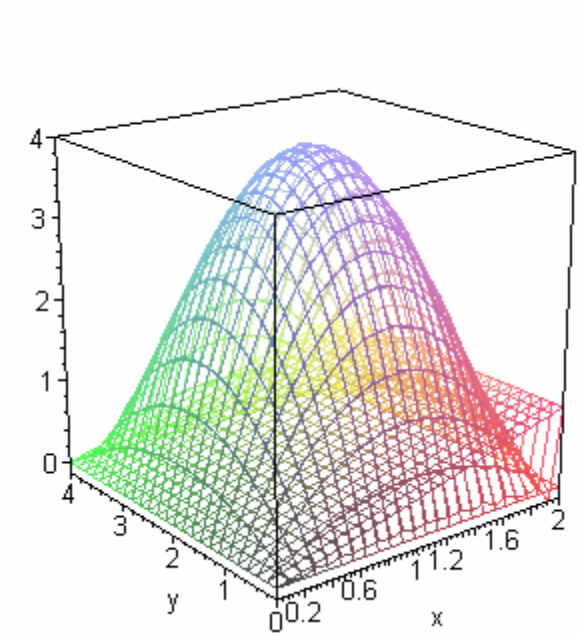
Chapter 4

Partial Differential Equations



Chapter 4 Partial Differential Equations

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4.1 Fundamental Principles of Engineering

Engineering models are built upon governing equations which usually are forms of partial differential equations. Derivation of these governing equations is based upon fundamental principles that have been developed through observation of natural phenomena. In this chapter we focus on five fundamental principles that are true for a Newtonian frame of reference. We present these principles to illustrate the process of deriving governing equations. The chapter then focuses on the solution techniques for partial differential equations.

We focus on five fundamental principles of engineering, upon which behavior of the physical world can be modeled. An effective engineer must have a theoretical and practical understanding of these principles. Rather than simply enumerating the principles, we develop them in some detail. We acknowledge that there are several different approaches to presenting these principles which are used in the literature. We choose our convention only for consistency.

A fundamental concept which is central to all five principles is that of a control volume. Since the universe is too large and complicated to model as a whole, subsets of the universe are carefully chosen to examine and model independently. These subsets are volumetric pieces of the universe without any particular shape. They can contain mass, energy, momentum, and entropy. Mass, heat, and work can cross the bounding surfaces, and forces can act on the bounding surfaces. These volumetric pieces are called ***control volumes***. There are different classes of control volumes depending upon what is allowed to flow across the boundaries. A general control volume referred to as an open system can have mass, heat, and work cross its boundaries. The mass can carry energy, momentum and entropy with it as it crosses the control volume boundaries. A second type of control volume referred to as a closed system can have heat and work cross its boundaries but not mass. The final type of control volume is an isolated system. Mass, heat, and work cannot cross the boundaries, but forces can act on these systems. Freebody diagrams are used to develop balances of static forces and time-rate-of-change of momentum (inertial forces). These three types of control volumes are used extensively to model and predict the behavior of natural phenomena. There are five basic principles common to all natural phenomena and control volumes, which are developed in detail in this section.

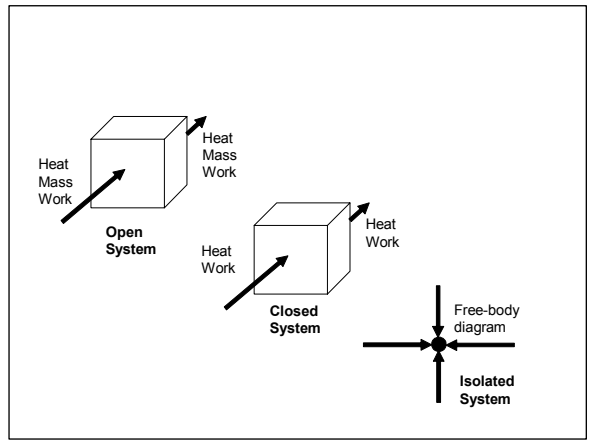


Figure 1. Control Volumes

The basic principles are developed as balances of quantities which are inside or are crossing the boundaries of a control volume. The first four basic principles may be stated in a common form in terms of the rate of creation:

$$\begin{bmatrix} Rate\ of \\ Creation \end{bmatrix} = \begin{bmatrix} Outflow \\ rate \end{bmatrix} - \begin{bmatrix} Inflow \\ rate \end{bmatrix} + \begin{bmatrix} Storage\ rate \\ increase \end{bmatrix} \tag{1}$$

The Outflow and Inflow terms refer to the transfer of quantities across the control volume boundaries either by mass flow or by direct transfer such as heat or work. The storage rate increase refers to changes in the quantity stored within the control volume boundaries. For conservation principles, the rate of creation of a quantity will be zero. These principles apply to all fields and natural systems and can be applied in many different contexts. Please note that lower case letters in the equations denote that the respective quantities are per unit mass.

4.1.1 Conservation of Mass

The first basic principle we present is the conservation of mass. Within a control volume, mass is neither created nor destroyed. The rate of creation is therefore equal to zero:

$$\begin{bmatrix} Rate\ of\ \ Creation \\ of\ Mass \end{bmatrix} = 0 \tag{2}$$

This means that for a control volume, mass can flow through the control volume or be stored in it, but cannot be created nor destroyed. We can represent this principle mathematically in several different forms. We choose to show only two, a discrete summation of mass quantities, and a continuous summation of mass quantities. The discrete sum of mass flowing in, out and being stored in a control volume is shown in equation (Eq 3):

$$\sum \dot{m}_{out} - \sum \dot{m}_{in} + \frac{d}{dt}\left(m_{cv}\right) = 0 \tag{3}$$

This equation states that the rate of change in mass flowing out of the control volume minus the rate of change of mass flowing into the control volume plus the rate of storage of mass must all balance to zero. Where $\dot{m}$ is the mass flow rate and m_{cv} is the amount of mass stored in the control volume.

The continuous form of this principle is given in an integral equation:

$$\left[\int_A \rho\left(\vec{v}\cdot\vec{n}\right)dA\right]_{out} - \left[\int_A \rho\left(\vec{v}\cdot\vec{n}\right)dA\right]_{in} + \frac{d}{dt}\left[\int_V \rho dV\right] = 0 \qquad (4)$$

This equation is the continuous analog of equation (3), ρ is density, or mass per unit volume, $\vec{v}\cdot\vec{n}$ is the velocity of the mass flow in the outward normal direction, A denotes a boundary integral where the boundary is the control volume surface and V denotes a domain integral where the domain is the volume of the control volume.

For a differential control volume, equation (3) is posed as a summation of differential quantities

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) \qquad (5)$$

u and v are velocity components in the x and y directions. The important concept is that the balance of mass flowing in and out versus the mass storage must always be zero for the conservation of mass principle.

4.1.2 Conservation of Energy (First Law of Thermodynamics)

This principle is similar to the conservation of mass, only energy is conserved. Again the rate of creation is zero:

$$\begin{bmatrix} Rate\ of \\ Creation \end{bmatrix} = 0 \qquad (6)$$

Stored energy is often classified into three common forms: potential energy (i.e. gravity), kinetic energy (i.e. motion), or thermal energy (i.e. temperature). Energy can also be transformed into heat and work. Energy can cross the boundary of the control volume with mass flow, or through heat or work. A quantifiable expression of the conservation of energy is in the form of equation (6). Again we use the discrete and continuous forms of this principle:

$$\dot{W} - \dot{Q} + \left[\sum \dot{m}e\right]_{out} - \left[\sum \dot{m}e\right]_{in} + \frac{d}{dt}\left(me_{cv}\right) = 0 \qquad (7)$$

The equation is more complicated because of the different forms that energy can assume. The rate of work crossing the boundaries (or power), Ý W , is traditionally considered going out of the control volume and the rate of heat transfer, Ý Q , is considered going into the control volume. In equation (Eq 7), the energy contained in the material entering and leaving the control volume takes the forms of:

$$e = h + \frac{v^2}{2} + gz \qquad (8)$$

Where h is enthalpy and is a combination of thermal energy, **u**, and flow work. Thermal energy can come from temperature, chemical energy, etc. The other two terms make up the rest of the types of energy that can be carried into or out of the control volume, where V is velocity, g is the gravity constant, and z is height above a convenient reference plane. These three terms represent the forms of energy: h (thermal energy + flow work), $v^2/2$ (kinetic energy), and gz (potential energy). The energy stored in the control volume takes the form of:

$$e_{cv} = u + \frac{v^2}{2} + gz \tag{9}$$

Where the storage can be in terms of thermal (**u**), kinetic ($v^2/2$), and potential (*gz*) forms of energy. The continuous form of equation (7) is:

$$\dot{W} - \dot{Q} + \left[\int_A \rho(\vec{v}\cdot\vec{n})edA\right]_{out} - \left[\int_A \rho(\vec{v}\cdot\vec{n})edA\right]_{in} + \frac{d}{dt}\left[\int_V \rho e_{cv} dV\right] = 0 \tag{10}$$

The differential form is shown in equation (11).

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho uh) + \frac{\partial}{\partial y}(\rho vh) = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} - \frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial x} - \sigma_x\frac{\partial u}{\partial x} - \sigma_y\frac{\partial v}{\partial y} - \tau_{xy}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{11}$$

where q is the heat transfer, P is the pressure, σ is the normal stress, and τ is the shear stress. The subscript denotes the direction of the quantity.

4.1.3 Momentum Principle

The rate of creation equation is not zero when dealing with momentum. Instead it is equal to the sum of the forces acting on the control volume.

$$\begin{bmatrix} \textit{Rate of Creation} \\ \textit{of Momentum} \end{bmatrix} = \begin{bmatrix} \textit{sum of forces acting} \\ \textit{on the control volume} \end{bmatrix} \tag{12}$$

Once again, there is a discrete form and a continuous form of the equation representing the momentum principle. Therefore equation (13) shows the discrete sum form of the principle where all the terms are as defined before, and $\vec{F}$ represents the force vectors acting upon the control volume.

$$\left[\sum \dot{m}\vec{v}\right]_{out} - \left[\sum \dot{m}\vec{v}\right]_{in} + \frac{d}{dt}(m\vec{v})_{cv} = \left(\sum \vec{F}\right)_{cv} \tag{13}$$

Momentum is not generally conserved but can be created or destroyed by forces. The balance of momentum within the control volume is offset by the forces acting upon the control volume. Once again equation (Eq 14) shows the continuous form of the momentum principle.

$$\left[\int_A \rho\vec{v}(\vec{v}\cdot\vec{n})edA\right]_{out} - \left[\int_A \rho\vec{v}(\vec{v}\cdot\vec{n})edA\right]_{in} + \frac{d}{dt}\left[\int_V \rho\vec{v}dV\right]_{cv} = \left[\sum \vec{F}\right]_{cv} \tag{14}$$

Again, all the terms are as before defined. It should be recognized by the student that this is a vector equation; therefore, it represents one scalar equation in each coordinate direction. The differential form for a two-dimensional system is,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) = -\frac{\partial p}{\partial x} + \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial x}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho vu) + \frac{\partial}{\partial y}(\rho vv) = -\frac{\partial p}{\partial y} + \rho g_y + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial y} \tag{15}$$

where $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$ represent pressure forces, ρg_x, ρg_x represent gravity forces, $\frac{\partial \sigma_x}{\partial x}$, $\frac{\partial \sigma_y}{\partial y}$ represent viscous normal forces, and $\frac{\partial \tau_{yx}}{\partial x}, \frac{\partial \tau_{xy}}{\partial y}$ represent shear forces.

4.1.4 Entropy Principle (Second Law of Thermodynamics)

Again the rate of creation equation does not equal zero. Entropy is not generally conserved but may be created by heat flow in or irreversibilities and may be destroyed by heat flow out. The general rate of creation equation is as follows:

$$\begin{bmatrix} \textit{Rate of Creation} \\ \textit{of Entropy} \end{bmatrix} = \begin{bmatrix} \textit{Creation or loss} \\ \textit{due to heat flow} \end{bmatrix} + \begin{bmatrix} \textit{The loss of available energy} \\ \textit{due to irreversibilities} \end{bmatrix} \quad (16)$$

The discrete and continuous forms are shown in equations (17) and (18) respectively.

$$\left[\sum \dot{m}s\right]_{out} - \left[\sum \dot{m}s\right]_{in} + \frac{d}{dt}(ms)_{cv} = \left(\sum \frac{\dot{Q}}{T}\right) + \dot{S}_{gen} \quad (17)$$

where $\dot{S}_{gen}$ is entropy generation from irreversible processes.

$$\left[\int_A \rho s(\vec{v}\cdot\vec{n})edA\right]_{out} - \left[\int_A \rho s(\vec{v}\cdot\vec{n})edA\right]_{in} + \frac{d}{dt}\left[\int_V \rho s dV\right]_{cv} = \left[\sum \frac{\dot{Q}}{T}\right] + \dot{S}_{gen} \quad (18)$$

The differential form is typically not used, since correctly specifying viscosity and thermal conductivity to be positive in the other differential equations forces the solution to satisfy the entropy principle.

4.1.5 Principles of State and Properties

Any given element of nature exists in a number of states. These states of matter are important in determining the behavior of natural systems. The principle of state and properties is defined as:
The state of a pure substance is determined by two independent properties
These relationships are often referred to as thermodynamic properties. These properties are articulated in several forms:

1. Tabulated: (Steam tables, Phase change, etc.)

2. Algebraic: (Ideal gas, etc.)

3. Graphical: (Temperature-entropy diagrams, etc.)

These relationships are usually expressed in empirical forms due to the difficulty of forming closed-form equations to represent them under all conditions. There are other properties that are of interest to engineering. These properties also define states and are used to determine behavior of materials. Yield strength and hardness are examples of these other types of properties. The principles of state and properties are fundamental to the understanding and modeling of natural systems.

4.2 Fundamental Phenomena in Engineering

Engineering is the art and science of designing and building mechanisms and prediction of the behavior of natural phenomena is key to designing and building effective mechanisms and systems. In this section we present nine specific phenomena which represent a majority of the phenomena encountered in engineering problems.

4.2.1 Fluid Flow

Fluid flow is the phenomena associated with the motion of fluids and gases. It involves the intermolecular forces and collective body forces that occur when materials in the fluid and gas states move. This phenomenon is important in aerodynamics, heating, cooling and ventilation, piping, casting, etc.

4.2.2 Heat Flow

Heat flow is a phenomena related to energy flow in the form of heat from one body to another through conduction, convection or radiation. It occurs when there are differences in energy states between two nearby bodies. Heat flow is important in chemical processes, heating, cooling, refrigeration, thermodynamic cycles, etc.

4.2.3 Friction

Friction is a phenomenon that occurs when two bodies contact each other while moving in relation to each other. It results in the transfer of energy in the forms of heat and noise and typically reduces the relative velocities of the two bodies. Friction is important in mechanisms, motion, etc.

4.2.4 Motion

The phenomena of motion can be studied relative to bodies in motion or on an absolute reference frame. This phenomenon is observed in planetary systems as well as molecular and atomic systems.

4.2.5 Elasticity/Plasticity

When material is deformed it can behave in an elastic manner; meaning it will return to its original configuration or a plastic manner; meaning it will not return completely to its original configuration. Elasticity and plasticity occur in all types of materials.

4.2.6 Electrical/Magnetic

Electrical and magnetic phenomena are related and are associated with the influence of charges on electrons and protons found in atoms. These forces influence motion and flow of heat and energy.

4.2.7 Thermal

Thermal phenomena deal with the exchange of heat, mass and work within systems. The rate at which energy is transferred and work is accomplished or mass is moved determines thermodynamic cycles.

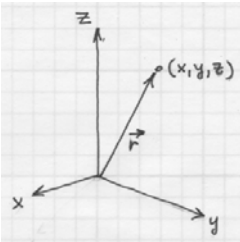
4.2.8 Gravity

Gravity is the attraction of mass to other mass. It is important in planetary motion as well as earth bound systems of motion and forces.

4.2.9 Behavior of materials

The behavior of materials is a phenomenon that involves states and properties, and how the materials react to energy, deformation, electricity, etc.

4.3.1 Domain



Consider the structure and notations of the domains of Euclidian space in which partial differential equations model some physical processes in continuous media. We need to recall some elementary topological definitions for its rigorous mathematical description.

Point in Euclidian space $\mathbb{R}^3$ is denoted by a position vector

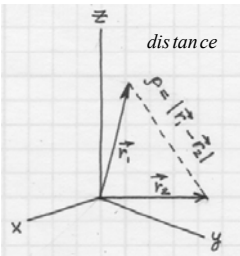
$$\mathbf{r} = (x, y, z) \in \mathbb{R}^3.$$

Scalar product of two vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is defined by

$$(\mathbf{r}_1, \mathbf{r}_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Then the **norm** of a vector $\mathbf{r} = (x, y, z)$ is defined as

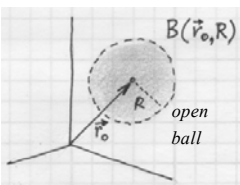
$$|\mathbf{r}| = \sqrt{(\mathbf{r}, \mathbf{r})} = \sqrt{x^2 + y^2 + z^2}$$



The **distance** between two vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is defined with the help of the norm

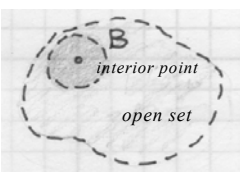
$$\rho(\mathbf{r}_1, \mathbf{r}_2) \equiv |\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

These definitions can be reduced to the cases of 2-dimensional $\mathbb{R}^2$ and 1-dimensional $\mathbb{R}^1$ Euclidian spaces.



Open ball in $\mathbb{R}^3$ with a center at $\mathbf{r}_0$ and radius R is defined as a set of points the distance from which to point $\mathbf{r}_0$ is less than R

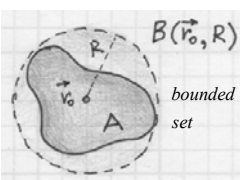
$$B(\mathbf{r}_0, R) = \left\{ \mathbf{r} \in \mathbb{R}^3 \,\middle|\, |\mathbf{r} - \mathbf{r}_0| < R \right\}$$



Point $\mathbf{r}_0$ is an **interior point** of the set $D \subset \mathbb{R}^3$ if it belongs to D with some open ball with a center at $\mathbf{r}_0$

$$B(\mathbf{r}_0, R) \subset D \qquad \text{for some radius } R > 0$$

The set is called **open** if all its points are interior.

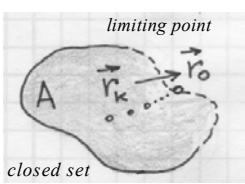


The set $A \subset \mathbb{R}^3$ is called **bounded** if there exists point $\mathbf{r}_0$ and radius R such that

$$B(\mathbf{r}_0, R) \supset A$$

A sequence of points $\mathbf{r}_k$ **converges** to point $\mathbf{r}$ (denoted $\mathbf{r}_k \to \mathbf{r}$ or $\lim_{k\to\infty} \mathbf{r}_k = \mathbf{r}$) if

$$\lim_{k\to\infty} |\mathbf{r}_k - \mathbf{r}| = 0$$



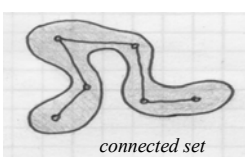
or, in other words, for any $\varepsilon > 0$ there exists a number $K \in \mathbb{N}$ such that

$$\mathbf{r}_k \in B(\mathbf{r}, \varepsilon) \text{ for all } k > K .$$

The point $\mathbf{r}_0$ is called a **limiting point** of set A if there exists a sequence

$$\mathbf{r}_k \in A \quad \text{such that } \mathbf{r}_k \to \mathbf{r}_0 .$$

The **closure** of set A is a set $\bar{A}$ to which consists of all limiting points of set A. If a set coincides with its closure then it is called **closed** (a closed set includes all its limiting points). A bounded closed set is called **compact**.



A set is called **connected** if any two points of the set can be connected by a piece-wise line belonged to the set.

Domain

A connected open set is called a **domain**.

Boundary

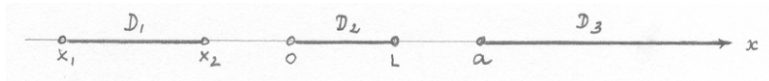
Let D be a domain. Then its **boundary** S is defined as a set of all points from its closure $\bar{D}$ which do not belong to D:

$$S = \left\{ \mathbf{r} \in \mathbb{R}^3 \,\middle|\, \mathbf{r} \notin D, \mathbf{r} \in \bar{D} \right\} = \bar{D} \setminus D$$

Initial boundary value problems for classical PDE's will be set in the domains of Euclidian space. Typical examples of such domains:

<u>1-dimensional space</u> $\mathbb{R}^1$ **:**

Intervals $D_1 = (x_1, x_2)$, $D_2 = (0, L)$, $D_3 = (a, \infty)$ are domains in $\mathbb{R}^1$. Their boundaries are sets which consist of points $S_1 = \{x_1, x_2\}$, $S_2 = \{0, L\}$, $S_3 = \{a\}$, consequently.



<u>2-dimensional space</u> $\mathbb{R}^2$ **:**

Open box $D_1 = (0, L) \times (0, M)$.

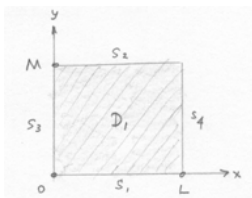
Boundary of D_1 consists of four segments

$$S_1 = \{(x, y) \mid y = 0, 0 \le x \le L\} \quad \text{or} \quad y = 0$$

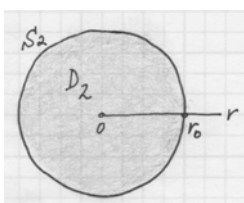
$$S_2 = \{(x, y) \mid y = M, 0 \le x \le L\} \quad \text{or} \quad y = M$$

$$S_3 = \{(x, y) \mid x = 0, 0 \le y \le M\} \quad \text{or} \quad x = 0$$

$$S_4 = \{(x, y) \mid x = L, 0 \le y \le M\} \quad \text{or} \quad x = L$$



then the whole boundary is the union $S = S_1 \cup S_2 \cup S_3 \cup S_4$

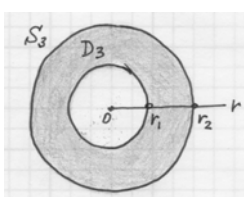


Circular domain (in polar coordinates):

$$D_2 = \{\mathbf{r} \in \mathbb{R}^2 \mid 0 \le \mathbf{r} < \mathbf{r}_0\}$$

with the boundary

$$S_2 = \{\mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = \mathbf{r}_0\} \text{ or just } \mathbf{r} = \mathbf{r}_0$$



Annular domain:

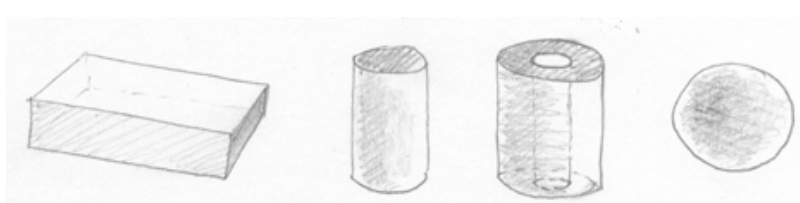
$$D_3 = \{\mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r}_1 < \mathbf{r} < \mathbf{r}_2\}$$

with the boundary

$$S_3 = \{\mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = \mathbf{r}_1\} \cup \{\mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = \mathbf{r}_2\} \text{ or just } \mathbf{r} = \mathbf{r}_0$$

<u>3-dimensional space</u> $\mathbb{R}^3$ **:**

Examples of domains in 3-dimensional space $\mathbb{R}^3$ are a parallelogram and an hollow parallelogram, a cylinder and an hollow cylinder, a sphere and an hollow sphere:



4.3.2 Heat Equation

1. Modeling of Heat Transport

In the philosophical treatise "On the Nature of the Universe", Roman scientist Titus Lucretius (100 B.C. – 55 B.C.) poetically presented the teachings of ancient Greek philosophers-atomists Democritus, Epicurus, Leucippus, and others, who lived in the 4-6 centuries B.C. They considered an example of heat or cold propagation for proof of some statements of their theory. Assuming that heat consists of tiny material particles, they end up with the following observations: any material is porous (includes voids) because heat penetrates to it; heat particles are extremely small because they are able to penetrate to very dense materials (like metals or stones); heat particles are practically weightless because a heated body does not change its weight noticeably; heat consists of not-rarified clusters of particles because the heating process is smooth and homogeneous.

We believe that there is no need to convince a modern reader that this theory is completely wrong. It looks very naïve to us. And we also are not going to present here the contemporary physical theory of heat transport. But, probably, some readers may be extremely surprised to discover that modern mathematical modeling of heat transport is based precisely on the statements of ancient Greek theory, and it is called ***thermodiffusion***. Moreover, in modeling of heat propagation in turbulent fluid flow, temperature is assumed to be an inert scalar specie. So, we still can benefit from the achievements of great Greek thinkers, which have not yet lost their value.

