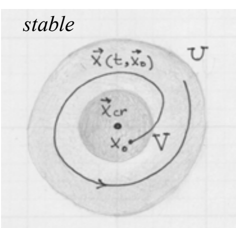
then there are infinitely many critical points which are located on the line

$$y = -\frac{a}{b}x$$

These critical points are not isolated.
For a non-linear dynamical system, the situation with the critical points is more complicated.

Stability of critical points:

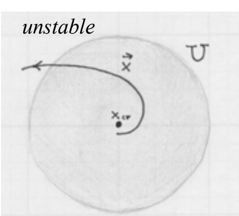
Let $\mathbf{x}_{cr}$ be the isolated critical point of the autonomous system (27): $\mathbf{f}(\mathbf{x}_{cr}) = \mathbf{0}$



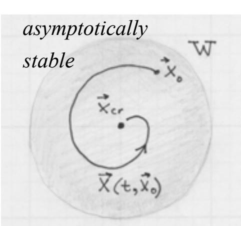
1. $\mathbf{x}_{cr}$ is said to be ***stable*** if for any neighborhood $U(\mathbf{x}_{cr})$ there exists a smaller neighborhood $V(\mathbf{x}_{cr})$ such that for any $\mathbf{x}_0 \in V(\mathbf{x}_{cr})$ the trajectory

$$\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0) = \begin{bmatrix} x(t, x_0) \\ y(t, y_0) \end{bmatrix} \in U(\mathbf{x}_{cr}) \text{ for all } t \geq 0$$

It means that the trajectory which starts in V remains completely in U.



2. $\mathbf{x}_{cr}$ is said to be ***unstable*** if it is not stable. It means that it does not matter how close to the critical point the starting point $\mathbf{x}_0$ will be, some trajectory will go outside of any neighborhood $U(\mathbf{x}_{cr})$ of the critical point.



3. $\mathbf{x}_{cr}$ is said to be ***asymptotically stable*** if
   a) $\mathbf{x}_{cr}$ is stable;
   b) there exists a neighborhood $W(\mathbf{x}_{cr})$ such that

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathbf{x}_{cr}$$

Here, we will investigate the stability of the plane dynamical systems (27) which can have only one isolated critical point $\mathbf{x}_{cr} = (0,0)$ depending on the matrix of coefficients $\mathbf{A}$.

Phase Portrait of the Plane System:

$$x' = ax + by \qquad y' = cx + dy \qquad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det\,\mathbf{A} = det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Eigenvalue Problem: The characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - Tr\mathbf{A}\lambda + det\,\mathbf{A} = 0$$

Eigenvalues: $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

The form of eigenvalues depends on the expression under the square roo which is called the discriminant:

Discriminant: $\Delta = (a+d)^2 - 4(ad-bc) = (Tr\mathbf{A})^2 - 4\,det\,\mathbf{A}$

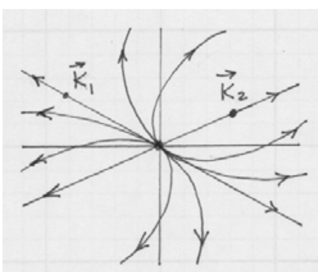
I) If $\Delta > 0$ then the eigenvalues are *real and distinct* $\lambda_1 \neq \lambda_2$

II) If $\Delta = 0$ then the eigenvalues are *rea land repeated* $\lambda_1 = \lambda_2 = \lambda$

III) If $\Delta < 0$ then the eigenvalues are *complex conjugates* $\lambda_{1,2} = \alpha \pm \beta i$

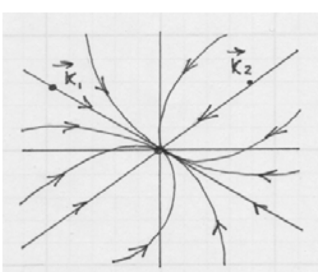
Consider the possible configurations of the plane phase portrait (for simplicity of presentation, the details of solution will be skipped; derivation of some of the results will be conducted in the examples and the exercises) :

I) $\Delta > 0$ General solution: $\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t}$



a) $\lambda_1 > 0,\ \lambda_2 > 0$ Both eigenvalues are positive

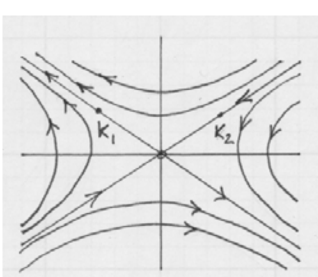
Unstable node



b) $\lambda_1 < 0,\ \lambda_2 < 0$ Both eigenvalues are negative

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty}\left[c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t}\right] = c_1\mathbf{k}_1 \lim_{t\to\infty} e^{\lambda_1 t} + c_2\mathbf{k}_2 \lim_{t\to\infty} e^{\lambda_2 t} = \mathbf{0}$$

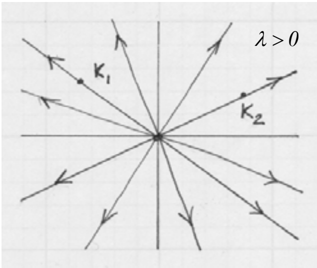
Stable node



c) $\lambda_1 > 0,\ \lambda_2 < 0$ The eigenvalues are of the opposite sign

Saddle point **(unstable)**

II) $\Delta = 0$ $\lambda_1 = \lambda_2 = \lambda$

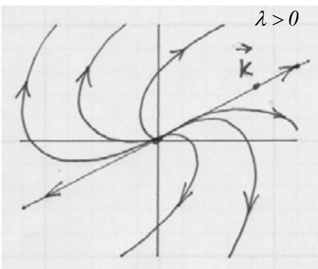


a) There are two linearly independent eigenvectors $\mathbf{k}_1$, $\mathbf{k}_2$