2. Physical Concepts

Heat is an internal energy contained in continuous media (which can be solids, fluids, or gases).
Temperature is a measure of heat (scalar quantity); it is used for the description of the heat distribution in the media (temperature field). Units for measurement of temperature are K, C, F, and R.

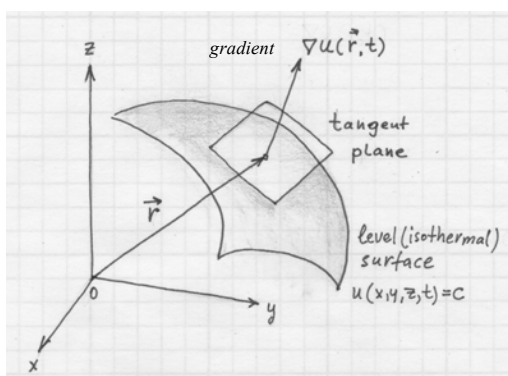
Notations for temperature: in domain D of $\mathbb{R}^3$, the non-stationary temperature field is defined by a function

$$u(x,y,z,t) \qquad (x,y,z) \in D \subset \mathbb{R}^3 \qquad t > 0$$

or vector notation may be used for space variables

$$u(\mathbf{r},t) \qquad \mathbf{r} \in D \subset \mathbb{R}^3 \qquad t > 0$$

We assume a temperature field to be a smooth continuous function of its variables in D.



Gradient is a vector defined by

$$\nabla u(\mathbf{r},t) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right), \qquad \mathbf{r} \in D \subset \mathbb{R}^3 \tag{1}$$

The set of points in R^3, which have the same temperature c is defined by the level surface $u(x,y,z,t) = c$. Here, the gradient vector $\nabla u(\mathbf{r},t)$ is orthogonal to the level surface.

In $\mathbb{R}^2$, for a fixed value of time t, a temperature field in a plane is represented by the surface

$$z = u(\mathbf{r}, t) \qquad \mathbf{r} \in D \subset \mathbb{R}^2$$

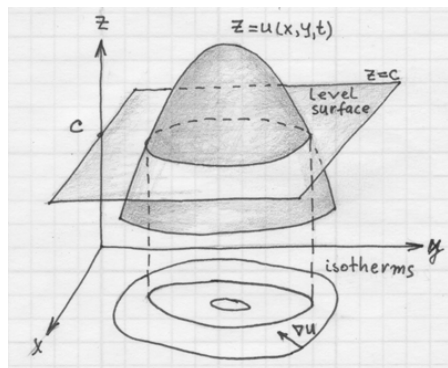
$$z = u(x, y, t) \qquad (x, y) \in D \subset \mathbb{R}^2$$

Plane temperature fields can be characterized by:

Level curves which are obtained as the intersection of the surface $z = u(x, y, t)$ with the planes $z = c$, $c \in \mathbb{R}$.

Isotherms are the projections of the level curves on the xy-plane. They constitute a set of points $(x, y) \in D$ satisfying the equation $u(x, y, t) = c$. The medium in which all points have the same temperature is called the ***isothermal***.

The gradient vector on the xy-plane ∇u is orthogonal to the isotherms, therefore, ∇u indicates the direction in which u increases most rapidly, and $-\nabla u$ indicates the direction in which u decreases most rapidly.

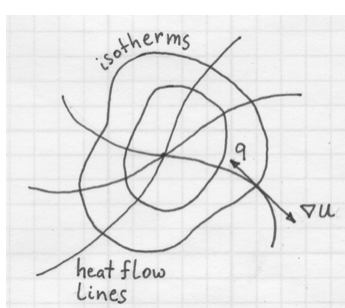


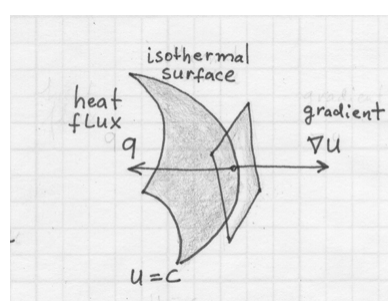
Thermal conduction (thermodiffusion) is a process of heat propagation due to the presence of the temperature gradient in a medium: heat tends to propagate from the regions with the higher temperature to the regions with the lower temperature; and there is no heat transfer in the isothermal media.

Fourier Law. Thermal conduction is characterized by the ***heat flux vector*** $\mathbf{q}(\mathbf{r}, t)$ which represents heat flow per unit time, per unit area of the isothermal surface in the direction of the decreasing temperature. For the qualitative description of the thermal diffusion, we will use the following empirical law formulated by the French scientist Joseph Fourier

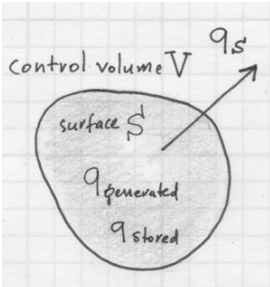
$$\mathbf{q}(\mathbf{r}, t) = -k \nabla u(\mathbf{r}, t) \tag{2}$$

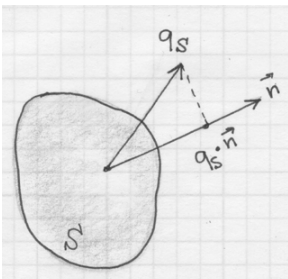
which assumes the linear dependence of heat flux on the temperature gradient with the constant of proportionality k, termed ***thermal conductivity***.





3. Heat Equation





Consider a point $\mathbf{r} \in D \subset \mathbb{R}^3$. Let V be an arbitrary small control volume containing point $\mathbf{r}$.
Application of the fundamental principles to a heat transfer system yields the following balance of conservation of energy for a control volume V with the surface boundary S:

$$\begin{bmatrix} \textit{rate of heat flow} \\ \textit{through the} \\ \textit{boundary } S \end{bmatrix} + \begin{bmatrix} \textit{rate of heat} \\ \textit{generation} \\ \textit{in volume } V \end{bmatrix} = \begin{bmatrix} \textit{rate of heat} \\ \textit{storage} \\ \textit{in volume } V \end{bmatrix}$$

The first term in this equation is caused by the diffusion of heat through the boundary of the control volume due to the presence of a temperature gradient, and it is defined by the heat flux through the surface (because **n** is the outward normal vector to the surface, the minus sign is used to insure that positive heat flux is into the control volume):

$$\begin{bmatrix} \textit{rate of heat flow} \\ \textit{through the} \\ \textit{boundary } S \end{bmatrix} = -\int_S \mathbf{q}(\mathbf{r},t) \cdot \mathbf{n} dS$$

The second term can be caused by a production of heat inside the control volume due to some source of energy $g(\mathbf{r},t)$: chemical, electrical, radiative etc.:

$$\begin{bmatrix} \textit{rate of heat} \\ \textit{generation} \\ \textit{in volume } V \end{bmatrix} = \int_V g(\mathbf{r},t) dV$$

The remaining term is evaluated as:

$$\begin{bmatrix} \textit{rate of heat} \\ \textit{storage} \\ \textit{in volume } V \end{bmatrix} = \int_V \rho c_p \frac{\partial u(\mathbf{r},t)}{\partial t} dV$$

Then the energy balance yields the equation

$$-\int_S \mathbf{q}(\mathbf{r},t) \cdot \mathbf{n} dS + \int_V g(\mathbf{r},t) dV = \int_V \rho c_p \frac{\partial u(\mathbf{r},t)}{\partial t} dV \tag{3}$$

Application of the divergence theorem to the first term gives us:

$$\int_S \mathbf{q}(\mathbf{r},t) \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{q}(\mathbf{r},t) dV$$

Then the surface integral in the equation can be replaced by the volume integral, and all terms can be combined in one expression

$$\int_V \left[-\nabla \cdot \mathbf{q}(\mathbf{r},t) + g(\mathbf{r},t) - \rho c_p \frac{\partial u(\mathbf{r},t)}{\partial t} \right] dV = 0$$

Because this equation has to hold for an arbitrary control volume for which **r** is an interior point, according to the theorem from vector calculus, the integrand should be identically equal to zero

$$-\nabla \cdot \mathbf{q}(\mathbf{r},t) + g(\mathbf{r},t) - \rho c_p \frac{\partial u(\mathbf{r},t)}{\partial t} = 0$$

Replacing the heat flux vector by the temperature gradient according to the Fourier law and moving the last term to the right hand side, one can get the differential equation of thermal diffusion

$$\nabla \cdot \left[k \nabla u(\mathbf{r},t)\right] + g(\mathbf{r},t) = \rho c_p \frac{\partial u(\mathbf{r},t)}{\partial t} \tag{4}$$

If thermal conductivity k is a constant, then the equation may be rewritten in the form

$$\nabla^2 u(\mathbf{r},t) + \frac{g(\mathbf{r},t)}{k} = a^2 \frac{\partial u(\mathbf{r},t)}{\partial t} \tag{5}$$

where the coefficient a is defined by

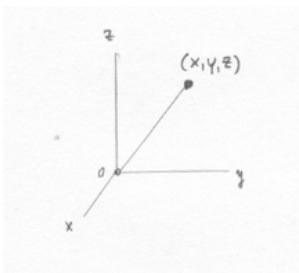
$$a^2 = \frac{1}{\alpha} = \frac{\rho c_p}{k} \tag{6}$$

If there are no heat sources in the considered domain, then the equation ( ) transforms to the homogeneous heat equation

$$\nabla^2 u(\mathbf{r},t) = a^2 \frac{\partial u(\mathbf{r},t)}{\partial t} \tag{7}$$

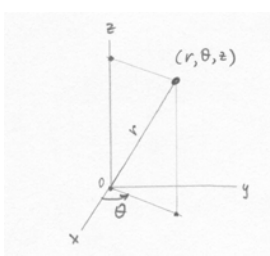
Since the heat equation was derived on the general assumption of propagation of some specie in the continuous media due to the concentration gradient, it is valid for any diffusion process where instead of temperature we use the other characterization of a specie's concentration.

Heat Equation in Cartesian Coordinates



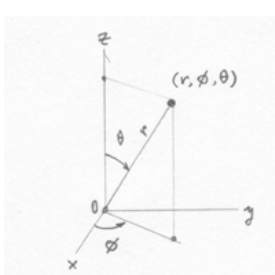
$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} + \frac{g(x,y,z,t)}{k} = a^2 \frac{\partial u(x,y,z,t)}{\partial t} \tag{8}$$

Heat Equation in Cylindrical Coordinates



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + \frac{g}{k} = a^2 \frac{\partial u}{\partial t} \tag{9}$$

Heat Equation in Spherical Coordinates



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \theta^2} + \frac{g}{k} = a^2 \frac{\partial u}{\partial t} \tag{10}$$

4. Thermophysical properties

Physical quantities involved in the Heat Equation have the following dimensions in SI units:

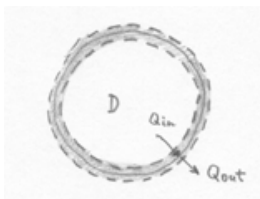
Temperature	u	$[K]$	
Heat flux	$q'' = -k\dfrac{\partial T}{\partial x}$	$\left[\dfrac{W}{m^2}\right]$	heat flux increases with increase of k
Thermal conductivity	k	$\left[\dfrac{W}{m \cdot K}\right]$	shows the ability of material to conduct heat
Density	ρ	$\left[\dfrac{kg}{m^3}\right]$	
Specific heat	c_p	$\left[\dfrac{J}{kg \cdot K}\right]$	specific heat at constant pressure
Thermal diffusivity	$\alpha = \dfrac{k}{\rho c_p}$	$\left[\dfrac{m^2}{s}\right]$	ratio of thermal conductivity to heat capacity; α compares ability of material to conduct energy relative to its ability to store energy: small α = slow change of temperature high α = quick change of temperature
Coefficient in Heat Equation	$a^2 = \dfrac{1}{\alpha} = \dfrac{\rho c_p}{k}$	$\left[\dfrac{s}{m^2}\right]$	small a = quick change of temperature high a = slow change of temperature

Typical properties of common materials at room temperature (*300K*)

	a	$\alpha \cdot 10^6$	ρ	c_p	k
Aluminum	100	80	2700	900	200
Copper	90	100	8900	390	400
Gold	90	130	19300	130	320
Steel	500	3	8000	500	15
Brick	1500	0.4	1900	835	0.7
Glass	1200	0.7	2500	800	1.4
Wood	2900	0.12	500	2500	0.15
Rock	900	1.3	2500	800	2.5
Sand	2000	0.25	1500	800	0.3
Water	2700	0.14	1000	4200	0.6
Beef	2774	0.13	1090	3540	0.47
Turkey	2774	0.13	1050	3540	0.5
Potato	2774	0.13	1055	3640	0.5

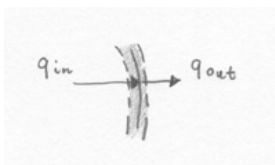
5. Modeling of Boundary Conditions

Heat transfer through the boundary S of the domain $D \subset \mathbb{R}^3$ is modeled by application of the conservation of energy law to the control surface (which can be described as a closure of the domain $C.S.$ which contains the boundary ($C.S. \supset S$) and which has negligible volume, such that in a control surface we can neglect volumetric storage and generation of the heat energy). A control surface allows us to distinguish heat fluxes crossing the boundary S and divide then into fluxes inside and outside the domain. Therefore, the rate of heat transfer which crosses the control surface inside of the domain is equal to the rate of heat transfer which crosses the control surface outside of the domain:



$$Q_{in} = Q_{out} \tag{1}$$

Locally, on a unit surface area basis, equation (1) results from conservation of heat flux through the control surface

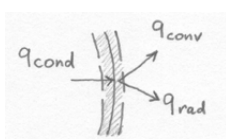


$$q_{in} = q_{out} \tag{2}$$

Consider the function $u(\mathbf{r},t)$ describing a temperature field in the domain $D \subset \mathbb{R}^3$ for $t > 0$ and let the surface S be the boundary of the domain D. Assume that heat transfer in the domain D is accomplished only by conduction with a coefficient of thermal conductivity k. Therefore, the heat flux q_{in} is described as a component of the flux vector in the normal direction to the surface S

$$q_n = q \cdot n = (-k\nabla u) \cdot n = -k \left.\frac{\partial u}{\partial n}\right|_S$$

where n is the outward unit normal vector to the surface S.
Let the media outside the domain D be characterized by the uniform temperature u_∞ of the transparent to thermal radiation fluid flow and the temperature u_{sur} of the large surroundings which is emitting thermal radiation as a black body. Then heat transfer from the surface S is accomplished by two modes of heat transfer – convection and radiation



$$q_{out} = q_{conv} + q_{rad} \tag{3}$$

Convective heat flux is described by Newton's law of cooling

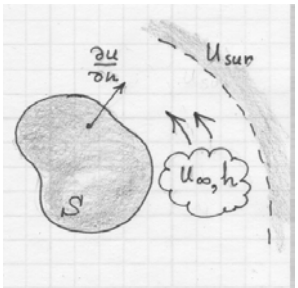
$$q_{conv} = h\left(u\big|_S - u_\infty\right) \tag{4}$$

where h is a coefficient of convective heat transfer and $u\big|_S$ is the surface temperature.

Net radiative heat flux at the surface is described by Stefan-Boltzmann Law

$$q_{rad} = \varepsilon\sigma\left(u\big|_S^4 - u_{sur}^4\right) \tag{5}$$

where ε is the total emissivity of the surface (physical property of the surface, $0 \le \varepsilon \le 1$), $\sigma = 5.67 \cdot 10^{-8}\left[\frac{W}{m^2 K^4}\right]$ is a Stefan-Boltzmann constant, and temperature is measured in the absolute temperature scale (Kelvin).



Then equations (2-5) yields a boundary condition

$$-k\left.\frac{\partial u}{\partial n}\right|_S = h\left(u\big|_S - u_\infty\right) + \varepsilon\sigma\left(u\big|_S^4 - u_{sur}^4\right) \tag{5}$$

This is the most general boundary condition for the heat transfer equation. Because in the right hand side of the equation, surface temperature appears in the 4th power, equation (5) is non-linear. It is more desirable, to have a linear boundary condition. Therefore, the radiation part of equation (5) either is neglected (if the contribution of radiation to the energy balance is small) or linearized, for example, on a basis of temperature u_{sur}

$$q_{rad} = \varepsilon\sigma\left(u\big|_S^4 - u_{sur}^4\right) = \varepsilon\sigma\left(u\big|_S u_{sur}^3 - u_{sur}^4\right) = \varepsilon\sigma u_{sur}^3\left(u\big|_S - u_{sur}\right) = h_{rad}\left(u\big|_S - u_{sur}\right)$$

If the fluid and surroundings have the temperature u_∞ then the linearized boundary condition can be written with some artificial coefficients $h = h_{conv} + h_{rad}$

$$-k\left.\frac{\partial u}{\partial n}\right|_S = h\left(u\big|_S - u_\infty\right) \tag{6}$$

This is the classical boundary condition of convective type for the Heat Equation in Heat Transfer Physics. It can be rewritten in the form used in the settings of a BVP for PDE as

$$\left[k\frac{\partial u}{\partial n} + hu\right]_S = hu_\infty$$

or, if the right hand side is denoted by a single function f (in general, f is a function of location on the boundary and time $f = f(\mathbf{r},t)$, $\mathbf{r} \in S$, $t > 0$), then

$$\left[k\frac{\partial u}{\partial n} + hu\right]_S = f \tag{7}$$

This is a classical boundary condition of convective type for the Heat Equation in the theory of Equations of Mathematical Physics (called a mixed boundary condition, boundary condition of the 3rd kind, or Robin boundary condition).

Special cases of boundary conditions:

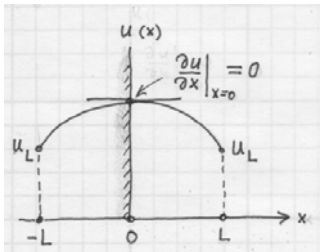
Homogeneous

1) $\left[k\frac{\partial u}{\partial n} + hu\right]_S = 0$ The right hand side is equal to zero. A boundary surface is exposed to the environment of zero temperature. Physically this condition is not often realized; but it can be obtained after the change of dependent variable, when in the heat equation, the temperature u is replaced by excess temperature $\theta = u - u_\infty$, or by $\theta = u - u_s$ where u_s is a steady state temperature satisfying the non-homogeneous boundary condition.

Thermostated boundary

2) $u\big|_S = const$ A thermostated boundary, for which temperature is supported by a very high value of the convective coefficient h (divide the equation by h and consider the limit when $h \to \infty$).

Insulated boundary



insulation = line of symmetry

3) $\left.\dfrac{\partial u}{\partial n}\right|_S = 0$ There is no heat flux through the boundary (this is accomplished by insulation of the boundary outside the domain with a material of negligible effective thermal conductivity). This condition also can be used at the surfaces of symmetry of the considered domain. Then it is sufficient to solve the problem in the symmetrical part of the domain where the surface of symmetry becomes a part of the boundary.

Typical values of physical parameters involved in boundary conditions

1) Coefficient of convective heat transfer $h\left[\dfrac{W}{m^2 K}\right]$:

	Gases	Liquids
Free convection	2-20	10-100
Forced convection	20-200	50-10,000
Convection with boiling or condensation	1000-50,000	

2) Total emissivity ε of some materials (at room temperature) is a surface property which can depend on material, surface temperature and surface roughness:

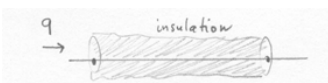
Polished or foil aluminum	0.04
Anodized aluminum	0.8
Polished steel	0.15
Brick (red)	0.95
Paints (from white to black)	0.9-0.98
Concrete	0.9
Wood	0.8-0.9

3) Typical values of conductivity k were presented in the section 4.3.2 4.

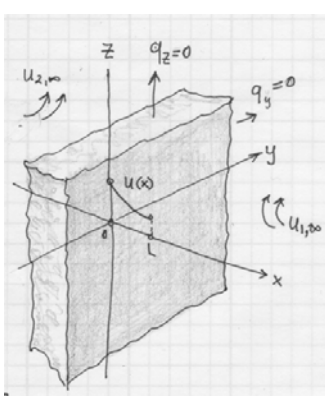
6. Mathematical dimension in modeling heat transfer

All physical models for heat transfer are in 3-dimensional Euclidean space. Mathematical dimension of the physical model determines the number of spatial variables in the function u describing the temperature field in the system; and it depends on the form of the domain and the choice of coordinate system. It is important to minimize mathematical dimension of the model because it can reduces the difficulty of solution.

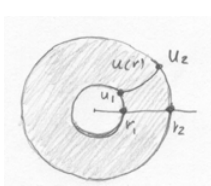
In many cases the "mathematical model" for a reduced dimension of the problem is not consistent with the physical sense of the model. For example, for adoption of a 2-dimensional model for heat transfer, the following explanation is given in a textbook of Engineering mathematics: "the two faces of sheet metal are insulated, and the sheet is so thin that heat flow in it can be regarded as two-dimensional. The edges of the sheet are maintained at constant temperature…" Or for a 1-dimensional model: "A rod has its lateral surface insulated against the flow of heat and is so thin that heat flow in the rod can be regarded as one-dimensional. Its left end is maintained at a constant temperature, and its right end radiates freely into air of constant temperature".



This approach to the dimension of problems in engineering mathematics textbooks is typical, but completely incorrect. First, it has the meaning of an approximation of the physical domain – no physical rod is a mathematical line. Second, it is inconsistent with the physical process of heat transfer. The thermal resistance of the material is increased with the decrease of cross-sectional area in the direction of heat flow – and therefore in the limit, there will be no heat flow along the line.. Also emission of radiation from the end of a rod of negligible cross-sectional area is also negligible. For the case of the sheet metal, thermal resistance in a direction of the sides is negligible compared to the direction along the infinitely thin layer – and no material can provide in this case a perfect insulation (even in a vacuum, there will be losses of energy due to thermal radiation from the surface).



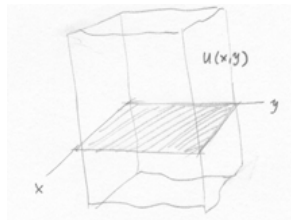
At the same time, there is the other approach to determination of the mathematical dimension of the physical domain, which is not an approximation and which is consistent with physical processes. For example, consider steady state heat conduction in a plane wall with dimensions L, W, H in the directions of the axes of the Cartesian coordinate system (x, y, z). Suppose that for a large wall, the thickness of the wall L is much smaller than other sizes W and H. Let the sides of the walls defined by planes $x = 0$ and $x = L$ be maintained at constant temperatures u_0 and u_L respectively , or be exposed to alarge convectional environment at uniform temperatures $u_{1,\infty}$ and $u_{2,\infty}$ respectively . The temperature gradients in the directions y and z are negligible, heat transfer occurs exclusively in the direction x, and, therefore, the Heat Equation includes only the derivative with respect to the variable x. It is sufficient to solve the problem as 1-dimensional; and the determined temperature profile will be the same along any line across the wall. Boundary conditions for this problem are set from the physically consistent heat transfer on the plane boundary.

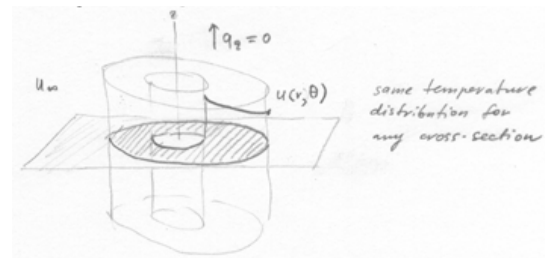


Mathematical dimension of the heat transfer problem can also be reduced by the appropriate choice of coordinate system. Thus, for heat conduction between two concentric spherical surfaces maintained at constant temperatures, the problem becomes 1-dimensional in the spherical coordinate system, because of the angular symmetry the derivatives with respect to angular variables disappear.

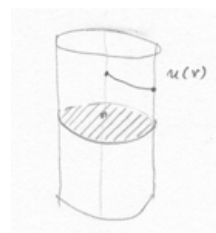
We list some other typical cases where mathematical dimension of the problem can be reduced:

A long rectangular column or cylinder of non-circular cross-section – to a 2-dimensional problem in the plane.

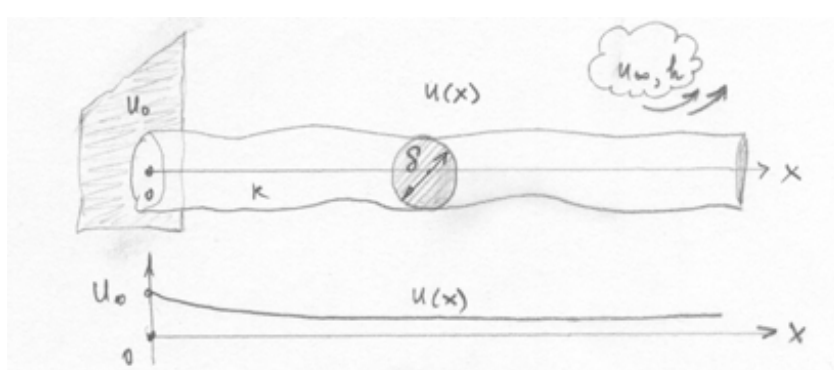




A long cylinder of circular cross section – to a 1-dimensional problem in polar coordinates.