General solution: $\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda t} + c_2\mathbf{k}_2 e^{\lambda t} = (c_1\mathbf{k}_1 + c_2\mathbf{k}_2)e^{\lambda t}$

i) $\lambda > 0$ ***degenerate (proper) unstable node***

ii) $\lambda < 0$ ***degenerate (proper) stable node***



b) There is one linearly independent eigenvector $\mathbf{k}$ (find $\mathbf{p}$, $\mathbf{q}$, ...).

General solution:

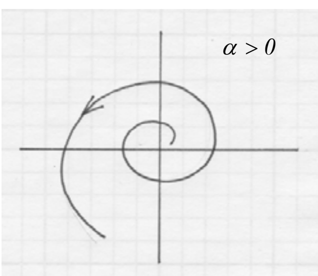
$$\mathbf{x}(t) = c_1\mathbf{k}e^{\lambda t} + c_2(\mathbf{k}t + \mathbf{p})e^{\lambda t} = \left(\frac{c_1}{t}\mathbf{k} + \frac{c_2}{t}\mathbf{p} + c_2\mathbf{k}\right)te^{\lambda t}$$

i) $\lambda > 0$ ***degenerate (improper) unstable node***

ii) $\lambda < 0$ ***degenerate (improper) stable node***

III) $\Delta < 0$ $\lambda_{1,2} = \alpha \pm \beta i$ eigenvectors $\mathbf{k}_1 = \mathbf{b}_1 + i\mathbf{b}_2$, $\mathbf{k}_2 = \mathbf{b}_1 - i\mathbf{b}_2$

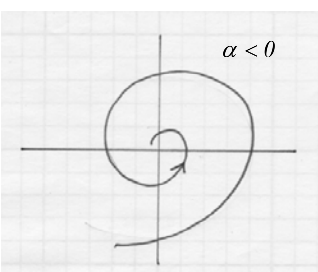
a) $\alpha \neq 0$



General Solution:

$$x = \left[c_1(\mathbf{b}_1 \cos\beta t - \mathbf{b}_2 \sin\beta t) + c_2(\mathbf{b}_1 \cos\beta t + \mathbf{b}_2 \sin\beta t)\right]e^{\alpha t}$$

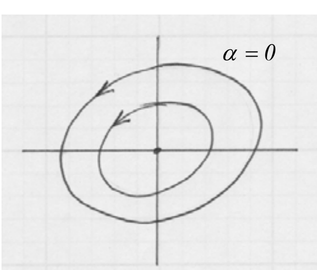
i) $\alpha > 0$ ***unstable focus (spiral point)***



ii) $\alpha < 0$ ***stable focus (spiral point)***

asymptotically stable

b) $\alpha = 0$**,** $\lambda_{1,2} = \pm\beta i$ (pure imaginary, when $a = -d$)

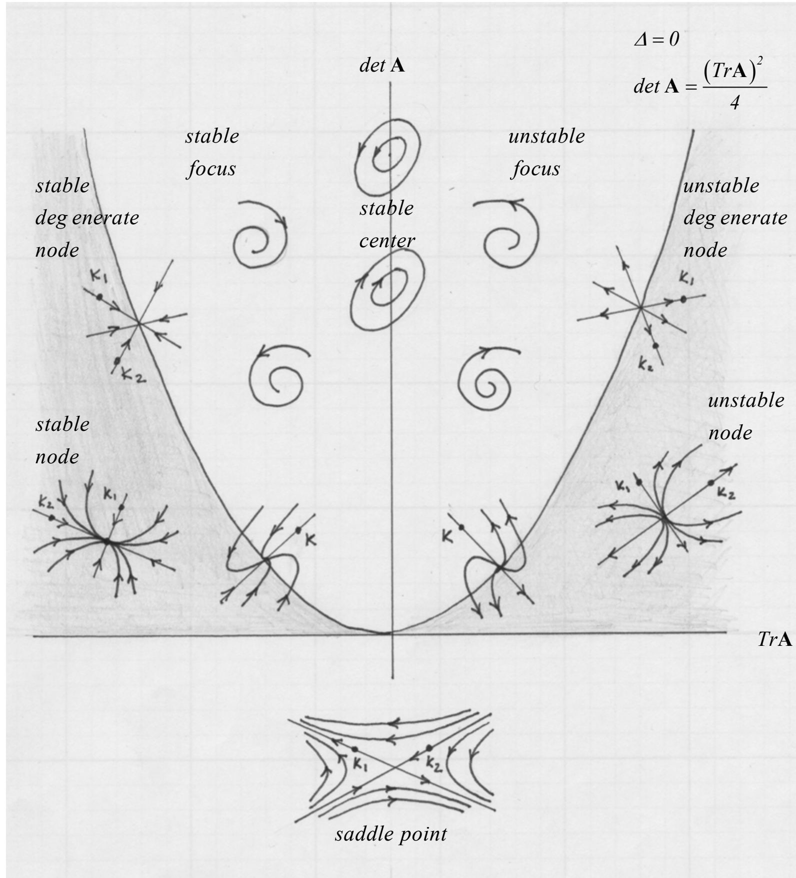


General Solution:

$$x = c_1(\mathbf{b}_1 \cos\beta t - \mathbf{b}_2 \sin\beta t) + c_2(\mathbf{b}_1 \cos\beta t + \mathbf{b}_2 \sin\beta t)$$

stable center (not asymptotically stable)

Classification of the critical points of the plane linear system



Procedure for Solution of the Linear System of 1st Order ODEs with Constant Coefficients:

1. Write the system in the normal matrix form (2):
$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

2. Solve the Eigenvalue Problem (17):
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{k} = \mathbf{0}$$
to find eigenvalues λ_i and eigenvectors $\mathbf{k}_i$.

3. Construct the fundamental matrix (8)
$$\mathbf{X}(t)$$
in correspondence with equations (19-22).

4. Calculate the general solution according to the variation of parameter formula (13a):
$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)\,dt$$

5. For solution of the IVP (4) with $\mathbf{x}(t_0) = \mathbf{x}_0$, use the variation of parameter formula (13b):
$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds$$

5. EXAMPLES:

1) *(reduction of the system of 1st order ODEs to a higher order ODE)*

Consider the system of two 1st order ODEs:

$$x' = 3x - 2y$$
$$y' = 2x - y$$

Reduction is performed by the differentiation of the equations and consecutive replacement of the unknown functions until a differential equation for a single unknown function is obtained.

Consider the second equation:

$$y' = 2x - y \qquad \Rightarrow \quad x = \frac{1}{2}y' + \frac{1}{2}y \qquad (☼)$$

Differentiate the second equation with respect to t

$$y'' = 2x' - y' \qquad \Rightarrow \quad x' = \frac{1}{2}y'' + \frac{1}{2}y'$$

Substitute expressions for x and x' into the first equation

$$x' = 3x - 2y$$
$$\left(\frac{1}{2}y'' + \frac{1}{2}y'\right) = 3\left(\frac{1}{2}y' + \frac{1}{2}y\right) - 2y$$

Rearrange it to the equation for the function y

$$y'' - 2y' + y = 0$$

This is a single 2nd order ODE, linear homogeneous with constant coefficients, which can be solved by the standard method:

auxiliary equation $\quad m^2 - 2m + 1 = 0 \quad \Rightarrow \quad m_{1,2} = 1$

general solution $\quad y = c_1 e^t + c_2 t e^t \qquad (♂)$

The function $x(t)$ can be found from equation (☼):

$$x = \frac{1}{2}y' + \frac{1}{2}y \quad = \frac{1}{2}\left(c_1 e^t + c_2 t e^t\right) + \frac{1}{2}\left(c_1 e^t + c_2 t e^t\right)$$
$$= \frac{1}{2}\left(c_1 e^t + c_2 t e^t\right)' + \frac{1}{2}\left(c_1 e^t + c_2 t e^t\right)$$
$$= \left[c_1 + c_2\left(\frac{1}{2} + t\right)\right]e^t$$

Therefore, the general solution of the system of equations is:

$$y(t) = c_1 e^t + c_2 t e^t$$
$$x(t) = \left[c_1 + c_2\left(\frac{1}{2} + t\right)\right]e^t$$

That can be verified by direct substitution into the original system of equations.

2) *(reduction of a higher order ODE to a system of 1^{st} order ODEs)*

Consider a normal n^{th} order linear ODE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = f(x) \quad a_0(x) \neq 0 \text{ for all } x \qquad (\Diamond)$$

Divide the equation by $a_0(x) \neq 0$:

$$y^{(n)} = -\frac{a_1(x)}{a_0(x)}y^{(n-1)} - \ldots - \frac{a_n(x)}{a_0(x)}y + \frac{f(x)}{a_0(x)} \qquad (\Diamond\Diamond)$$

Introduce the set of new functions:

$$x_1 = y$$
$$x_2 = y' \quad \Rightarrow \quad x_1' = x_2$$
$$x_3 = y'' \quad \Rightarrow \quad x_2' = x_3$$
$$\ldots$$
$$x_n = y^{(n-1)} \quad \Rightarrow \quad x_{n-1}' = x_n$$

Differentiate x_n

$$x_n' = y^{(n)}$$

With these notations, the equation ($\Diamond\Diamond$) can be rewritten as

$$x_n' = -\frac{a_1(x)}{a_0(x)}x_n - \ldots - \frac{a_n(x)}{a_0(x)}x_1 + \frac{f(x)}{a_0(x)}$$

Collect these in the normal linear system of 1st order ODEs:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\ldots \\ x_{n-1}' &= x_n \\ x_n' &= -\frac{a_1(x)}{a_0(x)}x_n - \ldots - \frac{a_n(x)}{a_0(x)}x_1 + \frac{f(x)}{a_0(x)} \end{aligned} \qquad (\Diamond\Diamond\Diamond)$$

3) Solve the 2nd order ODE

$$y'' - 2y' + y = 0$$

by reduction to a system of 1st order ODEs. Applying ($\Diamond\Diamond\Diamond$) for the 2nd order equation, we obtain

$$x_1' = x_2$$
$$x_2' = -\frac{a_1(x)}{a_0(x)}x_2 - \frac{a_2(x)}{a_0(x)}x_1 + \frac{f(x)}{a_0(x)} = 2x_2 - x_1$$