An appropriate 1-dimensional approximation can be made for elongated domains of finite cross-sectional area with characteristic length δ (diameter for circular, and average width for rectangular cross-section) exposed to a convective environment with the coefficient of convective heat transfer h. If the ratio $\frac{h\delta}{k} < 0.1$ (so called the non-dimensional Biot number), then the typical error in the heat transfer predictions is less than 1%. The lateral surface in this model is not considered as a boundary and heat transfer through it is included into the equation as a heat source. Boundary conditions are set at the ends of the domain in correspondence with physical heat transfer through the cross-section of the rod.



4.4 Classical Initial-Boundary Value Problems (IBVP)

Classical equations of mathematical physics are modelled by three major 2nd order PDE's with one unknown function $u(\mathbf{r},t)$, $\mathbf{r} \in D \subset \mathbb{R}^3$, $t > 0$:

Laplace's Equation (LE) $$\nabla^2 u = 0 \tag{1}$$

Heat Equation (HE) $$\nabla^2 u = a^2 \frac{\partial u}{\partial t} \tag{2}$$

Wave Equation (WE) $$\nabla^2 u = a^2 \frac{\partial^2 u}{\partial t^2} \tag{3}$$

defined in the domain of Euclidian space $\mathbf{r} \in D \subset \mathbb{R}^3$, and let S be a boundary of the domain D.
These equations can model the particular physical phenomena.
The mathematical problem which models a physical phenomena should be set in such a way that its solution has a proper physical meaning and it is unique (it also has to be continuously dependent on the boundary conditions). The classical equations of mathematical physics are solved subject to initial and boundary conditions set in the way to provide a physically reliable unique solution describing the model. In this case, the problem is said to be **well-set** (or correctly formulated. Otherwise, it is called an **ill-set** problem (although mathematicians still study the ill-set problems, in most cases, they have no practical interest).

We will consider only well-set IBVP's for classical PDE's which have the following formulation:

Initial-Boundary Value Problem

Find $u(\mathbf{r},t)$ $\mathbf{r} \in D \subset \mathbb{R}^3$, $t > 0$ such that

1. $u(\mathbf{r},t)$ satisfies the PDE in D for $t > 0$
2. $u(\mathbf{r},t)$ satisfies initial conditions in D for $t = 0$
3. $u(\mathbf{r},t)$ satisfies boundary conditions at $\mathbf{r} \in S$ for $t > 0$

Setting of the IBVP for classical PDE's is summarized in the following table:

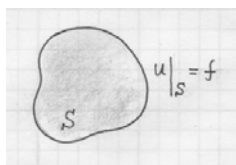
Note: If D is not a bounded domain and has no boundary (i.e. $D = \mathbb{R}^3$), then no boundary condition is set (however, some restrictions on the solution may be applied, such as a requirement of a bounded solution, etc.).

That is the most general set of IBVP for classical PDE's. Next, we will consider the specific details and treatment for individual equations.

Laplace's Equation	Heat Equation	Wave Equation
$\nabla^2 u = 0$ $\mathbf{r} \in D$	$\nabla^2 u = a^2 \dfrac{\partial u}{\partial t}$ $\mathbf{r} \in D\,,\ t > 0$	$\nabla^2 u = a^2 \dfrac{\partial^2 u}{\partial t^2}$ $\mathbf{r} \in D\,,\ t > 0$
Initial Conditions		
none	$u(\mathbf{r},0) = u_0(\mathbf{r})$ $\mathbf{r} \in D$	$u(\mathbf{r},0) = u_0(\mathbf{r})$ $\dfrac{\partial u(\mathbf{r},0)}{\partial t} = u_1(\mathbf{r})$ $\mathbf{r} \in D$
Boundary Conditions		

I Boundary condition of the Ist kind (**Dirichlet**)

for $t > 0$



$u|_S = f(t)$ value of unknown function u is specified at the boundary

$f(\mathbf{r},t) \quad \mathbf{r} \in S, t > 0$

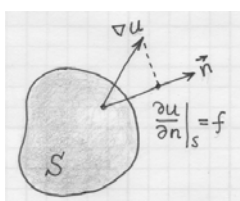
for one-dimensional case $x \in [0,L]$:

$u(0,t) = f_0(t)$

$u(0,t) = f_L(t)$

II Boundary condition of the IInd kind (**Neumann**)

for $t > 0$



$\left.\dfrac{\partial u}{\partial n}\right|_S = f(t) \qquad f(\mathbf{r},t) \quad \mathbf{r} \in S, t > 0$

for one-dimensional case $x \in [0,L]$:

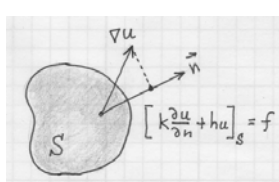
$\dfrac{\partial u(0,t)}{\partial x} = f_0(t)$

$\dfrac{\partial u(L,t)}{\partial x} = f_L(t)$

III Boundary condition of the IIIrd kind (**Robin**)

for $t > 0$

$\left.\left[k\dfrac{\partial u}{\partial n} + hu\right]\right|_S = f(t) \qquad f(\mathbf{r},t) \quad \mathbf{r} \in S, t > 0$

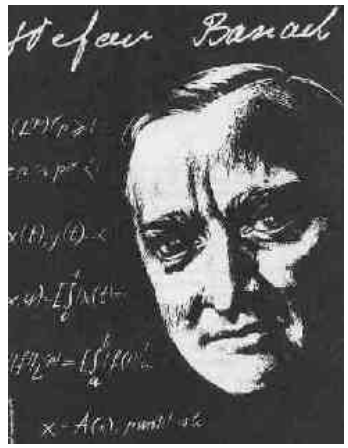


for one-dimensional case $x \in [0,L]$:

$-k\dfrac{\partial u(0,t)}{\partial x} + hu(0,t) = f_0(t)$

$k\dfrac{\partial u(L,t)}{\partial x} + hu(L,t) = f_L(t)$

4.5 Sturm-Liouville Theorem



4.5.1 Banach and Hilbert Spaces

The systematic presentation of the Banach spaces is given in the Chapter 10 "Banach Spaces". Here, only the necessary for our purpose material on the normed vector spaces is presented.

Let V be a vector space over field of real numbers $\mathbb{R}$.

1. Normed Space

Norm is a map $\|\cdot\| : V \to \mathbb{R}$ such that for all $u, v \in V$ and $c \in \mathbb{R}$

1. $\|u\| \geq 0$
   $\|u\| = 0$ if and only if $u = 0$
2. $\|cu\| = |c|\|u\|$
3. $\|u+v\| \leq \|u\| + \|v\|$ (triangle inequality)

Example: in the space $C[a,b]$ of all continuous functions defined in the closed interval $[a,b]$, the norm can be defined as

$$\|f\|_C = \max_{x\in[a,b]} |f(x)| \qquad \text{(it is called the maximum norm)}$$

2. Metric Space

Vector space V is a **metric** space if there exists a function $\rho : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$

1. $\rho(u,u) = 0$
   $\rho(u,v) > 0$ for $u \neq v$
2. $\rho(u,v) = \rho(v,u)$ (symmetry)
3. for all $\rho(u,v) \leq \rho(u,w) + \rho(w,v)$ (triangle inequality)

$\rho(u,v)$ is called the distance between $u, v \in V$.

Vector space with introduced metric is called a ***metric space***.

In the normed vector space the metric can be introduced as

$$\rho(u,v) = \|u - v\|$$

3. Inner Product

Inner product is a map $(,): V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$

1. $(u,v) = (v,u)$ (*for complex* $(u,v) = \overline{(v,u)}$)
2. $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$ $\alpha, \beta \in \mathbb{R}$
3. $(u,u) \geq 0$

   $(u,u) = 0$ if and only if $u = 0$

Vector space with introduced inner product is called an ***inner product space***. In inner product space the norm can be defined as

$$\|u\| = \sqrt{(u,u)} \qquad \text{for all } u \in V$$

4. Convergence

Let V be a normed (metric) space and let $f_k, f \in V$, $k = 1,2,...$
The sequence $f_1, f_2, ...$ converges to f if

$\lim\limits_{k\to\infty} \|f_k - f\| = 0$ or $\rho(f_k, f) \to 0$ as $k \to \infty$ for metric space.

The sequence $f_k \in V$ is called the **Cauchy sequence** (convergent in itself) if

$\lim\limits_{\substack{k\to\infty \\ m\to\infty}} \|f_k - f_m\| = 0$ or $\rho(f_k, f_m) \to 0$ as $k \to \infty$ and $m \to \infty$

The vector space V is called ***complete*** if all its Cauchy sequences are convergent in V .

A complete normed space is called the **Banach space**.
For example, $\mathbb{R}^n$ is a Banach space with $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots x_n^2}$.

A complete inner product space is called a **Hilbert space**.

5. Orthogonality

In the inner product space $u, v \in V$ are called ***orthogonal*** if $(u,v) = 0$.

If set $\{u_k\} \in V$ consists of mutually orthogonal vectors, $(u_k, u_m) = 0$ when $k \neq m$, then this set is called an ***orthogonal set***.

If in addition, $\|u_k\| = 1$, then set $\{u_k\} \in V$ is called ***orthonormal***.

Orthogonal set is linearly independent set (exercise).

If set $\{u_k\} \in V$ is linearly independent then it can be converted to the orthonormal set $\{v_k\} \in V$ with the help of the so called Gram-Schmidt orthogonalization process:

Gram-Schmidt process

$$\boxed{\begin{aligned} v_1 &= \frac{u_1}{\|u_1\|} \\ v_2 &= \frac{u_2-(u_2,v_1)v_1}{\|u_2-(u_2,v_1)v_1\|} \\ &\vdots \\ v_k &= \frac{u_k-(u_k,v_1)v_1-(u_k,v_2)v_2-\ldots-(u_k,v_{k-1})v_{k-1}}{\|u_k-(u_k,v_1)v_1-(u_k,v_2)v_2-\ldots-(u_k,v_{k-1})v_{k-1}\|} \end{aligned}}$$

This algorithm can be formalized with the help of Gram's determinant:

$$G_k = \begin{vmatrix} (u_1,u_1) & (u_2,u_1) & \cdots & (u_k,u_1) \\ (u_1,u_2) & (u_2,u_2) & \cdots & (u_k,u_2) \\ \vdots & \vdots & \ddots & \vdots \\ (u_1,u_k) & (u_2,u_k) & \cdots & (u_k,u_k) \end{vmatrix}, \quad G_0 = 1$$

Orthonormal vectors are determined by the formula

$$v_k = \frac{1}{\sqrt{G_k G_{k-1}}} \begin{vmatrix} (u_1,u_1) & (u_2,u_1) & \cdots & (u_k,u_1) \\ (u_1,u_2) & (u_2,u_2) & \cdots & (u_k,u_2) \\ \vdots & \vdots & \ddots & \vdots \\ (u_1,u_k) & (u_2,u_{k-1}) & \cdots & (u_2,u_{k-1}) \\ u_1 & u_2 & & u_k \end{vmatrix} \qquad k = 1,2,\ldots$$

The orthonormal set $\{u_k\} \in V$ is said to be **complete** if there does not exist a vector $v \neq \mathbf{0}$, $v \in V$ such that it is orthogonal to all vectors from $\{u_k\}$.

6. Fourier Series

Let $\{u_k\} \in V$ be an orthonormal set.

$\sum_{k=1}^{\infty}(f,u_k)u_k$ is called the Fourier series (**generalized Fourier series**)

(f,u_k) are called the Fourier coefficients, $c_k = (f,u_k)$

Theorem The Fourier series $\sum_{k=1}^{\infty}(f,u_k)u_k$ is convergent to the function $f \in L^2(a,b)$ if and only if

$$\sum_{k=1}^{\infty}(f,u_k)^2 = \|f\|^2 \qquad \text{(Parseval's equation)}$$

Proof: Let $f(x) = \sum_{k=1}^{\infty}(f,u_k)u_k$

$$\begin{aligned} \|f\|^2 &= (f,f) \\ &= \left(\sum_{k=1}^{\infty}(f,u_k)u_k, \sum_{k=1}^{\infty}(f,u_k)u_k\right) \end{aligned}$$

$$= \left(\sum_{k=1}^{\infty} c_k u_k , \sum_{k=1}^{\infty} c_k u_k \right)$$

$$= \sum_{k=1}^{\infty} c_k^2 \left(u_k , u_k \right) + 2 \sum_{k \neq m}^{\infty} c_k c_m \left(u_k , u_m \right)$$

$$= \sum_{k=1}^{\infty} c_k^2$$

$$= \sum_{k=1}^{\infty} \left(f , u_k \right)^2 \qquad \blacksquare$$

Let $\{u_k\} \in L^2(a,b)$ be an orthonormal set .

If for any $f \in L^2(a,b)$ its Fourier series

$$\sum_{k=1}^{\infty} \left(f, u_k \right)^2$$

converges to f in $L^2(a,b)$, then $\{u_k\}$ is said **complete** in $L^2(a,b)$.

7. Vector Space L^2

Consider a particular case of Equation 3.3 from Definition 3.13 (p.205), with $p = 2$ and interval $I = [a,b]$:

$$L^2(a,b) = \left\{ \varphi : (a,b) \to \mathbb{R} \middle| \int_a^b \varphi^2(x)\, dx < \infty \right\}$$

Inner product in vector space $L^2(a,b)$: For $u, v \in L^2(a,b)$ define:

$$(u,v) = \int_a^b u(x) v(x)\, dx \qquad \text{inner product in } L^2(a,b)$$

$$(u,v)_p = \int_a^b u(x) v(x)\, p(x)\, dx \qquad \text{weighted inner product in } L^2(a,b)$$

with the weight function $p(x) > 0$

Inner product vector space $L^2(a,b)$ belongs to the class of Hilbert spaces.

Introduced inner product induces the norm in $L^2(a,b)$:

$$\|u\| = \int_a^b u^2(x) dx$$

$$\|u\|_p = \int_a^b u^2(x)\, p(x) dx$$

Historically, the first complete set was used by Fourier set of trigonometric functions $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \sin kx \right\}$, $k = 1, 2, \ldots$ in the interval $(0, 2\pi)$.

The complete orthogonal sets used in the solution of PDE will be generated by the solution of the Sturm-Liouville problems.

8. Exercize: The set $\{1, x, x^2, x^3, \ldots\}$ is linearly independent in $L^2(-1,1)$.

a) Using the Gram-Schmidt orthogonalization algorithm with inner product

$$(u,v) = \int_{-1}^{1} u(x)v(x)dx$$

construct an orthonormal set in $L^2(-1,1)$
(the obtained set will be the set of the Legendre polynomials up to the scalar multiple).

b) Using the Gram-Schmidt orthogonalization algorithm with inner product

$$(u,v) = \int_{-1}^{1} \frac{u(x)v(x)}{\sqrt{1-x^2}}dx$$

construct an orthonormal set in $L^2(-1,1)$
(the obtained set will be the set of the Tchebyshev polynomials up the the scalar multiple).

c) Use the obtained orthonormal sets for generalized Fourier series expansion of the function:

$$f(x) = \begin{cases} -1 & x \in (-1,0) \\ 1 & x \in (0,1) \end{cases}$$

Compare the results for truncated series with 2,3,4 terms. Make some observations.

Lvov University where Stefan Banach worked in 1919-1945

9. Generalized Fourier Series

Recall from Chapter 3, that functions $u(x) \in L_2[x_1, x_2]$ can be represented by generalized Fourier series in some interval $[x_1, x_2]$

$$u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x) \tag{1}$$

Where coefficients are

$$u_n = \frac{\int_{x_1}^{x_2} u(x)\phi_n(x)p(x)dx}{\int_{x_1}^{x_2} \phi_n^2(x)p(x)dx}$$

and the set of functions $\{\phi_n(x)\}$ is complete in $L_2[x_1, x_2]$ the set of orthogonal functions over the inner product with a weight function $p(x)$:

$$\int_{x_1}^{x_2} \phi_m(x)\phi_n(x)p(x)dx = 0 \text{ if } m \neq n$$

Completeness of the set $\{\phi_n(x)\}$ in $L_2[x_1, x_2]$ means that any function $u(x) \in L_2[x_1, x_2]$ can be represented by generalized Fourier series. Conditions for that are established in Chapter 3 by the Dirichlet Theorem. In these equations we used the definition of a weighted inner product in the space $L_2(x_1, x_2)$

$$(u, v)_p = \int_{x_1}^{x_2} u(x)v(x)p(x)dx \tag{2}$$

which can be used to define the norm in space $L_2[x_1, x_2]$ as

$$\|u\|_p = \left[\int_{x_1}^{x_2} u^2(x)p(x)dx\right]^{1/2} \tag{3}$$

Analytical solution of IBVP's for PDE's will require the construction of such complete orthogonal sets of basis functions which are used for deriving solutions, which satisfies the differential equation and initial and boundary conditions. First, we will see the appearance of already known sets which are used in traditional Fourier series $\left\{1, cos\frac{n\pi}{L}x\right\}$, $\left\{sin\frac{n\pi}{L}x\right\}$ and sets for quarter-range expansions. Then novell sets will appear which happen to posses the same property of orthogonality and completeness. Generation of such orthogonal sets is provided by the solution associated with a PDE eigenvalue problem for the differential operator L acting on one of the space variables. This eigenvalue problem is formulated in traditional form (which we have already seen in linear algebra for eigenvalue problems for linear transformations defined by matrices):

Eigenvalue Problem

Find the values of parameter λ for which the operator equation

$$Lu = \lambda u \tag{4}$$

subject to boundary conditions has a non-trivial solution.

Under some conditions, this eigenvalue problem will generate the required set of orthogonal functions. These conditions are formulated in the fundamental form for analytical theory of PDE's as the regular Sturm-Liouville Theorem.

4.5.2 Regular Sturm-Liouville Problem

Consider the homogeneous differential equation

$$[r(x)u']' + [q(x) + \lambda p(x)]u = 0 \tag{5}$$

subject to boundary conditions

$$a_1 u(x_1) - b_1 u'(x_1) = 0 \tag{6}$$
$$a_2 u(x_2) + b_2 u'(x_2) = 0 \tag{7}$$

where coefficients $a_1, a_2, b_1, b_2 \geq 0$, and $a_1 + b_1 > 0$, $a_2 + b_2 > 0$ (8)

and functions $p(x) > 0$, $p(x) \in C[x_1, x_2]$ (9)

$r(x) > 0$, $r(x) \in C^1[x_1, x_2]$ (10)

$q(x) \in C(x_1, x_2)$ (11)

Find values of the parameter λ for which differential equation (5) subject to boundary conditions (6,7) has a non-trivial solution

$$u(x) \in C^2(x_1, x_2) \cup C^1[x_1, x_2],\ u''(x) \in L_2(x_1, x_2)$$

(Remark: coefficients in boundary conditions (6-7) are assumed to be non-negative $a_1, a_2, b_1, b_2 \geq 0$, because we want them to represent physical properties of the medium).

Sturm-Liouville Operator

Consider the differential operator $L : C^2(x_1, x_2) \cup C^1[x_1, x_2] \to L_2(x_1, x_2)$

$$Lu \equiv -\frac{1}{p}\left[(ru')' + qu\right] \tag{12}$$

which rewrites the Sturm-Liouville problem in the form

$$Lu = \lambda u \tag{13}$$

This is an eigenvalue problem for the differential operator L consisting in finding the values of parameter λ (eigenvalues) for which the operator equation (13) subject to boundary conditions (6-7) has non-trivial solutions (eigenfunctions).

4.5.3 Sturm-Liouville Theorem

The following statements hold for a Sturm-Liouville Problem:

1) The *differential operator L:*

if u_1 and u_2 are solutions of SLP (5-11), then

a) operator L is Hermitian (also called self-adjoint or self-conjugate)

$$(Lu_1, u_2)_p = (u_1, Lu_2)_p \tag{14}$$

b) operator L is positive

$$(Lu, u)_p \geq 0 \tag{15}$$

2) *Eigenvalues* have the following properties:

a) there are infinitely many values λ_n, $n = 0,1,2,...$ for which the Sturm-Liouville Problem has non-trivial solutions $u_n(x)$ (eigenfunctions);
b) all eigenvalues are real, $\lambda_n \in R$;
c) all eigenvalues are non-negative, $\lambda_n \geq 0$;
d) $\lambda_0 = 0$ only if $q = 0$ and $a_1 = a_2 = 0$ (Neumann boundary conditions)
e) eigenvalues can be organized in increasing order $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < ...$

3) *Eigenfunctions* have the following properties:

a) all eigenfunctions can be chosen real $u_n(x): R \to R$
b) if $u_n(x)$ is an eigenfunction then $cu_n(x)$ for any $c \in R$ is also an eigenfunction (eigenvalue λ_n generates a linear eigenspace);
c) dimension of each eigenspace is one (all λ_n are simple – there is only one linearly independent eigenfunction corresponding to λ_n);
d) the set of all eigenfunctions $\{u_n(x)\}$ is orthogonal with respect to the weight function $p(x)$
$$(u_m, u_n)_p = 0 \text{ if } m \neq n$$
e) the set $\{u_n(x)\}$ is complete in $L_2(x_1, x_2)$ i.e.
for any $f \in L_2(x_1, x_2)$ Fourier series $\sum_{k=0}^{\infty} c_k u_k(x) \to f(x)$
where coefficients $c_k = \dfrac{(f, u_k)_p}{(u_k, u_k)_p}$

f) if $u_n(x)$ is an eigenfunction corresponding to the eigenvalue λ_n, then $u_n(x)$ has exactly *n* zeroes in the open interval (x_1, x_2);
between two successive zeroes of $u_n(x)$ and also between x_1 and the first zero and between x_2 and the last zero there is exactly one zero of $u_{n+1}(x)$.

Proof:

1 a) Let u_1 and u_2 be two eigenfunctions. Therefore, they satisfy equation (13) with boundary conditions (6-7) in which assume that $a_1, a_2 > 0$ (a similar derivation is valid also in the assumption $b_1, b_2 > 0$ or their combination), then

$$Lu_1 = \lambda u_1 \quad a_1 u_{1,2}(x_1) - b_1 u'_{1,2}(x_1) = 0 \quad \Rightarrow \quad u_{1,2}(x_1) = \frac{b_1}{a_1} u'_{1,2}(x_1) \qquad (17)$$

$$Lu_2 = \lambda u_2 \quad a_2 u_{1,2}(x_2) - b_2 u'_{1,2}(x_2) = 0 \quad \Rightarrow \quad u_{1,2}(x_2) = \frac{b_2}{a_2} u'_{1,2}(x_2) \qquad (18)$$

Consider $\quad (Lu_1, u_2)_p - (u_1, Lu_2)_p \quad$ (use equation (12))

$$= \int_{x_1}^{x_2} \left(-\frac{1}{p}\left[(ru'_1)' + qu_1 \right] \right) u_2 p dx - \int_{x_1}^{x_2} \left(-\frac{1}{p}\left[(ru'_2)' + qu_2 \right] \right) u_1 p dx \quad \text{(expand integrands)}$$

$$= \int_{x_1}^{x_2} \left[-r'u_1'u_2 - ru_1''u_2 - qu_1u_2 + r'u_2'u_1 + ru_2''u_1 + qu_2u_1 \right] dx \qquad \text{(organize terms)}$$

$$= \int_{x_1}^{x_2} \left[r'\left(u_2'u_1 - u_1'u_2\right) + r\left(u_2''u_1 - u_1''u_2 + u_1'u_2' - u_2'u_1'\right) \right] dx \qquad \text{(add } \quad u_1'u_2' - u_1'u_2' \text{)}$$

$$= \int_{x_1}^{x_2} \left[r\left(u_2'u_1 - u_1'u_2\right) \right]' dx \qquad \text{(integrate expression)}$$

$$= \left[r\left(u_2'u_1 - u_1'u_2\right) \right]_{x_2} - \left[r\left(u_2'u_1 - u_1'u_2\right) \right]_{x_1} \qquad \text{(use equations (17,18))}$$

$$= -r(x_2)u_2'(x_2)\frac{b_2}{a_2}u_1'(x_2) + r(x_2)u_1'(x_2)\frac{b_2}{a_2}u_2'(x_2) - r(x_1)u_2'(x_1)\frac{b_1}{a_1}u_1'(x_1) + r(x_1)u_1'(x_1)\frac{b_1}{a_1}u_2'(x_1)$$

$$= 0$$

Therefore, $\left(Lu_1, u_2\right)_p = \left(u_1, Lu_2\right)_p$ and the operator L is Hermitian. ■

1 b) operator L is positive

$$\left(Lu_n, u_n\right)_p = \int_{x_1}^{x_2} u_n^2(x)p(x)dx \geq 0 \qquad \text{(because from condition (9), } p(x) > 0 \text{)} \ ■$$

2 b) Let λ_n be an eigenvalue and u_n be the corresponding eigenfunction, then $Lu_n = \lambda u_n$. Because the operator L is Hermitian, $\left(Lu_n, u_n\right)_p = \left(u_n, Lu_n\right)_p$. Then

$$\lambda_n\left(u_n, u_n\right)_p = \left(\lambda_n u_n, u_n\right)_p = \left(Lu_n, u_n\right)_p = \left(\lambda u_n, Lu_n\right)_p = \left(u_n, \lambda_n u_n\right)_p = \overline{\lambda}_n\left(u_n, u_n\right)_p$$

$\Rightarrow \ \lambda_n = \overline{\lambda}_n$. Therefore, eigenvalue λ_n is real, $\lambda_n \in R$. ■

2 c) Let λ_n be an eigenvalue and u_n be the corresponding eigenfunction, then $Lu_n = \lambda u_n$. Consider

$\left(Lu_n, u_n\right)_p = \left(\lambda_n u_n, u_n\right)_p = \lambda_n\left(u_n, u_n\right)_p = \lambda_n \left\|u_n\right\|^2$. Then

$$\lambda_n = \frac{\left(Lu_n, u_n\right)_p}{\left\|u_n\right\|^2} \geq 0 \qquad \text{(because the operator } L \text{ is positive (2b))} \ ■$$

3 d) Let u_m and u_n be two eigenfunctions: $Lu_m = \lambda u_m$ and $Lu_n = \lambda u_n$.

Consider

$$\begin{aligned} \lambda_m\left(u_m, u_n\right)_p &= \left(\lambda_m u_m, u_n\right)_p \\ &= \left(Lu_m, u_n\right)_p \\ &= \left(u_m, Lu_n\right)_p && \text{(operator is Hermitian)} \\ &= \left(u_m, \lambda_n u_n\right)_p \\ &= \lambda_n\left(u_m, u_n\right)_p && \text{(eigenvalues are real)} \end{aligned}$$

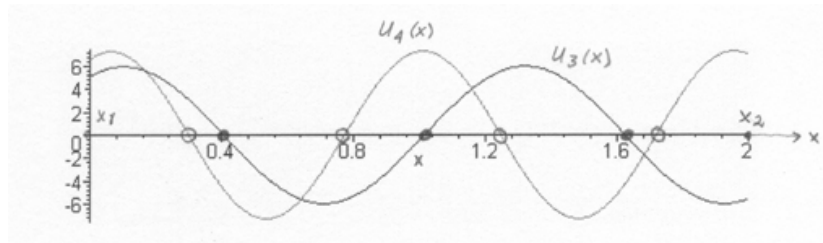
Therefore, $\left(\lambda_m - \lambda_n\right)\left(u_m, u_n\right)_p = 0$

If $m \neq n$, then $\lambda_m - \lambda_n \neq 0$ and

$$\left(u_m, u_n\right)_p = 0$$

Therefore, eigenfunctions u_m and u_n are orthogonal if $m \neq n$.

3 f) Illustration of this property with the eigenfunctions $u_3(x)$ and $u_4(x)$ obtained from the solution of equation (20) with Robin-Robin boundary conditions in the interval $[0,2]$



4.5.4 Reduction to self-adjoint form

If the linear differential equation with parameter λ is given in standard form

$$a_0(x)u'' + a_1(x)u' + \left[a_2(x) + \lambda\right]u = 0$$

then with the help of the multiplication factor

$$\mu(x) = \frac{e^{\int \frac{a_1(x)}{a_0(x)}dx}}{a_0(x)} \tag{19}$$

it can be reduced to the self-adjoint form

$$\left[a_0(x)\mu(x)u'\right]' + \left[a_2(x)\mu(x) + \lambda\mu(x)\right] = 0$$

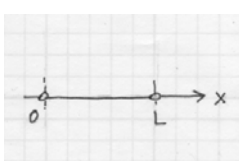
The corresponding coefficients of equation (5) can be identified as

$$r(x) = a_0(x)\mu(x) = e^{\int \frac{a_1(x)}{a_0(x)}dx} > 0$$

$$p(x) = \mu(x) = \frac{e^{\int \frac{a_1(x)}{a_0(x)}dx}}{a_0(x)} > 0 \quad \text{if } a_0(x) > 0$$

4.5.5 Sturm-Liouville problem for equation $\boxed{X'' - \mu X = 0}$

Consider a boundary value problem which is important for solution of classical PDE's in the Cartesian coordinate system:



$$X'' - \mu X = 0 \qquad x \in [0,L] \tag{20}$$

$$-k_1 X'(0) + h_1 X(0) = 0 \tag{21}$$

$$k_2 X'(L) + h_2 X(L) = 0 \tag{22}$$

Depending on coefficients, boundary conditions can be in one of the three classical types. There are nine possible different combinations of boundary conditions which yield different solutions.
We can see that equation (20) can be written in the Sturm-Liouville form (5) which produces non-negative eigenvalues only if the separation constant μ is assumed to be non-positive $\mu = -\lambda_n^2$:

$$-\left[X'\right]' = \lambda^2 X \tag{23}$$

In this equation, we identify: $r = 1 > 0$, $q = 0$, $p = 1 > 0$. The general solution of this 2nd order homogeneous linear ODE with constant coefficients is given by

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \tag{24}$$

The solution of Sturm-Liouville will consist in finding non-trivial solutions which satisfy boundary conditions (21-22). Consider some particular cases of boundary conditions:

1) **The case of Dirichlet-Dirichlet boundary conditions:**

$$X(0)=0 \quad \text{Dirichlet} \tag{25}$$

$$X(L)=0 \quad \text{Dirichlet} \tag{26}$$

(Remark: mathematically, Dirichlet boundary conditions are obtained from the general case, when in equations (21-22) coefficients $k_1 = k_2 = 0$; but, physically, it corresponds, for example, to zero thermal conductivity in the domain for heat transfer which is not acceptable for modeling. For physical consistency, we rewrite equations (21-22) in the form

$$-\frac{k_1}{h_1}X'(0)+X(0)=0$$

$$\frac{k_2}{h_2}X'(L)+X(L)=0$$

and assume that coefficients of convective heat transfer are very high, $h_1, h_2 \to \infty$, which corresponds to a physically acceptable assumption of negligible convective thermal resistance at the boundary of the domain. This case is treated as thermostating of the boundaries).

Substitution of solution (24) into the first boundary condition (25) yields

$$X(0)= \quad c_1 \cos\lambda 0 + c_2 \sin\lambda 0 = 0$$

$$c_1 \cdot 1 + c_2 \cdot 0 = 0$$

And from this the first coefficient $c_1 = 0$.

Then general solution (24) reduces to

$$X(x)=c_2 \sin\lambda x \tag{27}$$

Substitute it into the second boundary condition (26)

$$X(L)= c_2 \sin\lambda L = 0$$

Because one coefficient in the solution (24), c_1 is already assumed to be zero, for a non-trivial solution $X(x)$, the second coefficient should not be equal to zero. Therefore, the following equation should be satisfied

$$\sin\lambda L = 0 \tag{28}$$

This equation has infinitely many solutions

$$\lambda L = n\pi$$

where n is any integer. But we have to restrict ourself only to positive values of n (because negative values with odd function (27) do not satisfy equation (23), and zero yields the trivial solution). Therefore, the values of parameter λ for which we have non-trivial solutions of BVP (23,25,26) are eigenvalues

$$\lambda = \frac{n\pi}{L} \qquad n = 1,2,3,... \tag{29}$$

Then corresponding to these values of parameter solutions are eigenfunctions

$$X_n(x) = \sin\lambda_n x = \sin\frac{n\pi}{L}x \qquad n = 1,2,3,... \tag{30}$$

According to the Sturm-Liouville Theorem, this set of functions should be a complete set of functions orthogonal on the interval $[0,L]$ with the weight function $p = 1$, which yields already known Fourier sine series expansion

$$u(x)=\sum_{k=1}^{\infty} a_k \sin\frac{n\pi}{L}x \qquad \text{where } a_k = \frac{\int_0^L u(x)\sin\frac{n\pi}{L}x\,dx}{\int_0^L \left(\sin\frac{n\pi}{L}x\right)^2 dx} = \frac{2}{L}\int_0^L u(x)\sin\frac{n\pi}{L}x\,dx$$

The Sturm-Liouville Problem (20-22) with Neumann-Neumann boundary Conditions generates a set of eigenfunctions $\left\{1, \cos\frac{n\pi}{L}x\right\}$ which is known as a Fourier cosine series. The combinations of Neumann-Dirichlet and Dirichlet-Neumann boundary conditions generate sets of eigenfunctions familiar from quarter-range Fourier expansions. Other combinations of boundary conditions do not produce traditional sets. Consider, for example, Robin-Dirichlet boundary conditions:

2) **The case of Robin-Dirichlet boundary conditions** (*m10.mws*):

$-k_1 X'(0) + h_1 X(0) = 0$	Robin	(denote $H_1 = \frac{h_1}{k_1}$)	(31)
$X(L) = 0$	Dirichlet		(32)

Application of the second boundary condition (32) first, eliminates one of the constants immediately, if the general solution (24) is rewritten in equivalent shifted form:

$$X(x) = c_1 \cos\lambda(x-L) + c_2 \sin\lambda(x-L) \qquad (24')$$

Substitution of solution (24) into the second boundary condition (32) yields

$$X(L) = \quad c_1 \cos\lambda 0 + c_2 \sin\lambda 0 = 0$$
$$c_1 \cdot 1 + c_2 \cdot 0 = 0$$

from which yields that the first coefficient $c_1 = 0$. Then the solution reduces to

$$X(x) = c_2 \sin\lambda(x-L)$$

Application of the Robin boundary condition requires also its derivative

$$X'(x) = c_2\lambda \cos\lambda(x-L)$$

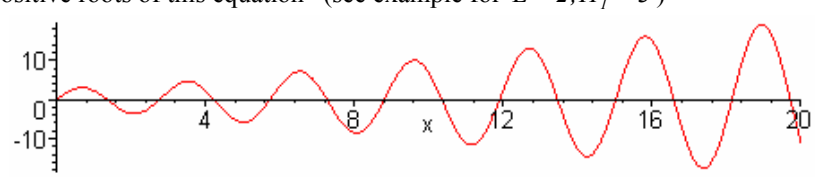
Then at $x = 0$

$$-X'(0) + H_1 X(0) \quad = -c_2\lambda \cos\lambda(0-L) + H_1 \sin\lambda(0-L)$$
$$= c_2\left(-\lambda\cos\lambda L - H_1 \sin\lambda L\right) = 0$$

For the non-trivial solution $c_2 \neq 0$, and, therefore

$$\lambda \cos\lambda L + H_1 \sin\lambda L = 0 \qquad (33)$$

is a characteristic equation for eigenvalues λ_n. There are infinitely many positive roots of this equation (see example for $L = 2, H_1 = 3$)



Though $\lambda_0 = 0$ is a root of the characteristic equation, it is not an eigenvalue, because it produces a trivial solution.

The set of eigenfunctions for this Sturm-Liouville Problem is

$$X_n(x) = \left\{\sin\lambda_n(x-L)\right\} \quad n = 1,2,3,... \qquad (34)$$

where eigenvalues λ_n are positive roots of the characteristic equation (33).

The norm (3) of eigenfunctions can be evaluated as

$$\|X_n\|_{p=1} = \left[\int_{x_1}^{x_2} X_n^2(x)dx\right]^{1/2} = \left[\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}\right]^{1/2} \qquad (35)$$

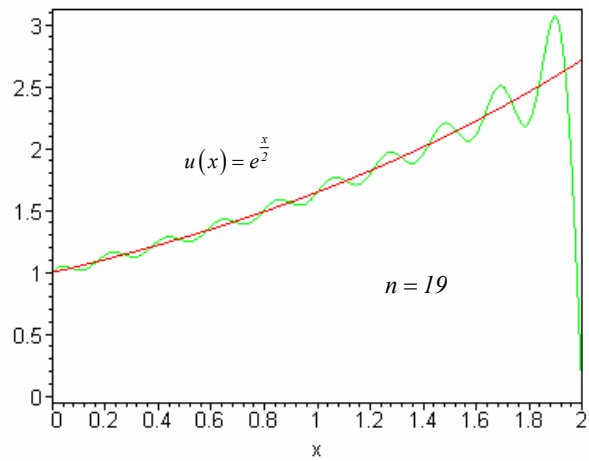
According to the Sturm-Liouville problem set (34) is complete and can be used for expansion of functions in a generalized Fourier series (1):

$$u(x) = \sum_{n=1}^{\infty} u_n X_n(x) = \sum_{n=1}^{\infty} u_n \sin\lambda_n(x-L) \qquad (36)$$

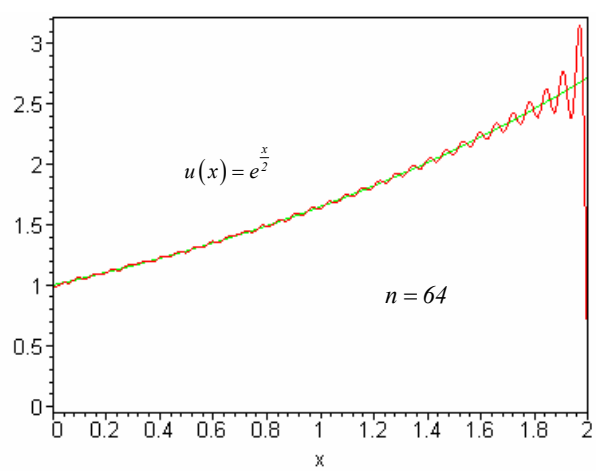
where

$$u_n = \frac{\int_0^L u(x) \sin \lambda_n (x-L)\, dx}{\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}$$

Example Consider expansion of the function $u(x) = exp(x/2)$ on the interval $[0,2]$. The graph shows the function $u(x)$ and its expansion in a Fourier series with *19* terms in summation:



With an increase of the number of terms (the next graph shows an approximation with *64* terms in the summation (36)), approximation by truncated Fourier series improves, there still is a presence of Gibb's phenomena at the right boundary because eigenfunctions satisfying homogeneous Dirichlet boundary conditions are zero at $x = L$ and there is a discontinuity with the non-zero value of the function $u(x)$ at this point.



Results for all combinations of types of boundary conditions for Sturm-Liouville problems (20-22) are collected in the table Sturm-Liouville Problems. The table includes the kernel of integral transforms based on the corresponding boundary value problem, which in Chapter 6 will be used for solution of IBVP for PDE in the finite domains. Notice, that $\lambda_0 = 0$ is an eigenvalue only for the case of Neumann-Neumann boundary conditions when both coefficients $h_1 = h_2 = 0$ (see 2d) of the Sturm-Liouville Theorem).

sturm-liouville problem

$H_1 = \frac{h_1}{k_1} \qquad H_2 = \frac{h_2}{k_2}$

$X'' - \mu X = 0 \qquad X(x) \quad x \in [0, L]$

boundary conditions	eigenvalues $\mu_n = -\lambda_n^2$	eigenfunctions X_n	norm $\|X_n\|^2 = \int_0^L X_n^2(x)dx$	kernel $K_n(x) = \frac{X_n(x)}{\|X_n\|}$
Dirichlet $X(0) = 0$ Dirichlet $X(L) = 0$	$\lambda_n = \frac{n\pi}{L}$ $\quad n = 1,2,\ldots$	$\sin\frac{n\pi}{L}x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$
Neumann $X'(0) = 0$ Dirichlet $X(L) = 0$	$\lambda_n = \left(n+\frac{1}{2}\right)\frac{\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\cos\left(n+\frac{1}{2}\right)\frac{\pi}{L}x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}}\cos\left(n+\frac{1}{2}\right)\frac{\pi}{L}x$
Dirichlet $X(0) = 0$ Neumann $X'(L) = 0$	$\lambda_n = \left(n+\frac{1}{2}\right)\frac{\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\sin\left(n+\frac{1}{2}\right)\frac{\pi}{L}x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}}\sin\left(n+\frac{1}{2}\right)\frac{\pi}{L}x$
Neumann $X'(0) = 0$ Neumann $X'(L) = 0$	$\lambda_n = \frac{n\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\cos\frac{n\pi}{L}x$	$L \quad n = 0$ $\frac{L}{2} \quad n = 1,2,\ldots$	$\frac{1}{\sqrt{L}}$ $\sqrt{\frac{2}{L}}\cos\frac{n\pi}{L}x$
Dirichlet $X(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	λ_n are positive roots of $\lambda\cos\lambda L + H_2\sin\lambda L = 0$ $\quad n = 1,2,\ldots$	$\sin\lambda_n x$	$\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\sin\lambda_n x}{\sqrt{\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Neumann $X'(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	λ_n are positive roots of $\lambda\sin\lambda L - H_2\cos\lambda L = 0$ $\quad n = 1,2,\ldots$	$\cos\lambda_n x$	$\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\cos\lambda_n x}{\sqrt{\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Dirichlet $X(L) = 0$	λ_n are positive roots of $\lambda\cos\lambda L + H_1\sin\lambda L = 0$ $\quad n = 1,2,\ldots$	$\sin\lambda_n(x - L)$	$\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\sin\lambda_n(x-L)}{\sqrt{\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Neumann $X'(L) = 0$	λ_n are positive roots of $\lambda\sin\lambda L - H_1\cos\lambda L = 0$ $\quad n = 1,2,\ldots$	$\cos\lambda_n(x - L)$	$\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\cos\lambda_n(x-L)}{\sqrt{\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	λ_n are positive roots of $(H_1H_2 - \lambda^2)\sin\lambda L + (H_1 + H_2)\lambda\cos\lambda L = 0$ $n = 1,2,\ldots$	$\lambda_n\cos\lambda_n x + H_1\sin\lambda_n x$	$\frac{(\lambda_n^2 + H_1^2)}{2}\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) + \frac{H_1}{2}$	$\frac{\lambda_n\cos\lambda_n x + H_1\sin\lambda_n x}{\sqrt{\frac{(\lambda_n^2 + H_1^2)}{2}\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) + \frac{H_1}{2}}}$

$\{X_n(x)\}$ is a complete set of orthogonal functions on [0,L]

$$\int_0^L X_n(x)X_m(x)dx = \begin{cases} \|X_n\|^2 & n = m \\ 0 & n \quad m \end{cases}$$

finite Fourier transform

$$F_n = \int_0^L K_n(x)f(x)dx$$

generalized Fourier series

$$f(x) = \sum_n a_n X_n(x) \qquad a_n = \frac{\int_0^L X_n(x)f(x)dx}{\|X_n\|^2}$$

inverse transform (Fourier series)

$$f(x) = \sum_n^\infty F_n K_n(x)$$

4.5.6 Sturm-Liouville problem for Bessel equation in the circular domain

Solution of these Sturm-Liouville Problems will be shown in detail in the Chapter 5 on Special Functions (Section 12). Analysis of corresponding problems will also be performed later in this chapter during the solution of PDE's in cylindrical coordinates. Here we indicate some general results.

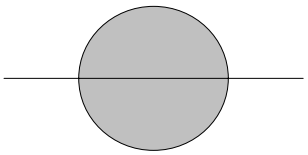
Consider a Bessel equation of order ν with parameter λ

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y = 0$$

With the help of the multiplying factor $\mu(x) = \dfrac{1}{x}$

It can be reduced to a self-adjoint form

$$\left[xy'\right]' + \left(-\frac{\nu^2}{x} + \lambda^2 x\right) y = 0 \qquad \text{(identify } p(x) = x\text{)}$$



Then, the Sturm-Liouville Problem in the circular domain, interval $x \in [0, L]$, produces infinitely many values of the parameter λ_n (eigenvalues) for which there exist non-trivial solutions $y_n(x)$ (eigenfunctions):

$$y_n(x) = \left\{J_\nu(\lambda_n x)\right\}$$

which are Bessel functions of the 1st kind of order ν (the Bessel functions of the 2nd kind Y_ν are not included in the solution because they are unbounded at $x = 0$). The characteristic equation for eigenvalues is determined by the boundary condition.

According to the Sturm-Liouville theorem, eigenfunctions are orthogonal with the weight function $p(x) = x$:

$$\int_0^L x y_n(x) y_m(x)\, dx = 0 \quad \text{for } n \neq m$$

Bessel-Fourier Series Obtained orthogonal systems can be used for expansion of functions in generalized Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n J_\nu(\lambda_n x)$$

where coefficients c_n are determined from the equation

$$a_n = \frac{\int_0^L x J_\nu(\lambda_n x) f(x)\, dx}{\int_0^L x J_\nu^2(\lambda_n x)\, dx} = \frac{\int_0^L x J_\nu(\lambda_n x) f(x)\, dx}{N_{\nu,n}^2}$$

The Chapter 5 on Special Functions includes also an analysis of the Sturm-Liouville problem for a Bessel Equation in an annular domain, and the Sturm-Liouville problem for the Legendre equation which yield complete orthogonal sets used for the solution of IBVP for PDE.

4.6 Method of Separation of Variables

4.6.1 Separation of variables

The classical analytical approach to the solution of initial-boundary value problems of equations of mathematical physics is based on the method of separation of variables. This method consists in building the set of basic functions which is used in developing solutions in the form of an infinite series expansion over the basic functions.

Thus for a 2-dimensional problem, the unknown function $u(x,y)$ is assumed to be represented as a product of two functions each of a single variable:

$$u(x,y) = X(x)Y(y)$$

where both $X(x)$ and $Y(y)$ are non-zero functions. Substitution of the assumed form of the solution into a differential equation (consider the Laplace Equation) yeilds a separated equation

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

where the left hand side and right hand side depend on different variables, and therefore, do not depend on either of them. Indeed, differentiate the equation with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{X''}{X}\right) = 0$$

then by integration we obtain

$$\frac{X''}{X} = \mu$$

with μ as a constant of integration.

Therefore,

$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y} = \mu}$$

where μ is called a *separation constant*. It yields two ordinary differential equations:

$$X'' - \mu X = 0$$

and

$$Y'' + \mu Y = 0$$

with corresponding boundary conditions. At this point the Sturm-Liouville Theorem will provide the existence of the set of eigenvalues μ_n and eigenfunctions $X_n(x)$ and $Y_n(y)$ which will be a basis for construction of the solution in the form of infinite series:

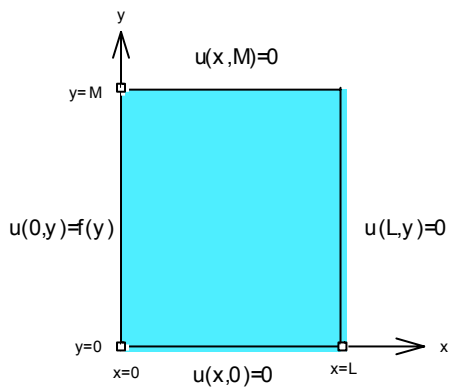
$$u(x,y) = \sum_n c_n X_n(x) Y_n(y)$$

where coefficients c_n should be determined in the process of solution.

But, first we will follow the traditional approach to investigate the solution set of appropriate boundary value problems. It will be shown why only the solution of the Sturm-Liouville problem yields the correct solution of the PDE which satisfies the corresponding initial and boundary conditions.

4.6.2 Laplace's Equation (LE) *Dirichlet Problem* *Basic Case: 3 homogeneous boundary conditions*

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0} \qquad u(x,y): \quad (x,y)\in D=(0,L)\times(0,M)$$



boundary conditions:

x=0	u(0,y) = f(y)
x=L	u(L,y) = 0
y=0	u(x,0) = 0
y=M	u(x,M) = 0

Hence LE is homogeneous, it obviously possesses a trivial solution $u(x,y)\equiv 0$. It has no interest for us, therefore, we will look for a non-trivial solution.

1. Separation of variables

We assume that the function $u(x,y)$ can be represented as a product of two functions each of a single variable:

$$u(x,y)=X(x)Y(y)$$

where both $X(x)$ and $Y(y)$ are non-zero functions.