(Surprisingly, this system is not identical to the system of Example 1)
In matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the fundamental set for a system with constant coefficients.
Characteristic equation (18):

$$det\begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = 0$$

There is only one eigenvalue $\lambda = 1$ of multiplicity 2.
Find eigenvectors by plugging $\lambda = 1$ in vector equation (17):

$$\begin{bmatrix} -1 & 1 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is a singular linear system of algebraic equations, it has only one independent solution:

$$\mathbf{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the vector $\mathbf{p}$ by solution of the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = \mathbf{k}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then p_1, p_2 can be found from the equation

$$-p_1 + p_2 = 1$$

One of the solutions can be

$$\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the fundamental matrix is:

$$\mathbf{X} = \begin{bmatrix} \mathbf{k}e^{\lambda t} & \mathbf{k}te^{\lambda t} + \mathbf{p}e^{\lambda t} \end{bmatrix}$$

$$\mathbf{X} = e^t \begin{bmatrix} \mathbf{k} & \mathbf{k}t + \mathbf{p} \end{bmatrix}$$

$$\mathbf{X} = e^t \begin{bmatrix} 1 & t \\ 1 & t+1 \end{bmatrix}$$

Then the general solution of the system is given by

$$\mathbf{x} = \mathbf{Xc}$$

or in the component form

$$x_1(t) = c_1e^t + c_2te^t$$

$$x_2(t) = c_1e^t + c_2(t+1)e^t$$

This is the solution of the system of ODEs to which the ODE was reduced.

Recall now that $x_1(t)$ was defined as $x_1 = y$, therefore, the general solution is

$$y(t) = x_1(t) = c_1e^t + c_2te^t$$

Which coincide with the previously obtained general solution (♂).

The second solution can be treated as

$$y' = x_2(t) = c_1e^t + c_2(t+1)e^t$$

Integration of this equation will duplicate the previous result.

4a) (*Linear System of 2 equations – General Solution*)

Find the general solution of the system of ODEs:

$$x_1' = x_1 + 2x_2 + 12e^{3t}$$
$$x_2' = 4x_1 + 3x_2 + 18e^{2t}$$

Solution:

1) Rewrite the given system in the matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 12e^{3t} \\ 18e^{2t} \end{bmatrix}$$

2) Solve the eigenvalue problem:

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic equation:

$$det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 5$ (real distinct)

Eigenvectors: $\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_1 = -1$ $\quad \left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{k}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$ $\quad \left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{k}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3) Fundamental matrix:

$$\mathbf{X}(t) = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}$$

$$|\mathbf{X}(t)| = det\begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix} = -2e^{4t} - e^{4t} = -3e^{4t} \neq 0$$

4) Variation of parameter formula (13a):

Inverse of the Fundamental matrix:

$$\mathbf{X}^{-1}(t) = \begin{bmatrix} -\frac{2}{3}e^{-t} & \frac{1}{3}e^{t} \\ \frac{1}{3}e^{-5t} & \frac{1}{3}e^{-5t} \end{bmatrix}$$

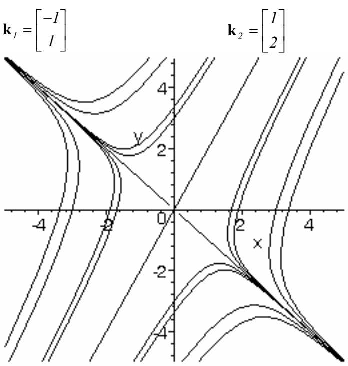
$$\mathbf{X}^{-1}(t)\mathbf{f}(t) = \begin{bmatrix} -\frac{2}{3}e^{-t} & \frac{1}{3}e^{t} \\ \frac{1}{3}e^{-5t} & \frac{1}{3}e^{-5t} \end{bmatrix}\begin{bmatrix} 12e^{3t} \\ 18e^{2t} \end{bmatrix} = \begin{bmatrix} -8e^{4t} + 6e^{3t} \\ 4e^{-2t} + 6e^{-3t} \end{bmatrix}$$

$$\int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt = \begin{bmatrix} -8\int e^{4t}dt + 6\int e^{3t}dt \\ 4\int e^{-2t}dt + 6\int e^{-3t}dt \end{bmatrix} = \begin{bmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} \end{bmatrix}$$

Particular solution:

$$\mathbf{x}_p = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}\begin{bmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -6e^{3t} - 2e^{2t} \end{bmatrix}$$

Complimentary Solution $\mathbf{x}_c$:



saddle point (eigenvalues of opposite sign)

Complimentary Solution (solution of the homogeneous system):

$$\mathbf{x}_c = \mathbf{X}(t)\mathbf{c} = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1e^{-t} + c_2e^{5t} \\ -c_1e^{-t} + 2c_2e^{5t} \end{bmatrix}$$

General solution:

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = \begin{bmatrix} -c_1e^{-t} + c_2e^{5t} - 4e^{2t} \\ -c_1e^{-t} + 2c_2e^{-3t} - 6e^{3t} - 2e^{2t} \end{bmatrix}$$

4b) (*Linear System of 2 equations – Initial Value Problem*)

Find the solution of the system of ODEs:

$$x_1' = x_1 + 2x_2 + 12e^{3t}$$
$$x_2' = 4x_1 + 3x_2 + 18e^{2t}$$

subject to the initial condition: $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Solution: Use the fundamental set of the previous example.