Differentiate the function $u(x,y)$ consequently with respect to x and y:

$$\frac{\partial u}{\partial x}=X'Y \qquad \frac{\partial^2 u}{\partial x^2}=X''Y$$

$$\frac{\partial u}{\partial y}=XY' \qquad \frac{\partial^2 u}{\partial y^2}=XY''$$

and substitute the second order derivatives into LE:

$$X''Y+XY''=0$$

Divide this equation by the product XY and separate the terms

$$\frac{X''}{X}=-\frac{Y''}{Y}$$

It yields an equation with separated variables: the left hand side of this equation is a function of independent variable x only, and the right hand side is a function of the independent variable y only. The equality for all values of x and y is possible only if both sides are equal to the same constant (call it μ). Indeed, differentiate the equation with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{X''}{X}\right)=0$$

then by integration we obtain

$$\frac{X''}{X}=\mu$$

Therefore,

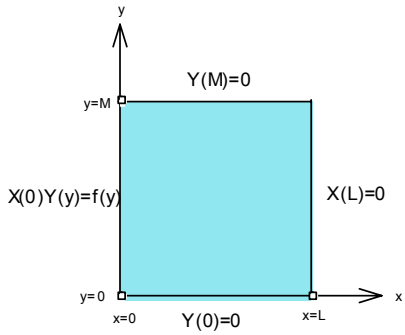
$$\boxed{\frac{X''}{X}=-\frac{Y''}{Y}=\mu}$$

where μ is called a *separation constant*. It yields two ordinary differential equations:

$$X''-\mu X=0 \text{ and } Y''+\mu Y=0$$

2. Boundary conditions

Boundary conditions are given for the function $u(x,y)$. Determine, what conditions should be satisfied by functions X and Y (assuming that both are non-trivial)



boundary conditions:

x=0	X(0)Y(y) = f(y)		
x=L	X(L)Y(y) = 0	$\Longrightarrow$	X(L) = 0
y=0	X(x)Y(0) = 0	$\Longrightarrow$	Y(0) = 0
y=M	X(x)Y(M) = 0	$\Longrightarrow$	Y(M) = 0

3. Solution of o.d.e.

Start with the equation for which <u>both boundary conditions are homogeneous:</u>

$$\boxed{Y'' + \mu Y = 0} \qquad Y(y) = \begin{cases} d_1 \cos\sqrt{\mu}\,y + d_2 \sin\sqrt{\mu}\,y & \lambda > 0 \\ d_1 + d_2 y & \lambda = 0 \\ d_1 \cosh\sqrt{-\mu}\,y + d_2 \sinh\sqrt{-\mu}\,y & \lambda < 0 \end{cases}$$

Consider the first boundary condition at $\underline{y = 0}$:

$$Y(0) = 0 = \begin{cases} d_1 \cos\sqrt{\mu}\,(0) + d_2 \sin\sqrt{\mu}\,(0) = d_1 & \Rightarrow \; d_1 = 0 \\ d_1 + d_2\,(0) = d_1 & \Rightarrow \; d_1 = 0 \\ d_1 \cosh\sqrt{-\mu}\,(0) + d_2 \sinh\sqrt{-\mu}\,(0) = d_1 & \Rightarrow \; d_1 = 0 \end{cases}$$

Therefore, the solution becomes

$$Y(y) = \begin{cases} d_2 \sin\sqrt{\mu}\,y \\ d_2 y \\ d_2 \sinh\sqrt{-\mu}\,y \end{cases}$$

Because we are looking for a non-trivial solution, the constant d_2 should not be equal to zero, $d_2 \neq 0$. Consider the second boundary condition at $\underline{y = M}$:

$$Y(M) = 0 = \begin{cases} d_2 \sin\sqrt{\mu}M = 0 & & \\ d_2 M = 0 & \Rightarrow & d_2 = 0 \Rightarrow d_1 \,\&\, d_2 = 0 \quad \textit{trivial solution} \\ d_2 \sinh\sqrt{-\mu}M = 0 & \Rightarrow & d_2 = 0 \Rightarrow d_1 \,\&\, d_2 = 0 \quad \textit{trivial solution} \end{cases}$$

$\Rightarrow \quad d_2 \sin\sqrt{\mu}M = 0$, where $d_2 \neq 0$ if we want only non-trivial solutions

$\Rightarrow \quad \sin\sqrt{\mu}M = 0 \quad \Rightarrow \quad \sqrt{\mu}M = n\pi$, where $n = 1,2,3,...$

$$\Rightarrow \quad \boxed{\mu_n = \frac{n^2\pi^2}{M^2}} \qquad n = 1,2,3,...$$

are values of the parameter μ for which the differential equation with boundary conditions has non-trivial solutions. They are said to be **eigenvalues**. Corresponding solutions are said to be **eigenfunctions**:

$$Y_1(y) = \sin\frac{\pi}{M}y$$

$$\vdots$$

$$Y_n(y) = \sin\frac{n\pi}{M}y$$

$$\vdots$$

(for simplicity, we choose $d_2 = 1$, using the fact that any multiple of an eigenfunction is also an eigenfunction).

solution for X

We already know that the separation constant which produces non-trivial solutions is positive, $\mu > 0$, therefore, the only solution for X is (choose the shifted form of solution with hyperbolic functions from the table special equation, because at the boundary $x=L$, we have a homogeneous boundary condition)

$$\boxed{X'' - \mu X = 0} \qquad X(x) = c_1 \cosh\sqrt{\mu}(x - x_0) + c_2 \sinh\sqrt{\mu}(x - x_0)$$

Consider the first homogeneous boundary condition at $\underline{x = L}$ (and choose $x_0 = L$):

$$X(L) = 0 = c_1 \cosh\sqrt{\mu}(0) + c_2 \sinh\sqrt{\mu}(0) = c_1 \Rightarrow \quad c_1 = 0$$

$$\Rightarrow \qquad X(x) = c_2 \sinh\sqrt{\mu}(x - L)$$

We already know values of μ from solution for Y (choose also $c_2 = 0$)

$$\Rightarrow \qquad X_n(x) = \sinh\sqrt{\mu_n}(x - L)$$

$$\Rightarrow \qquad X_n(x) = \sinh\left[\frac{n\pi}{M}(x - L)\right] \text{ where } n = 1,2,3,...$$

basic solutions

According to the assumed form of solution, we may construct a set of basic solutions:

$$u_1(x,y) = X_1(x)Y_1(y) = \sinh\left[\frac{\pi}{M}(x - L)\right]\sin\frac{\pi}{M}y$$

$$u_2(x,y) = X_2(x)Y_2(y) = \sinh\left[\frac{2\pi}{M}(x - L)\right]\sin\frac{2}{M}\pi y$$

$$\vdots$$

$$u_n(x,y) = X_n(x)Y_n(y) = \sinh\left[\frac{n\pi}{M}(x - L)\right]\sin\frac{n\pi}{M}y$$

All these solutions satisfy the LE and 3 homogeneous boundary conditions.
Any linear combination of the basic solutions is also a solution.
Construct a solution of LE in the form of a linear combination with coefficients a_n:

$$u(x,y) = X(x)Y(y) = \sum_{n=1}^{\infty} a_n u_n = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi}{M}(x - L)\right]\sin\frac{n\pi}{M}y$$

This solution also satisfies LE and 3 homogeneous boundary conditions. Determine coefficients a_n in such a way that the last boundary condition (non-homogeneous) is also satisfied (it will yield a solution of the problem).
Boundary condition at $x = 0$:

$$u(0,y) = X(0)Y(y) = -\sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi}{M}L)\sin\frac{n\pi}{M}y = f(y)$$

Rewrite it as:

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{M}y = f(y), \quad \text{where} \quad b_n = -a_n \sinh(\frac{n\pi}{M}L)$$

If we treat this sum as a Fourier sine expansion of the function $f(y)$ on the interval $y \in [0, M]$ with coefficients

$$b_n = \frac{2}{M}\int_0^M f(y)\sin\left(\frac{n\pi}{M}y\right)dy$$

then coefficients a_n are determined as:

$$a_n = -\frac{b_n}{\sinh(\frac{n\pi}{M}L)} = -\frac{\frac{2}{M}\int_0^M f(y)\sin\left(\frac{n\pi}{M}y\right)dy}{\sinh(\frac{n\pi}{M}L)}$$

and the solution now is completely determined.

4. Solution of BVP:

$$u(x,y) = -\frac{2}{M}\sum_{n=1}^{\infty}\left[\frac{\int_0^M f(t)\sin\left(\frac{n\pi}{M}t\right)dt}{\sinh\left(\frac{n\pi}{M}L\right)}\right]\sinh\left[\frac{n\pi}{M}(L-x)\right]\sin\frac{n\pi}{M}y$$

5. Example 1 *(010 heat1-1.mws)*

Let $f(y) = 1$, then

$$\int_0^M \sin\frac{n\pi}{M}t\,dt = -\cos\frac{n\pi}{M}t\Big|_0^M = -M\frac{\cos n\pi}{n\pi} + \frac{M}{n\pi} = -M\frac{(-1)^n - 1}{n\pi}$$

$$u(x,y) = \frac{2}{\pi M}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\frac{\sinh\left[\frac{n\pi}{M}(L-x)\right]}{\sinh\left(\frac{n\pi}{M}\right)}\sin\frac{n\pi}{M}y$$

6. Observations:

1. The solution is in the form of an infinite series. It represents the function from the functional vector space spanned by the obtained eigenfunctions $u_n(x,y)$ (basis for vector space).

2. We determined that the ODE $Y'' + \mu Y = 0$
with two homogeneous Dirichlet boundary conditions:

$$Y(0) = 0$$
$$Y(M) = 0$$

for values of the parameter μ:

$$\mu_n = \frac{n^2\pi^2}{M^2},\ n = 1,2,3,\ldots \quad \text{(eigenvalues)}$$

has non-trivial solutions:

$$Y_n(y) = \sin\frac{n\pi}{M}y \qquad \text{(eigenfunctions)}$$

Compare this result to the Sturm-Liouville Problem (20-22). the self-adjoint form of the equation for this case is

$$-\frac{1}{1}[Y']' = \mu Y$$

Positive values of the separation constant μ are consistent with (2 c) of the Sturm-Liouville Theorem.

3. Assumption of separation of variables was used to obtain the basic functions (eigenfunctions). But the obtained solution is not an approximation – it is an **exact solution** (this fact follows from the uniqness of the solution of the Dirichlet problem for Laplace's Equation). The same result may be obtained by the other methods without separation of variables (for example, using the finite integral transform).

4. Solution of the example problem may be treated:
- as a stationary temperature field in the rectangular domain with a fixed temperature at the boundaries;
- as equilibrium shape of a membrane stretched on the fixed frame;
- solution of the problem from differential geometry on optimization: find the surface with fixed boundaries, which has the minimal area etc.

4.6.2 **5. Example 1** *(heat1-1.mws)* **Laplace's equation, 3 homogenous boundary conditions (Dirichlet)**

```
> restart;
```

Non-homogeneous boundary condition:

```
> f(y):=1;
```

$$f(y) := 1$$

Fourier coefficients:

```
> b[n]:=2*int(f(y)*sin(n*Pi*y/M),y=0..M)/M;
```

$$b_n := -\frac{2(\cos(n\pi)-1)}{n\pi}$$

```
> b[n]:=factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b[n]));
```

$$b_n := -\frac{2((-1)^n-1)}{n\pi}$$

```
> a[n]:=-b[n]/sinh(n*Pi*L/M);
```

$$a_n := \frac{2((-1)^n-1)}{n\pi\sinh\left(\frac{n\pi L}{M}\right)}$$

```
> u[n](x,y):=a[n]*sinh(n*Pi*(x-L)/M)*sin(n*Pi*y/M);
```

$$u_n(x,y) := \frac{2((-1)^n-1)\sinh\left(\frac{n\pi(x-L)}{M}\right)\sin\left(\frac{n\pi y}{M}\right)}{n\pi\sinh\left(\frac{n\pi L}{M}\right)}$$

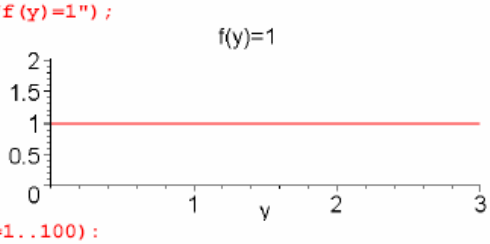
Solution:

```
> L:=2;M:=3;
```

$$L := 2$$
$$M := 3$$

Non-homogeneous boundary condition:

```
> plot(f(y),y=0..M,title="f(y)=1");
```

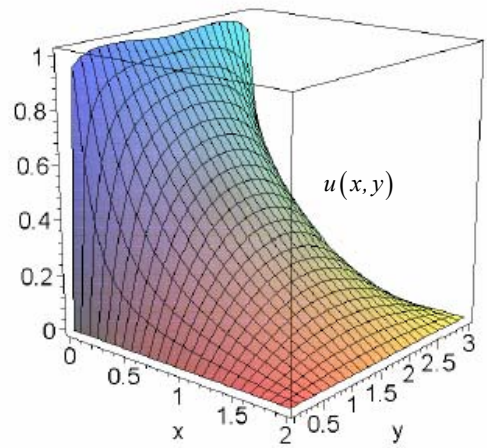



```
> u(x,y):=sum(u[n](x,y),n=1..100):
```

Vizualization of solution of Dirichlet problem for Laplace Equation:

```
> plot3d(u(x,y),x=0..L,y=0..M,axes=boxed, title="surface with fixed boundaries");
```

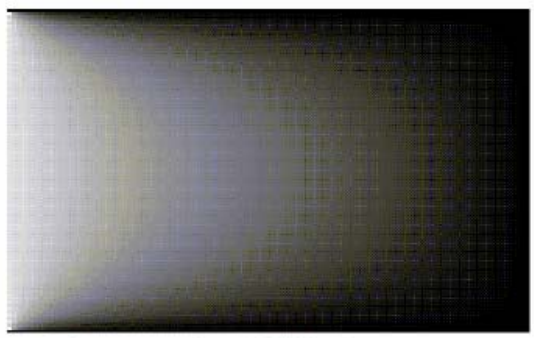
surface with fixed boundaries




```
> with(plots):
Warning, the name changecoords has been redefined
```

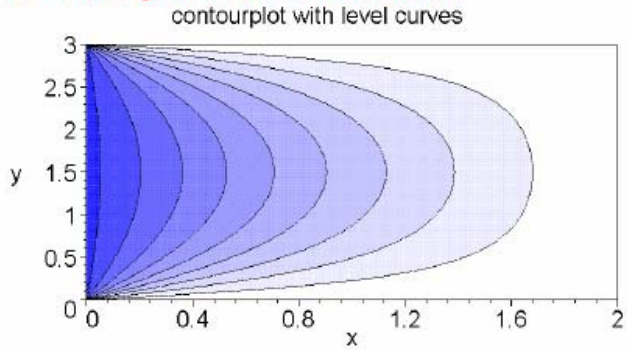
```
> densityplot(u(x,y),x=0..L,y=0..M,axes=none,grid=[100,100],style=patchnogrid,title="statio
  nary distribution of density field");
```

stationary distribution of density field

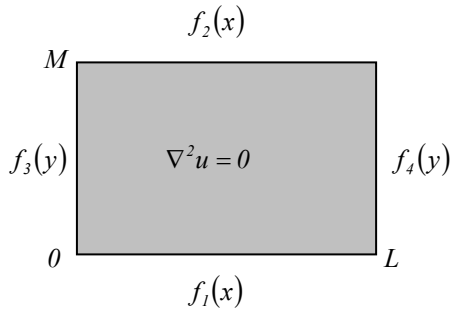


```
> contourplot(u(x,y),x=0..L,y=0..M,axes=boxed,filled=true,contours=10,coloring=[white,blue]
  ,grid=[100,100],title="contourplot with level curves");
```

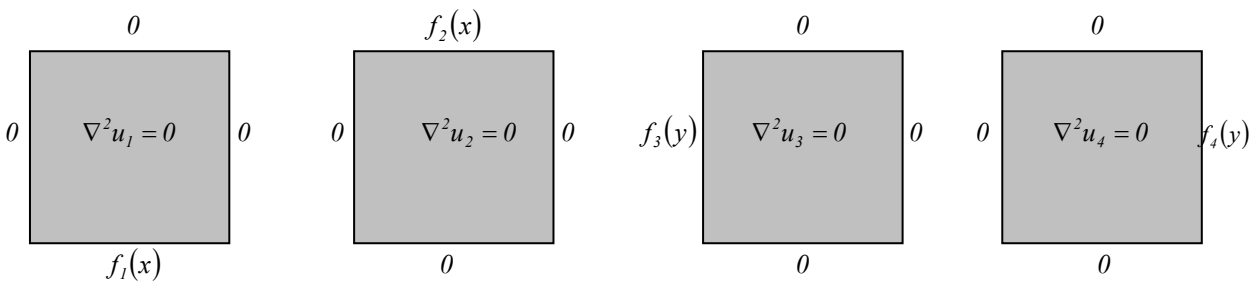
contourplot with level curves



7. Non-homogeneous boundary conditions (superposition principle)



Supplemental problems: ↓



Solution of supplemental problems:

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \sinh\frac{n\pi}{L}(y-M) \qquad a_n = \frac{-\frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi}{L}x\,dx}{\sinh\frac{n\pi}{L}M}$$

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \sinh\frac{n\pi}{L}y \qquad b_n = \frac{\frac{2}{L}\int_0^L f_2(x)\sin\frac{n\pi}{L}x\,dx}{\sinh\frac{n\pi}{L}M}$$

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi}{M}(x-L)\sin\frac{n\pi}{M}y \qquad c_n = \frac{-\frac{2}{M}\int_0^M f_3(y)\sin\frac{n\pi}{M}y\,dy}{\sinh\frac{n\pi}{M}L}$$

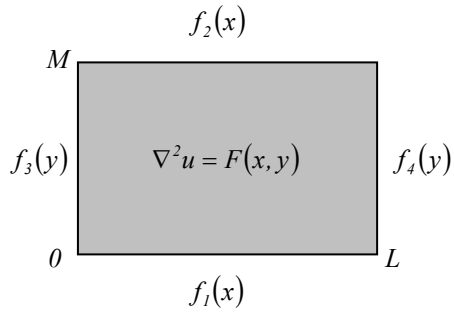
$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sinh\frac{n\pi}{M}x \sin\frac{n\pi}{M}y \qquad d_n = \frac{\frac{2}{M}\int_0^M f_4(y)\sin\frac{n\pi}{M}y\,dy}{\sinh\frac{n\pi}{M}L}$$

Solution of Dirichlet problem:

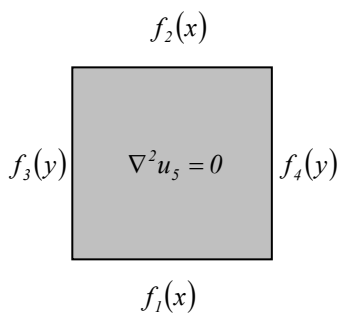
$$\boxed{u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)}$$

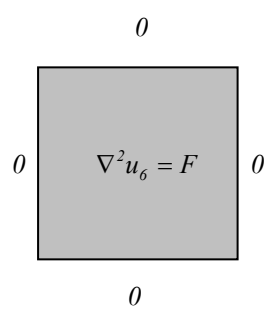
Examples: (*heat1-5.mws, heat1-5b.mws*)

8. Non-homogeneous equation (Poisson's Equation)



Supplemental problems: ↓





Solution of supplemental problems:

Solution of Dirichlet problem (Laplace's homogeneous equation):

$$u_5(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u_6(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\, sin\left(\frac{n\pi}{L}x\right) sin\left(\frac{m\pi}{M}y\right)$$

$$A_{nm} = \frac{-4}{\pi^2 LM\left(\frac{n^2}{L^2}+\frac{m^2}{M^2}\right)} \int_0^M\int_0^L F(x,y)\, sin\left(\frac{n\pi}{L}x\right) sin\left(\frac{m\pi}{M}y\right) dxdy$$

Solution of Poisson's Equation (superposition principle):

$$\boxed{u(x,y) = u_5(x,y) + u_6(x,y)}$$

Example 2: (*012 heat1-6.mws*)

012 **4.6.2 8. Poisson's Equation** **Example 2** *4 non-homogeneous boundary conditions (Dirichlet)*

```
> restart;
```

boundaries: y=0 f1, y=M f2, x=0 f3, x=L f4

```
> f[1](x):=x*(L-x);f[2](x):=x*(L-x);f[3](y):=y*(M-y);f[4](y):=y*(M-y);
```

$$f_1(x) := x\,(L-x)$$
$$f_2(x) := x\,(L-x)$$
$$f_3(y) := y\,(M-y)$$
$$f_4(y) := y\,(M-y)$$

Volumetric sorce function:

```
> F(x,y):=-x*y*(L-x)*(M-y)/50;
```

$$\mathrm{F}(x,y) := -\frac{x\,y\,(L-x)\,(M-y)}{50}$$

Fourier coefficients:

```
> b1[n]:=2/L*int(f[1](x)*sin(n*Pi*x/L),x=0..L);
```

$$b1_n := -\frac{2\,L^2\,(n\,\pi\,\sin(n\,\pi)+2\,\cos(n\,\pi)-2)}{n^3\,\pi^3}$$

```
> b2[n]:=2/L*int(f[2](x)*sin(n*Pi*x/L),x=0..L);
```

$$b2_n := -\frac{2\,L^2\,(n\,\pi\,\sin(n\,\pi)+2\,\cos(n\,\pi)-2)}{n^3\,\pi^3}$$

```
> b3[n]:=2/M*int(f[3](y)*sin(n*Pi*y/M),y=0..M);
```

$$b3_n := -\frac{2\,M^2\,(n\,\pi\,\sin(n\,\pi)+2\,\cos(n\,\pi)-2)}{n^3\,\pi^3}$$

```
> b4[n]:=2/M*int(f[4](y)*sin(n*Pi*y/M),y=0..M);
```

$$b4_n := -\frac{2\,M^2\,(n\,\pi\,\sin(n\,\pi)+2\,\cos(n\,\pi)-2)}{n^3\,\pi^3}$$

```
> b5[n,m]:=factor(int(int(F(x,y)*sin(n*Pi*x/L)*sin(m*Pi*y/M),x=0..L),y=0..M));
```

$$b5_{n,m} := -\frac{1}{50}\,\frac{M^3\,L^3\,(m\,\pi\,\sin(m\,\pi)-2+2\,\cos(m\,\pi))\,(n\,\pi\,\sin(n\,\pi)+2\,\cos(n\,\pi)-2)}{m^3\,\pi^6\,n^3}$$

```
> b5[n,m]:=factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n,sin(m*Pi)=0,cos(m*Pi)=(-1)^m},b5[n,m]))
);
```

$$b5_{n,m} := -\frac{2\,M^3\,L^3\,(-1+(-1)^m)\,(-1+(-1)^n)}{25\,m^3\,\pi^6\,n^3}$$

```
> b1[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b1[n])));
```

$$b1_n := -\frac{4\,L^2\,(-1+(-1)^n)}{n^3\,\pi^3}$$

```
> b2[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b2[n])));
```

$$b2_n := -\frac{4\,L^2\,(-1+(-1)^n)}{n^3\,\pi^3}$$

```
> b3[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b3[n])));
```

$$b3_n := -\frac{4\,M^2\,(-1+(-1)^n)}{n^3\,\pi^3}$$

```
> b4[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b4[n])));
```

$$b4_n := -\frac{4\,M^2\,(-1+(-1)^n)}{n^3\,\pi^3}$$

```
> a1[n]:=-b1[n]/sinh(n*Pi*M/L);
```

$$a1_n := \frac{4\,L^2\,(-1+(-1)^n)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,M}{L}\right)}$$

```
> a2[n]:=b2[n]/sinh(n*Pi*M/L);
```

$$a2_n := -\frac{4\,L^2\,(-1+(-1)^n)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,M}{L}\right)}$$

```
> a3[n]:=-b3[n]/sinh(n*Pi*L/M);
```

$$a3_n := \frac{4\,M^2\,(-1+(-1)^n)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,L}{M}\right)}$$

```
> a4[n]:=b4[n]/sinh(n*Pi*L/M);
```

$$a4_n := -\frac{4\,M^2\,(-1+(-1)^n)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,L}{M}\right)}$$

```
> a5[n,m]:=-4*b5[n,m]/Pi^2/(n^2/L^2+m^2/M^2);
```

$$a5_{n,m} := \frac{8\,M^3\,L^3\,(-1+(-1)^m)\,(-1+(-1)^n)}{25\,m^3\,\pi^8\,n^3\left(\frac{n^2}{L^2}+\frac{m^2}{M^2}\right)}$$

Fourier series terms:

```
> u1[n](x,y):=a1[n]*sin(n*Pi*x/L)*sinh(n*Pi*(y-M)/L);
```

$$u1_n(x,y) := \frac{4\,L^2\,(-1+(-1)^n)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\sinh\left(\frac{n\,\pi\,(-M+y)}{L}\right)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,M}{L}\right)}$$

```
> u2[n](x,y):=a2[n]*sin(n*Pi*x/L)*sinh(n*Pi*y/L);
```

$$u2_n(x,y) := -\frac{4\,L^2\,(-1+(-1)^n)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\sinh\left(\frac{n\,\pi\,y}{L}\right)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,M}{L}\right)}$$

```
> u3[n](x,y):=a3[n]*sinh(n*Pi*(x-L)/M)*sin(n*Pi*y/M);
```

$$u3_n(x,y) := \frac{4\,M^2\,(-1+(-1)^n)\,\sinh\left(\frac{n\,\pi\,(x-L)}{M}\right)\sin\left(\frac{n\,\pi\,y}{M}\right)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,L}{M}\right)}$$

```
> u4[n](x,y):=a4[n]*sinh(n*Pi*x/M)*sin(n*Pi*y/M);
```

$$u4_n(x,y) := -\frac{4\,M^2\,(-1+(-1)^n)\,\sinh\left(\frac{n\,\pi\,x}{M}\right)\sin\left(\frac{n\,\pi\,y}{M}\right)}{n^3\,\pi^3\,\sinh\left(\frac{n\,\pi\,L}{M}\right)}$$

```
> u5[n,m](x,y):=a5[n,m]*sin(n*Pi*x/L)*sin(m*Pi*y/M);
```

$$u5_{n,m}(x,y) := \frac{8}{25}\,\frac{M^3\,L^3\,(-1+(-1)^m)\,(-1+(-1)^n)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\sin\left(\frac{m\,\pi\,y}{M}\right)}{m^3\,\pi^8\,n^3\left(\frac{n^2}{L^2}+\frac{m^2}{M^2}\right)}$$

```
> L:=4;M:=3;
```

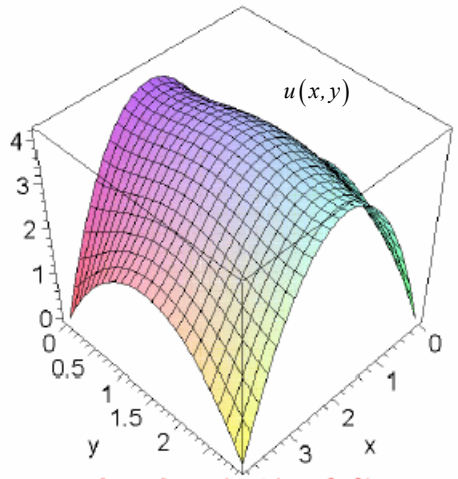
$$L := 4$$
$$M := 3$$

Solution of supplemental problems:

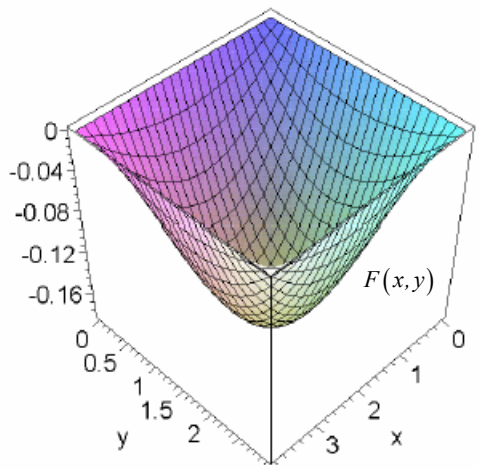
```
> u1(x,y):=sum(u1[n](x,y),n=1..40):
> u2(x,y):=sum(u2[n](x,y),n=1..40):
> u3(x,y):=sum(u3[n](x,y),n=1..40):
> u4(x,y):=sum(u4[n](x,y),n=1..40):
> u5(x,y):=sum(sum(u5[n,m](x,y),n=1..4),m=1..40):
```

SOLUTION:

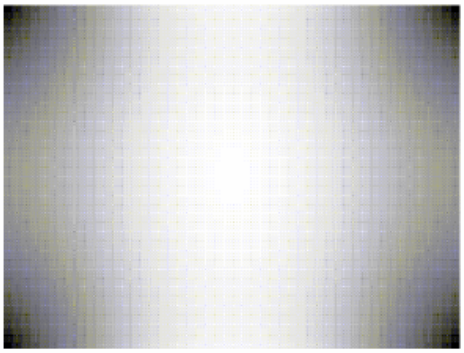
```
> u(x,y):=u1(x,y)+u2(x,y)+u3(x,y)+u4(x,y)+u5(x,y):
> plot3d(u(x,y),x=0..L,y=0..M,axes=boxed,projection=0.9);
```



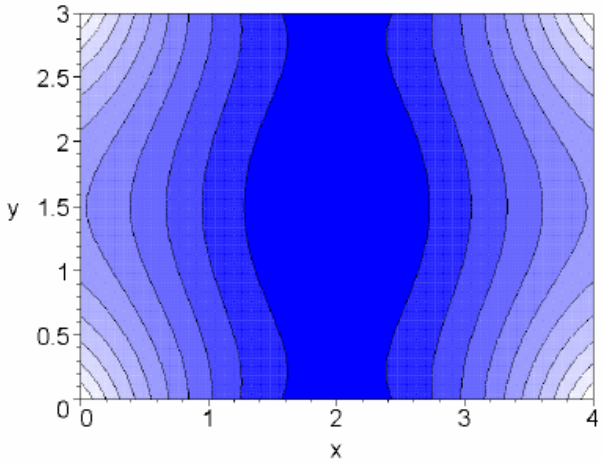

```
> plot3d(F(x,y),x=0..L,y=0..M,axes=boxed,projection=0.9);
```




```
> with(plots):
Warning, the name changecoords has been redefined
> densityplot(u(x,y),x=0..L,y=0..M,grid=[50,50],scaling=constrained,axes=none,style=patchno
  grid);
```



```
> contourplot(u(x,y),x=0..L,y=0..M,grid=[50,50],scaling=constrained,axes=boxed,filled=true,
  contours=10,coloring=[white,blue],grid=[100,100]);
```



9. The Maximum Principle for Laplace Equation

In this section, we will consider some important properties of solutions of the Laplace Equation, which have a special term. Let D be an open connected domain in $\mathbb{R}^3$

harmonic function

Definition The function $u(\mathbf{r}) \in C^2(D)$ is called a harmonic function if it satisfies the Laplace Equation in this open domain D.

The Maximum Principle is formulated in the following way:

Maximum Principle

Theorem If a harmonic function $u(\mathbf{r})$ is continuous in $\overline{D}$, then it cannot attain its minimum and maximum values in the domain D

$$\min_{\mathbf{r}\in S} u(\mathbf{r}) < u(\mathbf{r}) < \max_{\mathbf{r}\in S} u(\mathbf{r}) \tag{1}$$

This theorem is a direct corollary of the other property of harmonic functions (theorem of arithmetic mean): if the function $u(\mathbf{r})$ is harmonic in some open ball $B(\mathbf{r_0}, R)$ and continuous in the closed ball $\overline{B}(\mathbf{r_0}, R)$ then its value at the center of the ball $u(\mathbf{r}_0)$ is equal to its mean value over the sphere $S = \{\mathbf{r} \| \mathbf{r} - \mathbf{r_0}| = R\}$ (boundary of the ball $B(\mathbf{r_0}, R)$). For the whole closed domain $\overline{D}$, the Maximum Principle yields that

$$|u(\mathbf{r})| < \max_{\mathbf{r}\in S} |u(\mathbf{r})| \tag{2}$$

Dirichlet Problem

If, in particular, for the Dirichlet problem, $u(\mathbf{r})|_{\mathbf{r}\in S} = 0$ at all points of the boundary S of the domain D, then $u(\mathbf{r}) \equiv 0$ everywhere in the domain D. (3)

For the Dirichlet problem for the Laplace Equation $\nabla^2 u = 0$ in the rectangular domain $D = (x_1, x_2) \times (y_1, y_2)$ with boundary conditions:

$$\begin{aligned} u(x, y_1) &= f_1(x) & x &\in (x_1, x_2) \\ u(x, y_2) &= f_2(x) & x &\in (x_1, x_2) \\ u(x_1, y) &= f_3(x) & y &\in (y_1, y_2) \\ u(x_2, y) &= f_3(x) & y &\in (y_1, y_2) \end{aligned}$$

the Maximum Principle can be formulated in the following way

$$m < u(x, y) < M$$

where

$$m = \min\left\{ \min_{x\in(x_1,x_2)} \{f_1(x), f_2(x)\}, \min_{y\in(y_1,y_2)} \{f_3(y), f_4(y)\} \right\}$$

$$M = \max\left\{ \max_{x\in(x_1,x_2)} \{f_1(x), f_2(x)\}, \max_{y\in(y_1,y_2)} \{f_3(y), f_4(y)\} \right\}$$

It means that if the extremum of the solution of the Dirichlet problem occurs at the interior point of the domain, then the solution is constant everywhere in the domain. From the Maximum Principle, the uniqueness of the solution of the Dirichlet problem follows:

Uniqueness

Theorem Solution of the Dirichlet problem for the Laplace Equation is unique.

Proof: Suppose that u_1, u_2 are two solutions of the Dirichlet problem for the Laplace Equation: $\nabla^2 u = 0$, $u(\mathbf{r})|_{\mathbf{r}\in S} = f$. Then, because the Laplacian is linear, the function $U = u_1 - u_2$ is a solution of the Dirichlet problem $\nabla^2 U = 0$, $U(\mathbf{r})|_{\mathbf{r}\in S} = 0$, then, according to the Maximum Principle (2), the function U is identically equal to zero, $U \equiv 0$, and, therefore, $u_1 = u_2$. ■

10. Elimination of the Gibbs Effect

Eigenfunctions of the Dirichlet problem for the Laplace Equation ( ) were obtained with homogeneous boundary conditions. Therefore, unless in the considered Dirichlet problem for the Laplace Equation

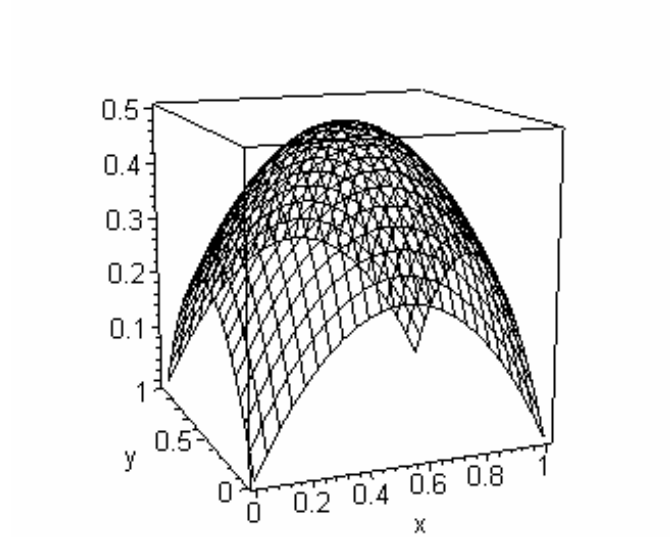
$$\nabla^2 u = 0 \qquad u\big|_S = f \tag{1}$$

the function f is equal to zero at the corners of the domain, there will be the presence of the Gibbs phenomenon (or the Gibbs effect). Thus, to obtain an accurate approximation to the solution one needs a large number of terms in the Fourier series solution. But even significant increase of number of terms does not eliminate the Gibbs effect – the amplitude of oscillation is not reduced. A simple trick can avoid this and greatly increase the speed of convergence.

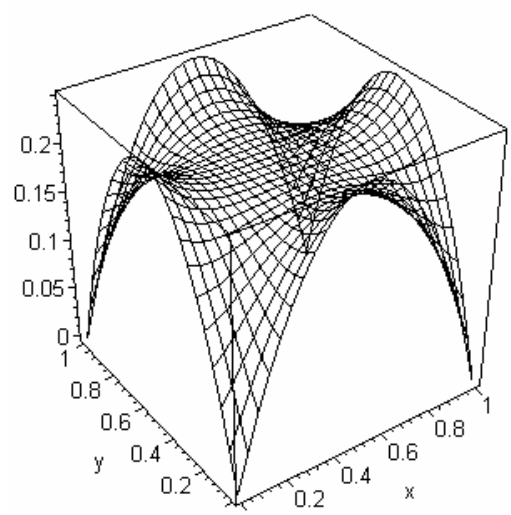
For example, consider the Dirichlet problem in the rectangular domain $D = (0, L) \times (0, M)$ with the boundary $S = \bar{D} \setminus D$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u\big|_S = x(L - x) + y(M - y) \equiv f(x) \tag{2}$$

at the corners of the rectangular S, the function f is zero,



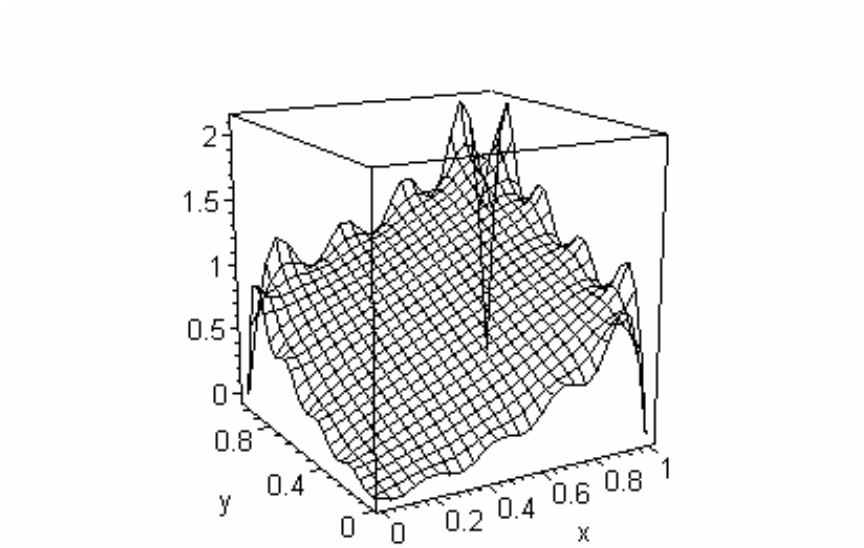
and there are no oscillations in the solution $u(x, y)$ due to the Gibbs effect:



In contrast, in the Dirichlet problem in the same domain:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad u\big|_S = x^2+y^2 \equiv f(x,y) \tag{3}$$

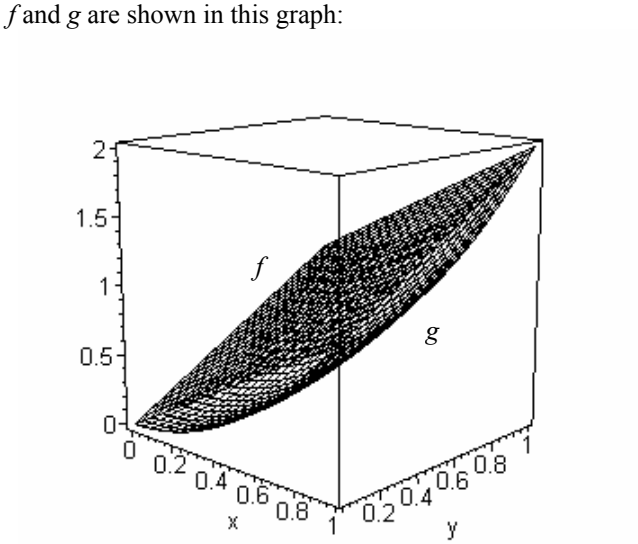
at the corners of the rectangular S, the function f is not equal to zero (except at the point $(0,0)$), and there are oscillations in the solution $u(x,y)$ due to the Gibbs effect:



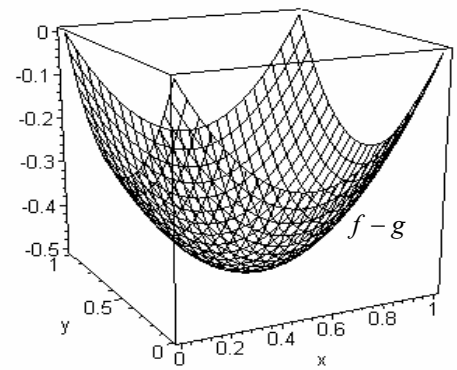
This effect can be eliminated by the following trick: consider a bilinear interpolation of the function f over $\overline{D}$

$$g(x,y)=\frac{f(0,0)}{LM}(L-x)(M-x)+\frac{f(L,0)}{LM}x(M-y)+\frac{f(0,M)}{LM}(L-x)y+\frac{f(L,M)}{LM}xy$$

Graphs of f and g are shown in this graph:



At the corners of the rectangular S, their values are the same. Therefore, function $f-g$ has the zero values at the corners:



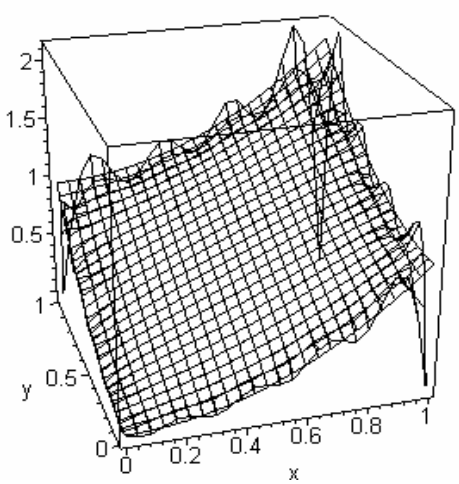
Consider the function $v(x,y)=u(x,y)-g(x,y)$, where $u(x,y)$ is a solution of BVP (3). The Laplacian of this function is equal to zero, because u is a solution of the LE and any bilinear function g satisfies LE. Therefore, the function v is a solution of the following Dirichlet problem:

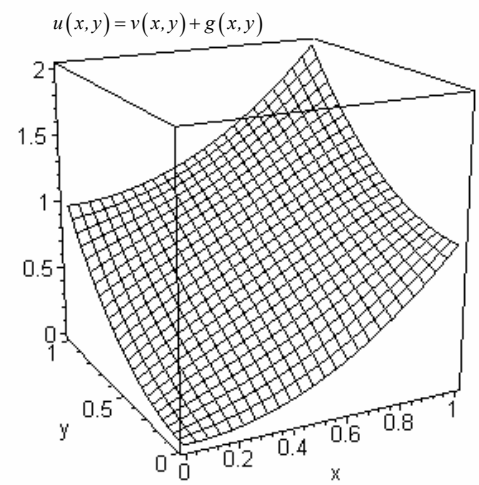
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad v\big|_S=f-g \tag{4}$$

Separation of variables solution for this problem does not have the Gibbs effect, and so then a solution for the original problem:

$$u(x,y)=v(x,y)+g(x,y)$$

On these graphs, solutions of BVP (3) are shown with an 8 term representation in the Fourier series solution with and without the Gibbs effect:





Elimination of the Gibbs effect due to discontinuity of the boundary condition in the interior point of the boundary interval $(x_0,y_0)\in S$, can be performed in the similar manner using the terms of the form $tan^{-1}\left(\dfrac{y-y_0}{x-x_0}\right)$.

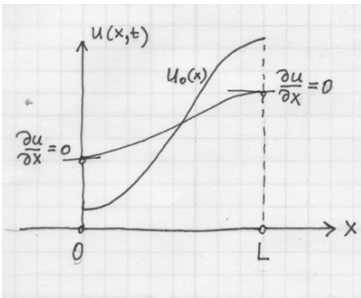
For more about the Gibbs phenomena see elsewhere:

T.E.Peterson *Eliminating Gibb's effect from separation of variables solutions.* SIAM (Society for Industrial and Applied Mathematics) Review, 1998, Vol.40, No.2, pp.324-326.

E.Hewitt, R.E.Hewitt *The Gibbs-Wilbraham Phenomenon: An Episode in Fourier Analysis.* Archive for History of Exact Sciences, 1979, Vol.21, No.2, pp.129-160.

4.6.3 Heat Equation

1) 1-D homogeneous equation and boundary conditions (Neumann-Neumann)



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in [0, L],\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $\dfrac{\partial u(0,t)}{\partial x} = 0\ ,\ t > 0$ (Neumann)

$\dfrac{\partial u(L,t)}{\partial x} = 0,\ \ t > 0$ (Neumann)

(both boundaries are insulated)

1. Separation of variables

We assume that the function $u(x,t)$ can be represented as a product of two functions each of a single variable:

$$u(x, y) = X(x)T(t)$$

Calculate the derivatives and substitute into the heat equation

$$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T \qquad \frac{\partial u}{\partial t} = XT'$$

$$X''T = a^2 XT'$$

Then separate the variables

$$\frac{X''}{X} = a^2 \frac{T'}{T}$$

The left hand side of this equation is a function of the independent variable *x* only, and the right hand side is a function of the independent variable *t* only. The equality is possible only if both of them are equal to the same constant (call it μ); μ is called a <u>*separation constant*</u>:

$$\frac{X''}{X} = a^2 \frac{T'}{T} = \mu$$

Therefore, it yields two ordinary differential equations:

$$X'' - \mu X = 0$$

$$T' - \frac{\mu}{a^2} T = 0$$

2. Boundary conditions

To avoid trivial solutions, we require

$$x = 0 \qquad \frac{\partial u(0,t)}{\partial x} = X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0$$

$$x = L \qquad \frac{\partial u(L,t)}{\partial x} = X'(L)T(t) = 0 \quad \Rightarrow \quad X'(L) = 0$$

3. Solution of o.d.e.

Start with the equation for a space variable

$$X'' - \mu X = 0$$

Solution of o.d.e. depends on the sign of constant μ :

$$X(x) = \begin{cases} c_1 \cosh\sqrt{\mu}x + c_2 \sinh\sqrt{\mu}x & \mu > 0 \quad \mu = \lambda^2 \\ c_1 + c_2 x & \mu = 0 \quad \mu = 0 \\ c_1 \cos\sqrt{-\mu}x + c_2 \sin\sqrt{-\mu}x & \mu < 0 \quad \mu = -\lambda^2 \end{cases}$$

Consider the first boundary condition at $\underline{x = 0}$:

$$X'(x) = \begin{cases} c_1\sqrt{\mu}\sinh\sqrt{\mu}x + c_2\sqrt{\mu}\cosh\sqrt{\mu}x & \mu > 0 \\ c_2 & \mu = 0 \\ -c_1\sqrt{-\mu}\sin\sqrt{-\mu}x + c_2\sqrt{-\mu}\cos\sqrt{-\mu}x & \mu < 0 \end{cases}$$

$$X'(0) = 0 = \begin{cases} c_1\sqrt{\mu}\sinh 0 + c_2\sqrt{\mu}\cosh 0 \\ c_2 \\ -c_1\sqrt{-\mu}\sin 0 + c_2\sqrt{-\mu}\cos 0 \end{cases} \Rightarrow c_2 = 0$$

Then the solution becomes:

$$X(x) = \begin{cases} c_1\cosh\sqrt{\mu}x & \mu > 0 \\ c_1 & \mu = 0 \\ c_1\cos\sqrt{-\mu}x & \mu < 0 \end{cases}$$

Consider the second boundary condition at $\underline{x = L}$:

$$X'(L) = 0 = \begin{cases} c_1\sqrt{\mu}\,sinh\sqrt{\mu}L & \mu > 0 \\ 0 & \mu = 0 \\ -c_1\,sin\sqrt{-\mu}L & \mu < 0 \end{cases}$$

For the solution to be non-trivial, a constant $c_1 \neq 0$.
It can be achieved for the case $\mu = 0$ in which any constant is a solution of the equation;
and in the case of $\mu < 0$ by requiring

$$sin\sqrt{-\mu}L = 0$$

that yields

$$\sqrt{-\mu}L = \lambda L = n\pi \quad n = 1,2,...$$

(again, 0 is excluded to avoid a trivial solution). Then the values of the separation constant for which there exist non-trivial solutions are defined by

$$\lambda_n = \frac{n\pi}{L} \qquad \mu_n = -\lambda_n^2 = -\frac{n^2\pi^2}{L^2}$$

These values are called ***eigenvalues***. The corresponding solutions which are called ***eigenfunctions*** are

$$X_0 = 1 \qquad n = 0$$

$$X_n = \cos\frac{n\pi}{L}x \qquad n = 1,2,...$$

They can be combined now in a single expression

$$X_n = \cos\frac{n\pi}{L}x \qquad n = 0,1,2,...$$

The values of the separation constant for which there exist non-trivial solutions satisfying the boundary conditions are

$$\mu_n = -\lambda_n^2 = -\frac{n^2\pi^2}{L^2} \quad n = 0,1,2,...$$

4. Solution for *T*

With determined eigenvalues, solutions of the equation for T

$$T' - \frac{\mu_n}{a^2}T = 0$$

become

$$T_n(t) = e^{\frac{\mu_n}{a^2}t} = e^{-\frac{n^2\pi^2}{a^2L^2}t} \qquad n = 0,1,2,...$$

5. Basic solutions

Recalling the assumed form of the solution, one gets

$$
\begin{aligned}
u_0(x,t) = X_0(x)T_0(t) &= 1 \\
u_1(x,t) = X_1(x)T_1(t) &= \cos\frac{\pi}{L}x e^{-\frac{\pi^2}{a^2L^2}t} \\
u_2(x,t) = X_2(x)T_2(t) &= \cos\frac{2\pi}{L}x e^{-\frac{4\pi^2}{a^2L^2}t} \\
&\vdots \\
u_n(x,t) = X_n(x)T_n(t) &= \cos\frac{n\pi}{L}x e^{-\frac{n^2\pi^2}{a^2L^2}t} \\
&\vdots
\end{aligned}
$$

All these solutions satisfy the Heat Equation and both boundary conditions.
Any linear combination of the basic solutions is also a solution. The idea is to find such a combination that the initial condition is also satisfied. So, we are looking for the function

$$u(x,t) = \sum_{n=0}^{\infty} b_n X_n(x)T_n(t) = b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{L}x e^{-\frac{n^2\pi^2}{a^2L^2}t}$$

such that at $t = 0$ $\quad u(x,0) = b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{L}x = u_0(x)$

If the infinite series converges, then under the known conditions, the function $u(x,t)$ is a solution of the Heat Equation. The last equation is an expansion of the function $u_0(x)$ in the Fourier cosine series in the interval $(0,L)$ with coefficients b_n defined by the equation

$$b_0 = \frac{1}{L}\int_0^L u_0(x)dx$$

$$b_n = \frac{2}{L}\int_0^L u_0(x)\cos\frac{n\pi}{L}x dx$$

Then solution of the given IBVP for the heat equation is:

6. Solution of IBVP

$$\boxed{u(x,t) = \frac{1}{L}\int_0^L u_0(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L u_0(x)\cos\frac{n\pi}{L}x dx\right]\cos\frac{n\pi}{L}x e^{-\frac{n^2\pi^2}{a^2L^2}t}}$$

7. Examples:

(*heat2-2.mws*) $u_0(x) = 10000\left(x - \frac{L}{3}\right)^2 + 100$, ^oC

Material: stainless steel, $a^2 = 500^2 \left[\frac{s}{m^2}\right]$, $L=0.1\ m$

8. Comments:

1) the solution is in the form of an infinite series. The obtained solution is formal, because the convergence of the infinite series was not investigated. But if the initial temperature distribution given by the function $u_0(x)$ satisfies the Dirichlet conditions, then the Fourier series is convergent and the function $u(x,t)$ satisfies the Heat Equation and initial and boundary conditions. Therefore, it is a solution of the given IBVP.