3) Fundamental matrix:

$$\mathbf{X}(t) = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix}$$

4) Variation of parameter formula (13b):

$$\mathbf{X}^{-1}(t) = \begin{bmatrix} -\frac{2}{3}e^{-t} & \frac{1}{3}e^{t} \\ \frac{1}{3}e^{-5t} & \frac{1}{3}e^{-5t} \end{bmatrix}$$

$$\mathbf{X}^{-1}(0) = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

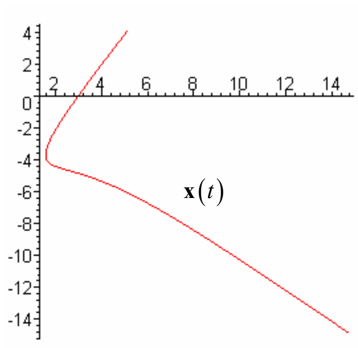
$$\mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + 2e^{5t} \end{bmatrix}$$

$$\mathbf{X}^{-1}(t)\mathbf{f}(t) = \begin{bmatrix} -\frac{2}{3}e^{-t} & \frac{1}{3}e^{t} \\ \frac{1}{3}e^{-5t} & \frac{1}{3}e^{-5t} \end{bmatrix} \begin{bmatrix} 12e^{3t} \\ 18e^{2t} \end{bmatrix} = \begin{bmatrix} -8e^{4t} + 6e^{3t} \\ 4e^{-2t} + 6e^{-3t} \end{bmatrix}$$

$$\int_0^t \mathbf{X}^{-1}(s)\mathbf{f}(s)ds = \begin{bmatrix} -8\int_0^t e^{4s}ds + 6\int_0^t e^{3s}ds \\ 4\int_0^t e^{-2s}ds + 6\int e^{-3s}ds \end{bmatrix} = \begin{bmatrix} -2e^{4t}\Big|_0^t + 2e^{3t}\Big|_0^t \\ -2e^{-2t}\Big|_0^t - 2e^{-3t}\Big|_0^t \end{bmatrix} = \begin{bmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} + 4 \end{bmatrix}$$

$$\mathbf{X}(t)\int_0^t \mathbf{X}^{-1}(s)\mathbf{f}(s)ds = \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} + 4 \end{bmatrix} = \begin{bmatrix} -4e^{2t} + 4e^{5t} \\ -6e^{3t} - 2e^{2t} + 8e^{5t} \end{bmatrix}$$

Solution of IVP:



$$\mathbf{x}(t) = \begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + 2e^{5t} \end{bmatrix} + \begin{bmatrix} -4e^{2t} + 4e^{5t} \\ -6e^{3t} - 2e^{2t} + 8e^{5t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - 4e^{2t} + 5e^{5t} \\ -2e^{-t} - 6e^{3t} - 2e^{2t} + 10e^{5t} \end{bmatrix}$$

5) (stability of autonomous system)

Investigate for stability the equilibrium point and sketch the phase portrait of the following autonomous system:

$$x' = -2x + 3y$$
$$y' = -3x - 2y$$

Solution:

1) Rewrite the given system in matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2) Solve the eigenvalue problem:

$$\left(\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic equation:

$$det\left(\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{vmatrix} -2-\lambda & 3 \\ -3 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 13 = 0$$

Eigenvalues: $\lambda_1 = -2 + 3i$, $\lambda_2 = -2 - 3i$ (complex)

Eigenvectors: $\left(\begin{bmatrix} -2 & 3 \\ -3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_1 = -2 + 3i$ $\quad \left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} -2+3i & 0 \\ 0 & -2+3i \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{k}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\mathbf{k}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3) Fundamental matrix (use equation (22)):

$$\mathbf{x}_1 = \left(\mathbf{b}_1 \cos \beta t - \mathbf{b}_2 \sin \beta t \right) e^{\alpha t} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 3t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t \right) e^{-2t}$$

$$\mathbf{x}_2 = \left(\mathbf{b}_2 \cos \beta t + \mathbf{b}_1 \sin \beta t \right) e^{\alpha t} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 3t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 3t \right) e^{-2t}$$

$$\mathbf{X}(t) = \begin{bmatrix} \cos(3t) e^{-2t} & \sin(3t) e^{-2t} \\ -\sin(3t) e^{-2t} & \cos(3t) e^{-2t} \end{bmatrix}$$

4) General solution:

$$\mathbf{x}(\mathbf{t}) = \mathbf{X}(t)\mathbf{c} \quad = \begin{bmatrix} \cos(3t) e^{-2t} & \sin(3t) e^{-2t} \\ -\sin(3t) e^{-2t} & \cos(3t) e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \cos(3t) e^{-2t} + c_2 \sin(3t) e^{-2t} \\ -c_1 \sin(3t) e^{-2t} + c_2 \cos(3t) e^{-2t} \end{bmatrix}$$

In parametric form:

$$x(t) = c_1 \cos(3t)e^{-2t} + c_2 \sin(3t)e^{-2t}$$
$$y(t) = -c_1 \sin(3t)e^{-2t} + c_2 \cos(3t)e^{-2t}$$

According to case III) a) ii) this is the asymptotically stable focus. Sketch the phase portrait:

For the particular curve, choose $c_1 = 1,\ c_2 = 0$, then

$$x(t) = \cos(3t)e^{-2t}$$
$$y(t) = -\sin(3t)e^{-2t}$$

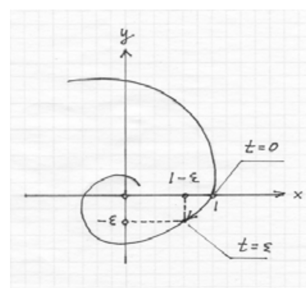
A graphing calculator can be used for sketching the graph of this curve, but it is important to know how to sketch the graph just from the parametric equation – we can perform it qualitatively in the following way:

For $t = 0$ $\quad x(t) = 1$, $\ y(t) = 0$

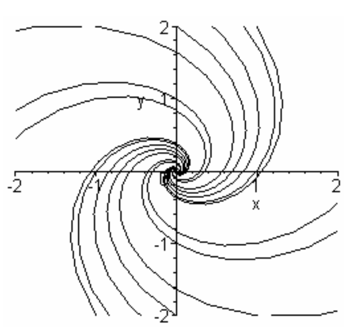
The starting point is defined. Now let us see where the curve will go under a small increase of time

$t = \varepsilon$ $\quad x(t) = 1 - \varepsilon$, $\ y(t) = -\varepsilon$

Then continue the curve as a shrinking spiral in the determined direction:



All other trajectories will be of the same shape, covering the entire plane without intersections. Here is the graph generated by Maple:



Conclusion: the equilibrium point is the asymptotically stable focus (spiral point).

6. REVIEW QUESTIONS AND EXERCISES:

1) What is a system of differential equations?
2) What type of systems did we study in this section?
3) How many solutions of a normal system of 1st order ODEs can go through an arbitrary point of the plane?
4) How many solutions of a homogeneous system of 1st order ODEs can go through the point $(0,0)$?
5) Why is uniqueness not violated for the saddle equilibrium point?
6) Why is the system of two 1st order ODEs called a dynamical system?
7) What is the stability of an equilibrium point? What does it mean?

EXERCISES:

1) Reduce the following ODEs to a system of 1st order ODEs:

a) $y''' + 5y'' + 3y' - 6y = te^{2t}$

b) $2y^{(iv)} - 6y''' + 2y'' - y' - 3y = t\cos t$

Reduce the system of 1st order ODEs to a higher order ODE:

c) $x_1' = 4x_1 + x_2 + t$
$x_2' = 2x_1 + 3x_2 - 2t$

2) Matrix exponential:

a) Using the definition of matrix exponential, verify the differentiation rule:
$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

b) Show that if $\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ then $e^{\mathbf{A}t} = \begin{bmatrix} e^{a_{11}t} & 0 & \cdots & 0 \\ 0 & e^{a_{22}t} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_{nn}t} \end{bmatrix}$

c) Consider IVP: solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$ subject to $\mathbf{x}(0) = \mathbf{x}_0$.
Show that $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$, where $\mathbf{X}(t)$ is the fundamental matrix.

3) Find the general solution of the following systems and sketch the solution curves of the homogeneous part of the systems:

a) $x_1' = x_2$
$x_2' = -x_1 + \sec x$

b) $x_1' = -5x_1 + 3x_2 + e^{-2t}$
$x_2' = 2x_1 - 10x_2 + 1$

c) $x_1' = 4x_1 + x_2 + t$
$x_2' = 2x_1 + 3x_2 - 2t$

d) $x_1' = -2x_1 + x_2 + \sin t$
$x_2' = -x_1 + \cos t$

e) $x_1' = -x_1 + x_2$
$x_2' = -x_2 + 4x_3$
$x_3' = x_1 - 4x_3$

f) $x_1' = 3x_1 - x_2 - x_3$
$x_2' = x_1 + x_2 - x_3 + t$
$x_3' = x_1 - x_2 + x_3 + 2e^t$

4) Find the solution of the following Initial Value Problems and sketch the graph of the solution:

a) $x_1' = 3x_1 - x_2 + 4e^{2t}$
$x_2' = -x_1 + 3x_2 + 4e^{4t}$
subject to $x_1(0) = 1,\ x_2(0) = 2$

b) $x_1' = 3x_1 - 2x_2 + 2$
$x_2' = 5x_1 - 3x_2 + 3$
$x_1(\pi/2) = 0,\ x_2(\pi/2) = 0$

c) $x_1' = 3x_1 - x_2 - x_3$
$x_2' = x_1 + x_2 - x_3 + t$
$x_3' = x_1 - x_2 + x_3 + 2e^t$
$x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = 0$

c) $x_1' = 3x_1 - x_2 - x_3$
$x_2' = x_1 + x_2 - x_3 + t$
$x_3' = x_1 - x_2 + x_3 + 2e^t$
$x_1(0) = 0,\ x_2(0) = 2,\ x_3(0) = 0$

5) Investigate for stability the equilibrium point and sketch the phase portrait of the following autonomous systems:

a) $x' = 2x + 3y$
$y' = -3x + 2y$

b) $x' = x + 4y$
$y' = x + y$

c) $x' = 2x + 5y$
$y' = x - 2y$

d) $x' = x - 4y$
$y' = x + 5y$

e) $x' = 2x + 5y$
$y' = -x + 5y$

f) $x' = 2x + y$
$y' = -x + 2y$

g) $x' = 4x - y$
$y' = -9x + y$

h) $x' = -x - \alpha y$
$y' = x - y$

7. LINEAR SYSTEMS OF ODEs WITH MAPLE

1. a) Find the general solution of the homogeneous system and sketch the phase portrait:

$$x_1' = 2x - x_2$$
$$x_2' = 3x_1 - 2x_2 + 4t$$

```
> with(linalg):
> k:=matrix(2,1,[[2],[0]]);
```

$$k := \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

```
> f:=matrix(2,1,[[0],[4*t]]);
```

$$f := \begin{bmatrix} 0 \\ 4\,t \end{bmatrix}$$

```
> C:=matrix(2,1,[[c[1]],[c[2]]]);
```

$$C := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Eigenvalue Problem:

```
> A:=matrix(2,2,[[2,-1],[3,-2]]);
```

$$A := \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

```
> eigenvects(A);
```

$$[1, 1, \{[1, 1]\}], [-1, 1, \{[1, 3]\}]$$

Fundamental matrix:

```
> X:=matrix(2,2,[[exp(t),exp(-t)],[exp(t),3*exp(-t)]]);
```

$$X := \begin{bmatrix} e^t & e^{(-t)} \\ e^t & 3\,e^{(-t)} \end{bmatrix}$$

Complimentary Solution - General Solution of Homogeneous System:

```
> Xc:=evalm(X&*C);
```

$$Xc := \begin{bmatrix} e^t c_1 + e^{(-t)} c_2 \\ e^t c_1 + 3\,e^{(-t)} c_2 \end{bmatrix}$$

-Phase Portrait:

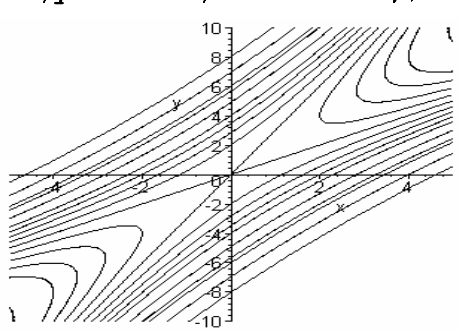
```
> x(t):=exp(t)*c[1]+exp(-t)*c[2];
```

$$x(t) := e^t c_1 + e^{(-t)} c_2$$

```
> y(t):=exp(t)*c[1]+3*exp(-t)*c[2];
```

$$y(t) := e^t c_1 + 3\,e^{(-t)} c_2$$

```
> p:={seq(seq(subs({c[1]=i/2*2,c[2]=j/2*2},
          [x(t),y(t),t=-4..4]),i=-4..4),j=-4..4)}:
> plot(p,x=-5..5,y=-10..10,color=black);
```



b) Find the general solution of the non-homogeneous system:

Particular Solution - Variation of Parameter:

```
> Xinv:=simplify(inverse(X));
```

$$Xinv := \begin{bmatrix} \frac{3}{2}e^{(-t)} & -\frac{1}{2}e^{(-t)} \\ -\frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix}$$

```
> simplify(evalm(Xinv&*f));
```

$$\begin{bmatrix} -2\,e^{(-t)}\,t \\ 2\,e^{t}\,t \end{bmatrix}$$

```
> map(int,%,t);
```

$$\begin{bmatrix} 2\,e^{(-t)}\,t + 2\,e^{(-t)} \\ 2\,e^{t}\,t - 2\,e^{t} \end{bmatrix}$$

```
> Xp:=simplify(evalm(X&*%));
```

$$Xp := \begin{bmatrix} 4\,t \\ 8\,t - 4 \end{bmatrix}$$

General Solution:

```
> GS:=evalm(Xc+Xp);
```

$$GS := \begin{bmatrix} e^{t}\,c_1 + e^{(-t)}\,c_2 + 4\,t \\ e^{t}\,c_1 + 3\,e^{(-t)}\,c_2 + 8\,t - 4 \end{bmatrix}$$

Solution Curves:

```
> xn(t):=exp(t)*c[1]+exp(-t)*c[2]+4*t;
```

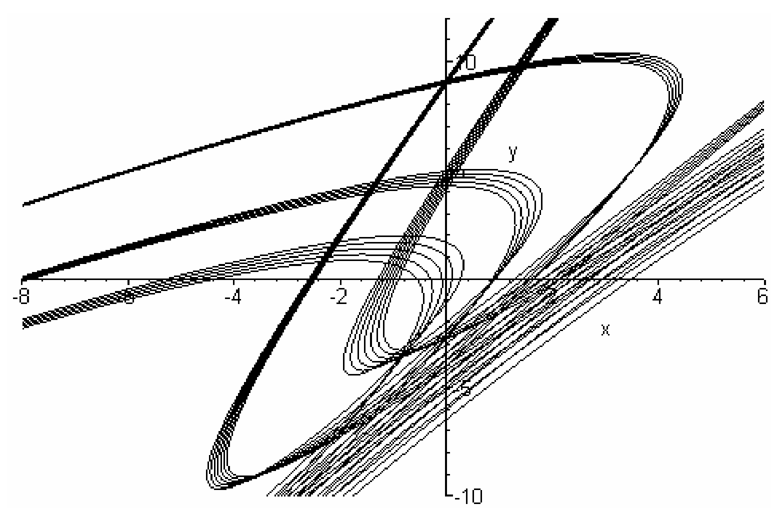
$$\mathrm{xn}(t) := e^{t}\,c_1 + e^{(-t)}\,c_2 + 4\,t$$

```
> yn(t):=exp(t)*c[1]+3*exp(-t)*c[2]+8*t-4;
```

$$\mathrm{yn}(t) := e^{t}\,c_1 + 3\,e^{(-t)}\,c_2 + 8\,t - 4$$

```
> pn:={seq(seq(subs({c[1]=i/2,c[2]=j/2},
        [xn(t),yn(t),t=-20..20]),i=-3..3),j=-2..2)}:
```

```
> plot(pn,x=-8..6,y=-10..12,color=black, numpoints=500);
```



c) Find the solution of the Initial Value Problem:

Solution of IVP - Variation of parameter formula (13b):

```
> Xinv0:=simplify(subs(t=0,evalm(Xinv)));
```

$$Xinv0 := \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

```
> X1:=evalm(evalm(X&*Xinv0)&*k);
```

$$X1 := \begin{bmatrix} 3\,e^t - e^{(-t)} \\ 3\,e^t - 3\,e^{(-t)} \end{bmatrix}$$

```
> X2:=simplify(evalm(Xinv&*f));
```

$$X2 := \begin{bmatrix} -2\,e^{(-t)}\,t \\ 2\,e^t\,t \end{bmatrix}$$

```
> X3:=subs(t=s,evalm(X2));
```

$$X3 := \begin{bmatrix} -2\,e^{(-s)}\,s \\ 2\,e^s\,s \end{bmatrix}$$

```
> X4:=simplify(map(int,X3,s=0..t));
```

$$X4 := \begin{bmatrix} 2\,e^{(-t)}\,t + 2\,e^{(-t)} - 2 \\ 2\,e^t\,t - 2\,e^t + 2 \end{bmatrix}$$

```
> X5:=simplify(evalm(X&*X4));
```

$$X5 := \begin{bmatrix} 4\,t - 2\,e^t + 2\,e^{(-t)} \\ 8\,t - 4 - 2\,e^t + 6\,e^{(-t)} \end{bmatrix}$$

```
> XS:=evalm(X1+X5);
```

$$XS := \begin{bmatrix} e^t + e^{(-t)} + 4\,t \\ e^t + 3\,e^{(-t)} + 8\,t - 4 \end{bmatrix}$$

Graph of the solution of IVP:

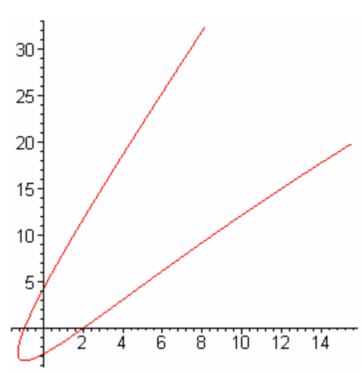
```
> u:=exp(t)+exp(-t)+4*t;
```

$$u := e^t + e^{(-t)} + 4\,t$$

```
> v:=exp(t)+3*exp(-t)+8*t-4;
```

$$v := e^t + 3\,e^{(-t)} + 8\,t - 4$$

```
> plot([u,v,t=-3..2]);
```



Application of the standard Maple procedure for solution of the system of ODEs.

2. Investigate for stability the equilibrium point and sketch the phase portrait of the following autonomous system:

$$x' = 2x + 3y$$
$$y' = -3x + 2y$$

```
> eq1:=diff(x(t),t)=-4*x(t)+3*y(t);
```

$$eq1 := \frac{d}{dt}\mathrm{x}(t) = -4\,\mathrm{x}(t) + 3\,\mathrm{y}(t)$$

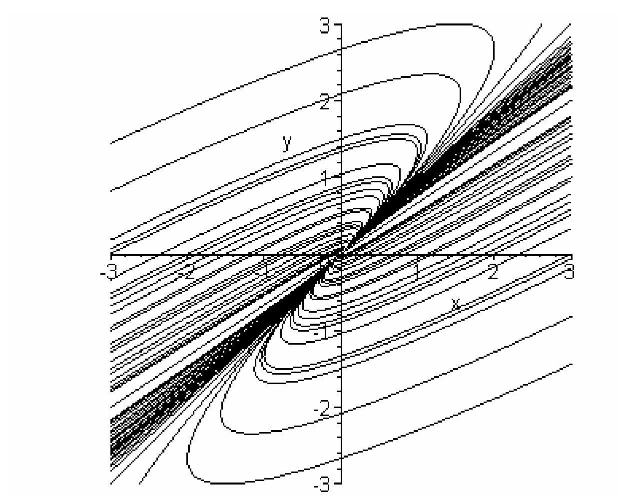
```
> eq2:=diff(y(t),t)=-2*x(t)+1*y(t);
```

$$eq2 := \frac{d}{dt}\mathrm{y}(t) = -2\,\mathrm{x}(t) + \mathrm{y}(t)$$

```
> Solution:=dsolve({eq1,eq2},{x(t),y(t)});
```

$$Solution := \{\mathrm{x}(t) = \_C1\,\mathrm{e}^{(-2\,t)} + \_C2\,\mathrm{e}^{(-t)},\ \mathrm{y}(t) = \frac{2}{3}\_C1\,\mathrm{e}^{(-2\,t)} + \_C2\,\mathrm{e}^{(-t)}\}$$

```
> assign(Solution):
> p:={seq(seq(subs({_C1=i/2,_C2=j/2},
        [x(t),y(t),t=-2..2]),i=-5..5),j=-5..5)}:
> plot(p,x=-3..3,y=-3..3,color=black,scaling=constrained);
```



The matrix of coefficients has two real distinct negative eigenvalues $\lambda_1 = -2,\ \ \lambda_2 = -1$.
The equilibrium point is **a stable node**.