2) We determined that the ODE

$$X'' - \mu X = 0 \qquad \text{or} \qquad -X'' = (-\mu)X \quad \text{(self-adjoint form)}$$

with two homogeneous Neumann boundary conditions:

$$X'(0) = 0 \qquad a_1 = 0 \quad \text{(in SLP)}$$

$$X'(L) = 0 \qquad a_2 = 0$$

for values of the parameter μ :

$$-\mu_n = \lambda_n^2 = \frac{n^2\pi^2}{L^2}, \; n = 0,1,2,3,... \qquad \text{(eigenvalues)}$$

has non-trivial solutions:

$$X_n(x) = \cos\frac{n\pi}{L}x \qquad \text{(eigenfunctions)}$$

According to 2 d) of the Sturm-Liouville Theorem, for this problem, $\lambda_0 = 0$ is an eigenvalue with the corresponding eigenfunction $X_0 = 1$.

3) With the increase of time, the solution approaches the steady state (the averaged temperature in the slab). Boundaries are insulated, and there are no heat sources. As a result, no heat escapes into the surroundings. The driving force – temperature gradient – is directed toward the areas with lower temperature. There exists a process of redistribution of heat energy that produces the uniform temperature in the slab.

4) Basic functions consist of the product

$$u_n(x,t) = cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{a^2L^2}t}$$

where the cosine function provides the spatial shape of the temperature profile; and the exponential function is responsible for decay of the temperature profile in time.

5) The rate of change of temperature depends on the thermophysical property *a*.

6) Very often, a 1-D Heat equation is treated as a model for heat transfer in a long very thin rod of constant cross-section whose surface, except for the ends, is insulated against the flow of heat Although, it is a correct model, the practical application of it is very limited. But there is another interpretation of a 1-D model, which is more reliable.
Consider a 3-D wall with finite dimension in the x-direction (within $x = 0$ and $x = L$) and elongated dimensions (may be infinite) in y- and z-directions. If the conditions at the walls $x = 0$ and $x = L$ are uniform, and the initial condition is independent of variables *y* and *z*, then the variation of temperature in the *y*- and *z*-directions is negligible (no heat flux in these directions)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

and the heat equation becomes 1-D

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial u}{\partial t}$$

It describes the variation of temperature along any line perpendicular to the wall.

016 **4.6.3** (heat2-2.mws) **7. Example:** **1-D Heat Equation** plane steel wall - both sides are insulated (Neumann BC) L=0.1 m

```
> restart;
> u[0](x):=10000*(x-L/3)^2+100;
```

$$u_0(x) := 10000\left(x - \frac{L}{3}\right)^2 + 100$$

Eigenfuctions:

```
> X[n]:=cos(n*Pi*x/L);
```

$$X_n := \cos\left(\frac{n\,\pi\,x}{L}\right)$$

```
> T[n]:=exp(-n^2*Pi^2*t/L^2/a^2);
```

$$T_n := \mathrm{e}^{\left(-\frac{n^2\,\pi^2\,t}{L^2\,a^2}\right)}$$

Fourier coefficients:

```
> b[0]:=simplify(factor(int(u[0](x),x=0..L)/L));
```

$$b_0 := \frac{10000\,L^2}{9} + 100$$

```
> b[n]:=2*int(u[0](x)*X[n],x=0..L)/L;
```

$$b_n := \frac{200}{9}\,\frac{-1800\,L^2\sin(n\,\pi) + 9\,n^2\,\pi^2\sin(n\,\pi) + 400\,n^2\,\pi^2\sin(n\,\pi)\,L^2 + 1200\,n\,\pi\cos(n\,\pi)\,L^2 + 600\,L^2\,n\,\pi}{n^3\,\pi^3}$$

```
> b[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},b[n])));
```

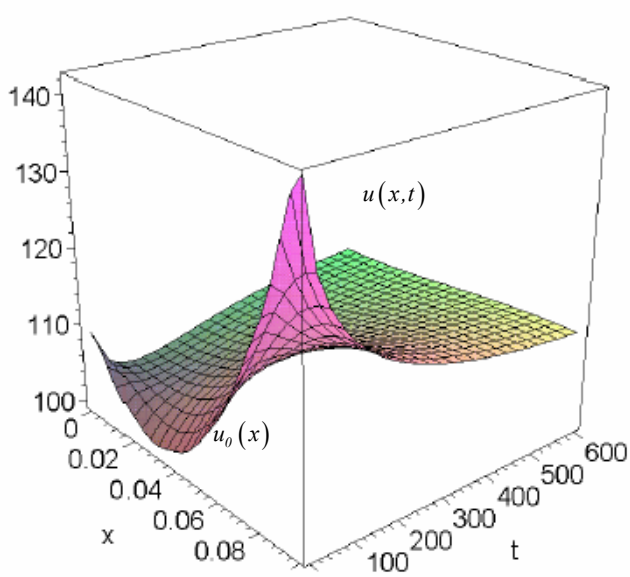
$$b_n := \frac{40000\,L^2\,(2\,(-1)^n + 1)}{3\,n^2\,\pi^2}$$

```
> L:=0.1;a:=500;
```

$$L := 0.1$$

$$a := 500$$

```
> u(x,t):=b[0]+sum(b[n]*X[n]*T[n],n=1..10):
> plot3d(u(x,t),x=0..L,t=0..600,axes=boxed,projection=0.85);
```



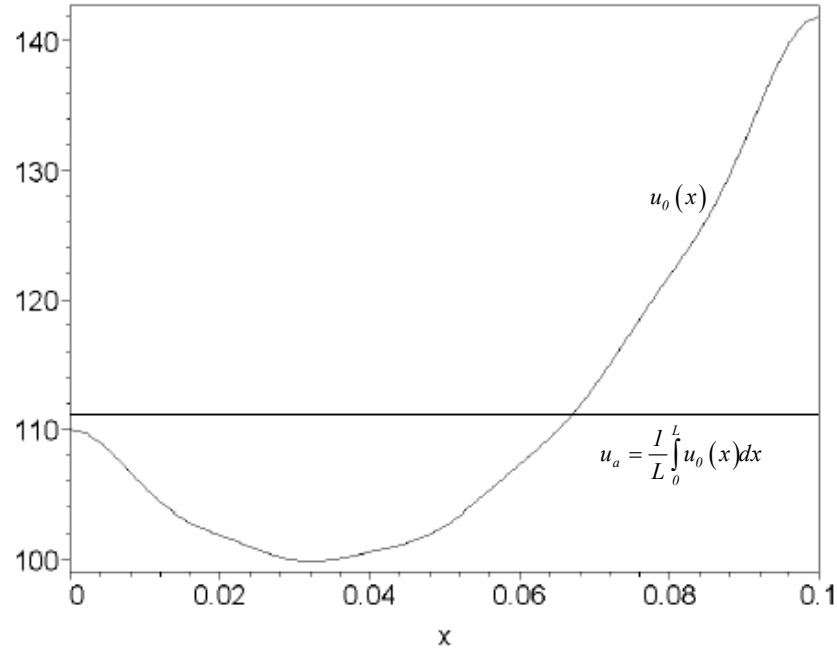

```
> with(plots):
Warning, the name changecoords has been redefined
```

ua = veraged temperature of the wall:

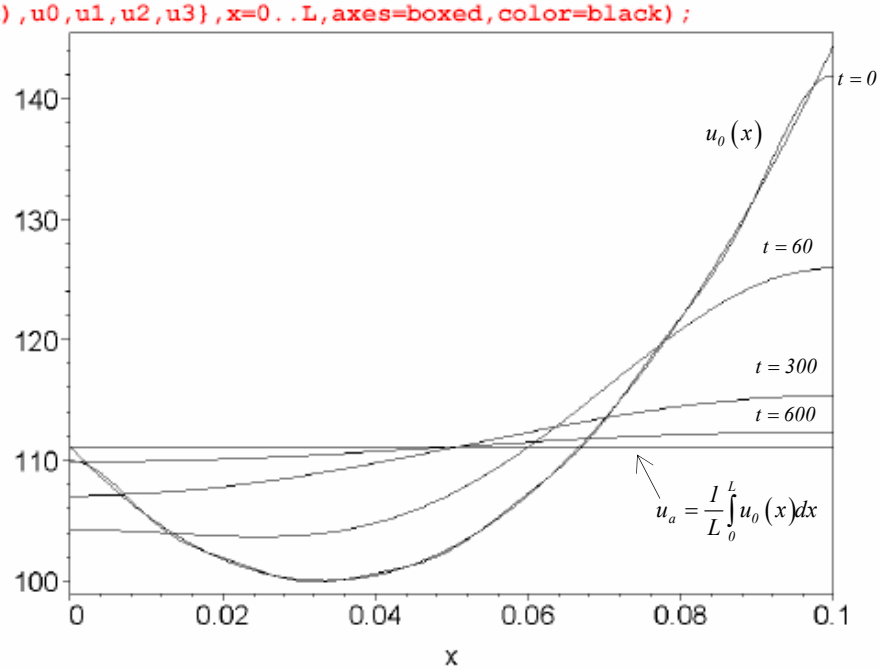
```
> ua:=value(int(u[0](x),x=0..L)/L);
```

$$ua := 133.3333333 - 333.3333334\,L + 1111.111111\,L^2$$

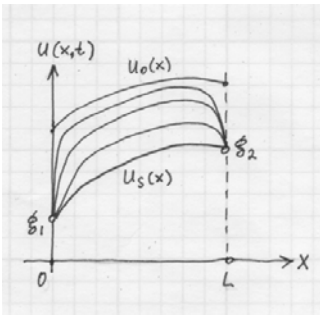
```
> animate({ua,u(x,t)},x=0..L,t=0..1200,frames=500,axes=boxed,color=black);
```




```
>
> u0:=subs(t=0,u(x,t)):
> u1:=subs(t=60,u(x,t)):
> u2:=subs(t=300,u(x,t)):
> u3:=subs(t=600,u(x,t)):
> plot({ua,u[0](x),u0,u1,u2,u3},x=0..L,axes=boxed,color=black);
```



4.6.3 2) non-homogeneous equation and non-homogeneous boundary conditions, reduction to homogeneous



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + b \qquad u(x,t),\ \ x \in (0,L),\ t > 0 \quad b(x,t) = \frac{\dot{g}(x,t)}{k}$$

initial condition: $u(x,0) = u_0(x)$

boundary conditions: $u(0,t) = g_1 \quad t > 0$ (Dirichlet)

$u(L,t) = g_2 \quad t > 0$ (Dirichlet)

1. Steady State Solution

> *Definition* A time-independent function which satisfies the heat equation and boundary conditions obtained as
>
> $$u_s(x) = \lim_{t\to\infty} u(x,t)$$
>
> is called a ***steady state solution***

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial x^2} = b \quad \text{subject to boundary conditions:} \quad u_s(0) = g_1, \quad u_s(L) = g_2$$

Let $b = const$, then integrating the equation twice, we come up with the following solution:

$$\frac{\partial u_s}{\partial x} = bx + c_1$$

$$u_s = \frac{b}{2}x^2 + c_1 x + c_2$$

Apply boundary conditions to determine the constants of integration:

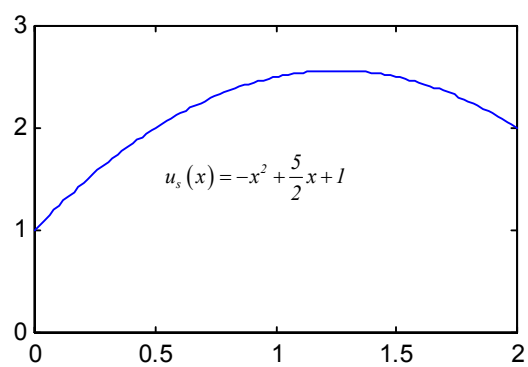
$$\underline{x = 0} \quad \Rightarrow \quad c_2 = g_1$$

$$\underline{x = L} \quad \Rightarrow \quad \frac{b}{2}L^2 + c_1 L + g_1 = g_2$$

$$\Rightarrow \quad c_1 = \frac{g_2 - g_1}{L} - \frac{bL}{2}$$

$$\boxed{u_s(x) = \frac{b}{2}x^2 + \left(\frac{g_2 - g_1}{L} - \frac{bL}{2}\right)x + g_1}$$

Example $b = -2$, $g_1 = 1$, $g_2 = 2$, $L = 2$



2. Change of variable

$U(x,t)=u(x,t)-u_s(x)$ or $u(x,t)=U(x,t)+u_s(x)$

$\frac{\partial u}{\partial t}=\frac{\partial U}{\partial t}$ $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 u_s}{\partial x^2}$ substitute into equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 u_s}{\partial x^2}=a^2\frac{\partial U}{\partial t}+b$$

Since $\frac{\partial^2 u_s}{\partial x^2}=b$, $\frac{\partial^2 U}{\partial x^2}=a^2\frac{\partial U}{\partial t}$

We obtained the equation for the new unknown function $U(x,t)$ which has homogeneous boundary conditions:

$x=0$ $U(0,t)=u(0,t)-u_s(0)=g_1-g_1=0$

$x=L$ $U(L,t)=u(L,t)-u_s(L)=g_2-g_2=0$

As a result, we reduced the non-homogeneous problem to a homogeneous equation for $U(x,t)$ with homogeneous Dirichlet boundary conditions.

Initial condition for function $U(x,t)$:

$$U(x,0)=u(x,0)-u_s(x)=u_0(x)-u_s(x)$$

3. Solution of heat equation for *U(x,t)*

We consider the following initial boundary value problem:

$$\frac{\partial^2 U}{\partial x^2}=a^2\frac{\partial U}{\partial t} \qquad U(x,t),\ x\in[0,L],\ t>0$$

initial condition: $U(x,0)=u_0(x)-u_s(x)$

boundary conditions: $U(0,t)=0$, $t>0$ (Dirichlet)

$U(L,t)=0$, $t>0$ (Dirichlet)

We already know a solution of this homogeneous Dirichlet problem obtained by separation of variables (exercise):

$$U(x,t)=X(x)T(t)=\sum_{n=1}^{\infty}d_n\sin\frac{n\pi}{L}x e^{\frac{-n^2\pi^2}{a^2L^2}t}$$

where coefficients d_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

$$d_n=\frac{2}{L}\int_0^L\left[u_0(x)-u_s(x)\right]\sin\frac{n\pi}{L}x\,dx$$

4. Solution of Non-homogeneous Heat Equation:

Return to the original function $u(x,t)$:

$$u(x,t)=U(x,t)+u_s(x)=u_s(x)+\sum_{n=1}^{\infty}d_n\sin\frac{n\pi}{L}x e^{-\frac{n^2\pi^2}{a^2L^2}t}$$

Then the solution of the non-homogeneous heat equation with non-homogeneous Dirichlet boundary conditions becomes:

$$u(x,t)=\left[\frac{b}{2}x^2+\left(\frac{g_2-g_1}{L}-\frac{bL}{2}\right)x+g_1\right]+\frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L\left[u_0(x)-u_s(x)\right]\sin\frac{n\pi}{L}x\,dx\right\}\sin\frac{n\pi}{L}x e^{-\frac{n^2\pi^2}{a^2L^2}t}$$

5. Example Maple solution: (017 *heat3-1.mws)*

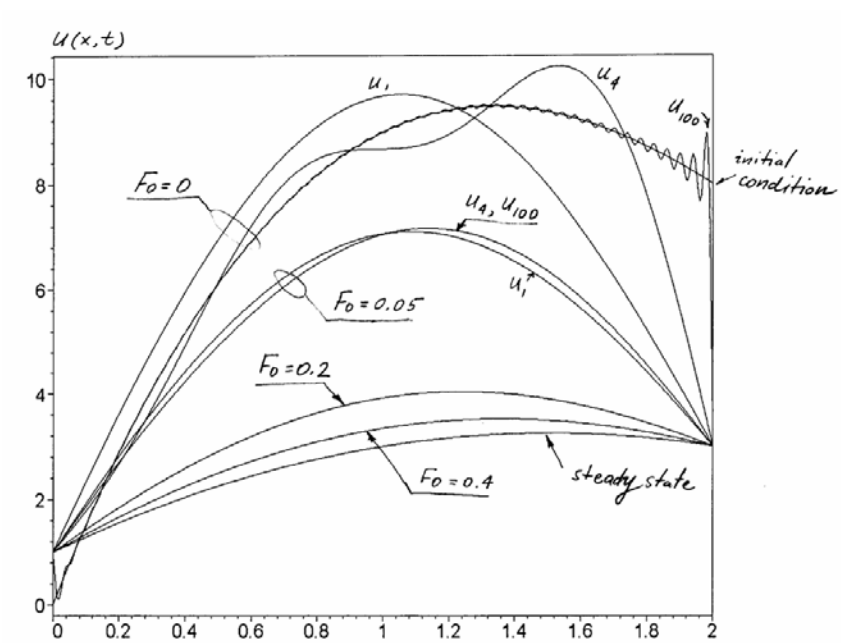
Remark: In practice, instead of the exact solution defined by the infinite series given in section 4, the truncated series is used for calculation of the approximate solution. How many terms are needed in the truncated series for the accurate approximation? Comparison of the exact solution (which is also a truncated series but with a very large number of terms, which we assume, provides an accurate result) with the calculation with a small number of terms in a truncated series shows that the accuracy depends on time: the further we proceed in time, the more accurate becomes an approximate solution (why?). For uniform characterization of physical processes, the non-dimensional parameters are used in engineering. In heat transfer, non-dimensional time is defined by the Fourier number:

$$Fo = \frac{\alpha t}{L^2}$$

where $\alpha = \dfrac{1}{a^2}$ is a thermal diffusivity (see section 4.3.2 4).

In engineering heat transfer analysis, a 4 term approximation is considered as an accurate approximation for all values of the Fourier number. For simplicity, very often even a 1 term approximation is used, which is considered to be accurate for $Fo > 0.2$ (error in most cases does not exceed *1%,* and this is a convention in engineering heat transfer).

Consider comparison of the exact solution (100 terms) with 1 and 4 terms approximations. Results are calculated for $Fo = 0.0$, $Fo = 0.05$, $Fo = 0.2$, $Fo = 0.4$. The lowest curve is a steady state solution. As can be seen from the figure, for $Fo > 0.2$, all results coincide.



017 **4.6.3 2)** **5. Example** *(heat3-1.mws)* **Heat Equation** non-homogeneous equation and boundary conditions (Dirichlet-Dirichlet)

```
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> L:=2;g1:=1;g2:=3;b:=-2;a:=0.5;
```

$$L := 2$$
$$g1 := 1$$
$$g2 := 3$$
$$b := -2$$
$$a := 0.5$$

Initial Condition:

```
> u[0](x):=x*(2*L-x)^2;
```

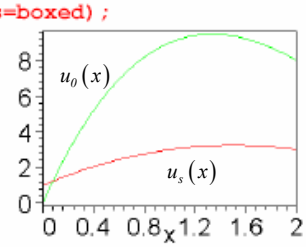
$$u_0(x) := x\,(4-x)^2$$

Steady State solution:

```
> us(x):=b*x^2/2+((g2-g1)/L-b*L/2)*x+g1;
```

$$\mathrm{us}(x) := -x^2 + 3\,x + 1$$

```
> plot({us(x),u[0](x)},x=0..L,axes=boxed);
```



Eigenfunctions:

```
> X[n]:=sin(n*Pi/L*x);
```

$$X_n := \sin\left(\frac{n\,\pi\,x}{2}\right)$$

```
> T[n]:=exp(-n^2*Pi^2/a^2/L^2*t);
```

$$T_n := \mathrm{e}^{(-1.000000000\,n^2\,\pi^2\,t)}$$

Fourier Coefficients:

```
> d[n]:=2/L*int((u[0](x)-us(x))*X[n],x=0..L);
```

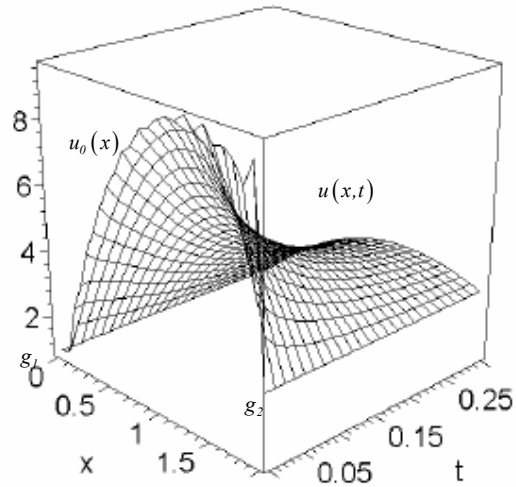
$$d_n := -\frac{2\,(6\,n^2\,\pi^2\sin(n\,\pi) + 8\,n\,\pi\cos(n\,\pi) + 5\,n^3\,\pi^3\cos(n\,\pi) + 48\sin(n\,\pi) + n^3\,\pi^3 - 56\,n\,\pi)}{n^4\,\pi^4}$$

```
> d[n]:=simplify(factor(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},d[n])));
```

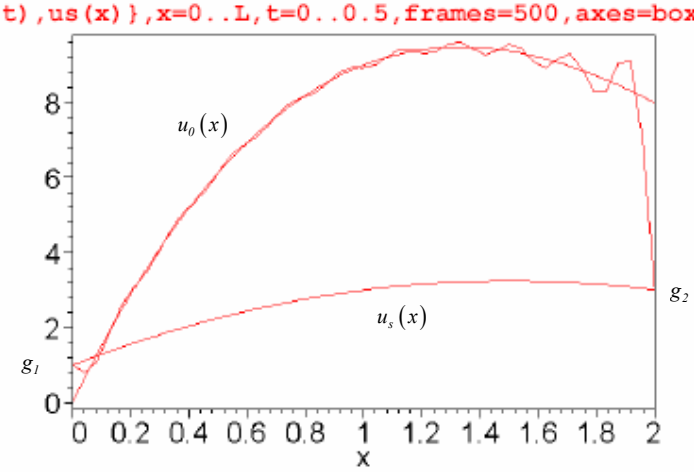
$$d_n := -\frac{2\,(8\,(-1)^n + 5\,n^2\,\pi^2\,(-1)^n + n^2\,\pi^2 - 56)}{n^3\,\pi^3}$$

Solution:

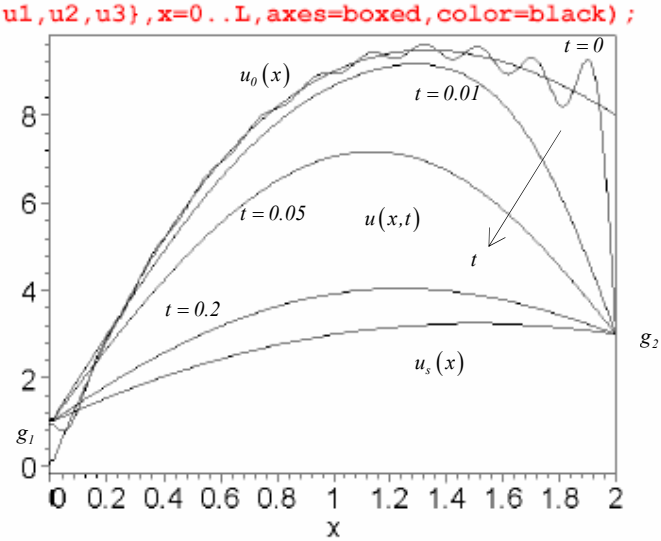
```
> u(x,t):=us(x)+sum(d[n]*X[n]*T[n],n=1..20):
> plot3d(u(x,t),x=0..L,t=0..0.3,axes=boxed,projection=0.9,style=wireframe,color=black);
```




```
> animate({u[0](x),u(x,t),us(x)},x=0..L,t=0..0.5,frames=500,axes=boxed);
```



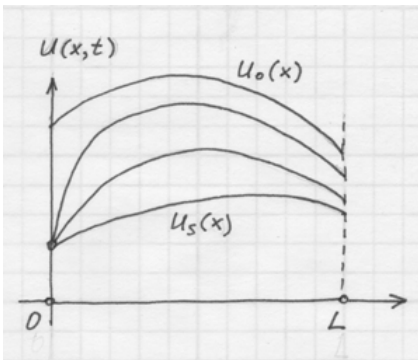

```
> u0:=subs(t=0,u(x,t)):
> u1:=subs(t=0.01,u(x,t)):
> u2:=subs(t=0.05,u(x,t)):
> u3:=subs(t=0.2,u(x,t)):
> plot({us(x),u[0](x),u0,u1,u2,u3},x=0..L,axes=boxed,color=black);
```



Note: the Gibbs effect is observed only for initial temperature profile

Accuracy of truncated solution (**see on web-site**):

4.6.3 *3) Heat Equation with Dirichlet and Robin boundary conditions – application of Sturm-Liouville Theorem*



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = 0,\qquad t > 0$ (Dirichlet)

$k\dfrac{\partial u(L,t)}{dx} + hu(L,t) = 0,\qquad t > 0$ (Robin)

Rewrite second condition as

$$\frac{\partial u(L,t)}{dx} + Hu(L,t) = 0, \text{ where } H = \frac{h}{k},\ \ H > 0$$

1. Separation of variables

We assume that the function $u(x,t)$ can be represented as a product of two functions each of a single variable
$u(x,y) = X(x)T(t)$
From the analysis of the heat transfer equation, we know that it leads to a separated equation
$\dfrac{X''}{X} = a^2 \dfrac{T'}{T} = \mu$ where μ is a separation constant.
That yields two ordinary differential equations:

$$X'' - \mu X = 0 \quad \text{and} \quad T' - \frac{\mu}{a^2} T = 0$$

2. Boundary conditions

$\underline{x = 0}$ $\quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$

$\underline{x = L}$ $\quad X'(L)T(t) + HX(L)T(t) = 0 \quad \Rightarrow \quad X'(L) + HX(L) = 0$

3. Solution of Sturm-Liouville problem

Solutions of the first o.d.e. is determined by (depending on the form of separation constant μ):

$$X(x) = \begin{cases} c_1 \cosh\sqrt{\mu}x + c_2 \sinh\sqrt{\mu}x & \mu > 0 \\ c_1 + c_2 x & \mu = 0 \\ c_1 \cos\sqrt{-\mu}x + c_2 \sin\sqrt{-\mu}x & \mu < 0 \end{cases}$$

Apply first boundary condition:

$$X(0) = 0 = \begin{cases} c_1 \cosh 0 + c_2 \sinh 0 \\ c_1 + c_2 0 \\ c_1 \cos\sqrt{-\mu}0 + c_2 \sin 0 \end{cases} \quad \Rightarrow \quad c_1 = 0$$

So, the solution and its derivative are written as:

$$X(x) = \begin{cases} c_2 \sinh\sqrt{\mu}x \\ c_2 x \\ c_2 \sin\sqrt{-\mu}x \end{cases} \quad X'(x) = \begin{cases} c_2\sqrt{\mu}\cosh\sqrt{\mu}x & \mu > 0 \\ c_2 & \mu = 0 \\ c_2\sqrt{-\mu}\cos\sqrt{-\mu}x & \mu < 0 \end{cases}$$

Consider the second boundary condition

$$\begin{cases} c_2\sqrt{\mu}\cosh\sqrt{\mu}L + c_2 H\sinh\sqrt{\mu}L = 0 \\ c_2 + c_2 HL = 0 \\ c_2\sqrt{-\mu}\sin\sqrt{-\mu}L + c_2 H\cos\sqrt{-\mu}L = 0 \end{cases}$$

Since we are looking for a non-trivial solution, we require $c_2 \neq 0$.
Consider the first equation (for $\mu > 0$)

$$c_2\sqrt{\mu}\cosh\sqrt{\mu}L + c_2 H\sinh\sqrt{\mu}L = 0$$

division by c_2 yields

$$\sqrt{\mu}\cosh\sqrt{\mu}L + H\sinh\sqrt{\mu}L = 0$$

This equation does not have a solution for positive μ.

The second equation (for $\mu = 0$)

$$c_2 + c_2 HL = 0$$

leads immediately to $c_2 = 0$.

The third equation ($\mu < 0$, denote for convenience $\mu = -\lambda^2$) is

Characteristic equation

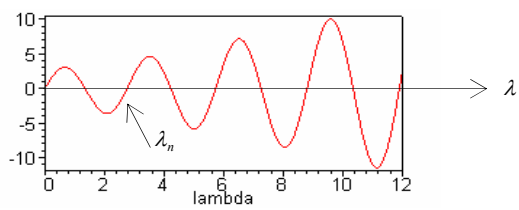
$$\lambda\cos\lambda L + H\sin\lambda x = 0$$

There are infinitely many positive roots of this equation λ_n, $n = 1,2,\ldots$.

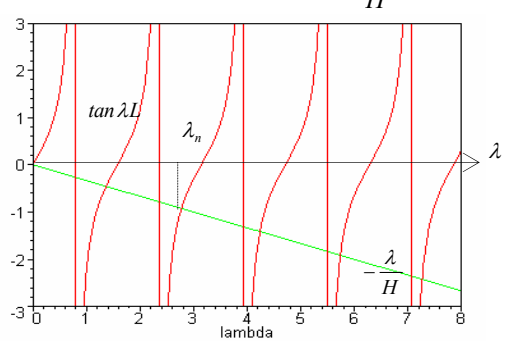
Graphically, they can be shown as intersections of the graph of function

$$w(\lambda) = \lambda\cos\lambda L + H\sin\lambda L$$

with the λ -axis:



or, if we rewrite the equation in the form which is traditionally used in the textbooks, $\tan\lambda L = -\dfrac{\lambda}{H}$, they are shown as the abscises of intersection of the graph $\tan\lambda L$ with the graph of $-\dfrac{\lambda}{H}$:



Positive roots λ_n are called to be ***eigenvalues***.
The corresponding solution functions

$$X_n = \sin\lambda_n x$$

are called to be ***eigenfunctions***.

4. Solution for *T(t)*

With determined eigenvalues, the solution for T becomes:

$$T_n(t) = e^{\frac{-\lambda_n^2}{a^2}t}$$

5. Basic solutions

Recalling the assumed form of solution, we construct the basic solution

$$u_n(x,t) = X_n T_n = \sin\lambda_n x e^{\frac{-\lambda_n^2}{a^2}t}$$

Then the solution of the given IBVP is in the vector space spanned by the defined above basic functions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\lambda_n x e^{\frac{-\lambda_n^2}{a^2}t}$$

This solution satisfies the heat equation and boundary conditions. We want to define coefficients a_n in a such a way that the obtained solution satisfies also the initial condition at $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\lambda_n x = u_0(x)$$

In our problem, functions $\sin\lambda_n x$ are obtained as eigenfunctions of the Sturm-Liouville problem for the equation $-X'' = \lambda^2 X$; therefore, the set of all eigenfunctions is a complete system of functions orthogonal with respect to the weight function $p = 1$. Then, the last equation is an expansion of the function $u_0(x)$ in a generalized Fourier series over the interval $(0,L)$ with coefficients defined by

$$a_n = \frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}$$

6. Solution

Then, the solution of the initial-boundary value problem is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}\right]\sin\lambda_n x e^{\frac{-\lambda_n^2}{a^2}t}$$

where the squared norm of eigenfunctions may be evaluated after integration as

$$\|X_n\|^2 = \int_0^L \sin^2\lambda_n x dx = \frac{L}{2} - \frac{\sin 2\lambda_n L}{4\lambda_n}$$

Finally, the solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\frac{L}{2} - \frac{\sin 2\lambda_n L}{4\lambda_n}}\right]\sin\lambda_n x e^{\frac{-\lambda_n^2}{a^2}t}}$$

7. Example

(018 *Heat 4-1a.mws*) Let $L = 2$, $f(x) = x(2-x)$, $a = 0.5$

018 **4.6.3 3) 7. Example** *(heat4-1a.mws)* **Heat Equation** (Dirichlet-Robin)

```
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> L:=2;H:=3;A:=4;
```

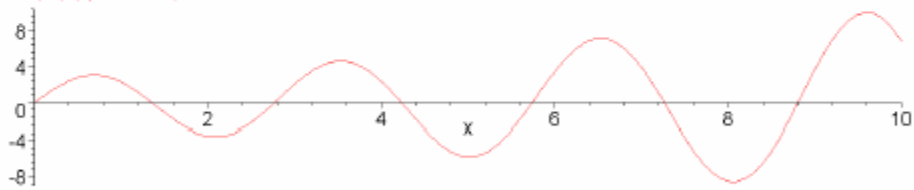
$$L := 2$$
$$H := 3$$
$$A := 4$$

Characteristic equation:

```
> w(x) :=x*cos(x*L)+H*sin(x*L);
```

$$w(x) := x\cos(2\,x) + 3\sin(2\,x)$$

```
> plot(w(x),x=0..10);
```



Eigenvalues:

```
> lambda:=array(1..50);
```

$$\lambda := \mathrm{array}(1..50, [\ ])$$

```
> n:=1: for m from 1 to 500 do z:=fsolve(w(x)=0,x=m/10..(m+1)/10): if type(z,float) then
  lambda[n]:=z: n:=n+1 fi od:
> for i to 6 do lambda[i] od;
```

$$1.358229874$$
$$2.768911636$$
$$4.235147453$$
$$5.738636645$$
$$7.264403196$$
$$8.803594038$$

```
> N:=n-1;
```

$$N := 32$$

```
> n:='n':i:='i':
```

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*x);
```

$$X_n := \sin(\lambda_n\, x)$$

Squared-norm:

```
> NX[n]:=int(X[n]^2,x=0..L);
```

$$NX_n := \frac{1}{2}\,\frac{-\cos(2\,\lambda_n)\sin(2\,\lambda_n) + 2\,\lambda_n}{\lambda_n}$$

GENERALIZED FOURIER SERIES

Function (initial condition):

```
> u0(x) :=x*(L-x)+1;
```

$$u0(x) := x\,(2 - x) + 1$$

Fourier coefficients:

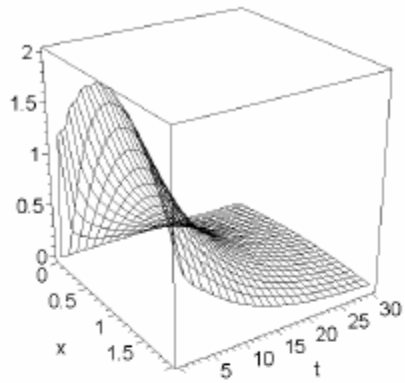
```
> a[n]:=simplify(int(u0(x)*X[n],x=0..L)/NX[n]);
```

$$a_n := -\frac{2\,(2\,\lambda_n\sin(2\,\lambda_n) + \lambda_n^{\,2}\cos(2\,\lambda_n) + 2\cos(2\,\lambda_n) - 2 - \lambda_n^{\,2})}{\lambda_n^{\,2}\,(-\cos(2\,\lambda_n)\sin(2\,\lambda_n) + 2\,\lambda_n)}$$

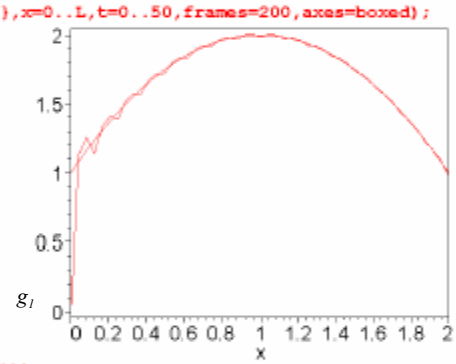
Generalized Fourier series:

```
> u(x,t) :=sum(a[n]*X[n]*exp(-lambda[n]^2*t/A^2),n=1..N):
```

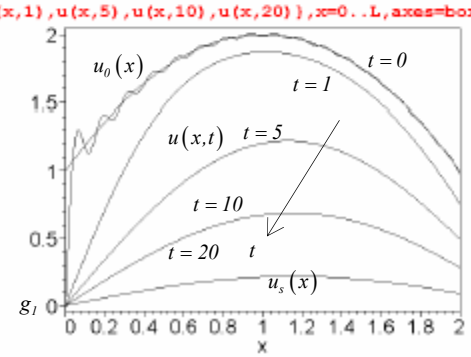
```
> plot3d(u(x,t),x=0..L,t=0..30,axes=boxed,projection=0.9,color=black,style=wireframe);
```



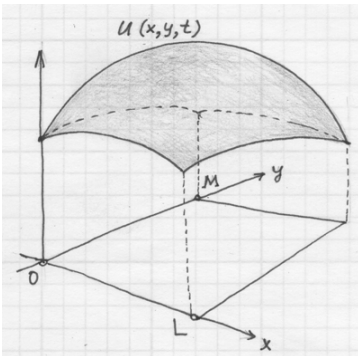

```
> animate((u0(x),u(x,t)},x=0..L,t=0..50,frames=200,axes=boxed);
```




```
> u(x,0):=subs(t=0,u(x,t)):
> u(x,1):=subs(t=1,u(x,t)):
> u(x,5):=subs(t=5,u(x,t)):
> u(x,10):=subs(t=10,u(x,t)):
> u(x,20):=subs(t=20,u(x,t)):
> plot({u0(x),u(x,0),u(x,1),u(x,5),u(x,10),u(x,20)},x=0..L,axes=boxed,color=black);
```



4.6.3 *4) Heat Equation in 2-D rectangular domain*



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=a^2\frac{\partial u}{\partial t} \qquad u(x,y,t),\quad (x,y)\in(0,L)\times(0,M),\quad t>0$$

initial condition: $u(x,y,0)=u_0(x,y)$

boundary conditions:

$x=0 \quad u(0,y,t)=0 \quad y\in(0,M),\, t>0$ (Dirichlet)

$x=L \quad u(L,y,t)=f(y) \quad y\in(0,M),\, t>0$ (Dirichlet)

$y=0 \quad \frac{\partial u}{\partial y}(x,0,t)=0 \quad x\in(0,L),\ t>0$ (Neumann)

$y=M \quad \frac{\partial u}{\partial y}(x,M,t)=0 \quad x\in(0,L),\ t>0$ (Neumann)

1. Steady State Solution

Although all boundary conditions except for one and the differential equation are homogeneous, for the case of an equation with three independent variables, it is useful to find the first time-independent steady state solution $u_s(x,y)$ and then to use it for a change of dependent variable. Thus, we are looking for a steady state solution which satisfies the differential equation

$$\frac{\partial^2 u_s}{\partial x^2}+\frac{\partial^2 u_s}{\partial y^2}=0$$

and boundary conditions:

$x=0 \quad u_s(0,y)=0 \quad y\in(0,M)$ (Dirichlet)

$x=L \quad u_s(L,y)=f(y) \quad y\in(0,M)$ (Dirichlet)

$y=0 \quad \frac{\partial u_s}{\partial y}(x,0)=0 \quad x\in(0,L)$ (Neumann)

$y=M \quad \frac{\partial u_s}{\partial y}(x,M)=0 \quad x\in(0,L)$ (Neumann)

This is the basic case of BVP for Laplace's Equation when all but one boundary conditions are homogeneous (if the equation is a non-homogeneous Poisson's Equation or more boundary conditions are non-homogeneous, the superposition principle should be used to reduce the problem to the set of supplemental basic problems).

Separation of variables Assume that the steady state solution can be written as a product of two functions $u_s(x,y)=X(x)Y(y)$, substitute it into the differential equation, and separate the variables:

$$-\frac{X''}{X}=\frac{Y''}{Y}=\mu$$

Choose the first equation for Y, because both conditions for Y are homogeneous. Then we have the following Sturm-Liouville problem:

$$Y''-\mu Y=0 \qquad Y'(0)=0 \qquad Y'(M)=0$$

According to the table with the results for the Sturm-Liouville problem, it has the following eigenvalues and eigenfunctions:

$$\mu_m=-\lambda_m^2=-\frac{m^2\pi^2}{M^2} \qquad m=0,1,2,\ldots$$

$$Y_0=1 \qquad m=0 \qquad \|Y_0\|^2=M$$

$$Y_m=\cos\frac{m\pi}{M}y \qquad m=1,2,\ldots \qquad \|Y_m\|^2=\frac{M}{2}$$

Then the second ODE in the separated equation is determined according to the form of the separation constant

$$X'' - \frac{m^2\pi^2}{M^2}X = 0 \qquad X(0) = 0 \qquad m = 0,1,2,...$$

But it has only one boundary condition because only one boundary condition at $x = 0$ is homogeneous. The general solution can be written with the hyperbolic functions:

$$X_0 = d_1 + d_2 x$$

$$X_m = d_1 \cosh\frac{m\pi}{M}x + d_2 \sinh\frac{m\pi}{M}x \quad m = 1,2,...$$

Apply the boundary condition at $x = 0$ and require a non-zero solution:

$$X_0(0) = d_1 + d_2 \cdot 0 = 0 \qquad \Rightarrow \qquad d_1 = 0$$

$$X_m(0) = d_1 \cosh\frac{m\pi}{M}0 + d_2 \sinh\frac{m\pi}{M}0 = 0 \qquad \Rightarrow \qquad d_1 = 0$$

Then solutions for X can be chosen as

$$X_0 = x$$

$$X_m = \sinh\frac{m\pi}{M}x \qquad m = 1,2,...$$

Therefore, the basic functions for steady state solution are

$$u_{s,0}(x,y) = X_0 Y_0 = x$$

$$u_{s,m}(x,y) = X_m Y_m = \sinh\frac{m\pi}{M}x \cos\frac{m\pi}{M}y \qquad m = 1,2,...$$

Then the steady state solution can be written in the form of infinite series

steady state solution

$$\boxed{u_s(x,y) = a_0 x + \sum_{m=1}^{\infty} a_m \sinh\frac{m\pi}{M}x \cos\frac{m\pi}{M}y}$$

where coefficients a_m have to be determined using the non-homogeneous boundary condition

$$x = L \qquad u_s(L,y) = f(y) = a_0 L + \sum_{m=1}^{\infty}\left(a_m \sinh\frac{m\pi}{M}L\right)\cos\frac{m\pi}{M}y$$

If this infinite series is treated as the generalized Fourier series expansion of the function $f(y)$, then coefficients a_m can be determined as

$$a_0 L = \frac{1}{M}\int_0^M f(y)dy \qquad \Rightarrow \qquad a_0 = \frac{1}{LM}\int_0^M f(y)dy$$

$$a_m \sinh\frac{m\pi}{M}L = \frac{2}{M}\int_0^M f(y)\cos\left(\frac{m\pi}{M}y\right)dy \qquad \Rightarrow \qquad a_m = \frac{2}{M\sinh\frac{m\pi}{M}L}\int_0^M f(y)\cos\left(\frac{m\pi}{M}y\right)dy$$

2. Transient Solution

Change the unknown function to

$$U(x,y,t) = u(x,y,t) - u_s(x,y)$$

It can be easily verified that this function satisfies the unsteady Heat Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = a^2\frac{\partial U}{\partial t}$$

and four homogeneous boundary conditions:

$x = 0$	$U(0,y,t) = 0$	$y \in (0,M),\ t > 0$	(Dirichlet)
$x = L$	$U(L,y,t) = 0$	$y \in (0,M),\ t > 0$	(Dirichlet)
$y = 0$	$\frac{\partial U}{\partial y}(x,0,t) = 0$	$x \in (0,L),\ \ t > 0$	(Neumann)

$$y = M \quad \frac{\partial U}{\partial y}(x, M, t) = 0 \quad x \in (0, L), \quad t > 0 \qquad \text{(Neumann)}$$

and initial condition becomes

$$U(x, y, 0) = g(x, y) - u_s(x, y)$$

This IBVP Is now well suited for solution by separation of variables. We assume that the function $U(x, y, t)$ can be written as a product of three functions

$$U(x, y, t) = X(x)Y(y)T(t)$$

each of a single variable. Substitute it into the Heat Equation

$$X''YT + XY''T = a^2 XYT'$$

Rewrite this equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} + a^2 \frac{T'}{T}$$

Variables in this equation are not completely separated, but both sides are functions of different variables, and, therefore, do not depend on either of them and are equal to some constant

$$\frac{X''}{X} = -\frac{Y''}{Y} + a^2 \frac{T'}{T} = \mu$$

It yields a Sturm Liouville problem

$$X'' - \mu X = 0 \qquad X(0) = 0 \qquad X(L) = 0$$

which has the following non-trivial solutions (eigenfunctions) for the corresponding values of the separation constant

$$\mu_n = -\lambda_n^2 = -\frac{n^2\pi^2}{L^2} \qquad n = 1,2,...$$

$$X_n = sin\frac{n\pi}{L}x \qquad m = 1,2,... \qquad \|X_n\|^2 = \frac{L}{2}$$

Substitute determined values of the separation constant into the second part of the equation

$$-\frac{Y''}{Y} + a^2 \frac{T'}{T} = -\frac{n^2\pi^2}{L^2}$$

and separate variables

$$\frac{Y''}{Y} = a^2 \frac{T'}{T} + \frac{n^2\pi^2}{L^2}$$

By the same reasoning, both parts of the equation are just a constant

$$\frac{Y''}{Y} = a^2 \frac{T'}{T} + \frac{n^2\pi^2}{L^2} = \eta$$

Consider the equation for Y

$$Y'' - \eta Y = 0 \qquad Y'(0) = 0 \qquad Y'(M) = 0$$

which has the following solutions

$$\eta_m = -\frac{m^2\pi^2}{M^2} \qquad m = 0,1,2,...$$

$$Y_0 = 1 \qquad m = 0 \qquad \|Y_0\|^2 = M$$

$$Y_m = cos\frac{m\pi}{M}y \qquad m = 1,2,... \qquad \|Y_m\|^2 = \frac{M}{2}$$

Then the separated equation becomes an equation for the function T

$$a^2 \frac{T'}{T} + \frac{n^2\pi^2}{L^2} = -\frac{m^2\pi^2}{M^2}$$

Solutions of this equation are:

$$T_{mn} = e^{-\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t}$$

Then the basic functions for the transient solution are

$$U_{mn}(x, y, t) = X_n(x)Y_m(y)T_{mn}(t) = sin\left(\frac{n\pi}{L}x\right)cos\left(\frac{m\pi}{M}y\right)e^{-\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t}$$

All these basic functions satisfy the Heat Equation and all boundary conditions. Construct a transient solution in the form of a double infinite series

transient solution

$$\boxed{U(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\,sin\left(\frac{n\pi}{L}x\right)cos\left(\frac{m\pi}{M}y\right)e^{-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t}}$$

where coefficients A_{mn} are chosen such that the constructed function satisfies the initial condition:

$$U(x,y,0)=g(x,y)-u_s(x,y)\qquad=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\,sin\left(\frac{n\pi}{L}x\right)cos\left(\frac{m\pi}{M}y\right)$$

Basis functions in this expansion are solutions of the corresponding Sturm-Liouville problem, therefore, they are orthogonal functions used in a generalized Fourier series. If the equation above is treated as a double Fourier series then coefficients A_{mn} can be determined in two steps as follows:
rewrite the equation in the form

$$g(x,y)-u_s(x,y)\;=\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}A_{mn}\,sin\left(\frac{n\pi}{L}x\right)\right]cos\left(\frac{m\pi}{M}y\right)$$
$$=\sum_{m=0}^{\infty}B_m(x)cos\left(\frac{m\pi}{M}y\right)$$

where the following notation is used for coefficients

$$B_m(x)=\sum_{n=1}^{\infty}A_{mn}\,sin\left(\frac{n\pi}{L}x\right)$$

which are coefficients of a cosine Fourier series expansion of the initial condition

$$B_0(x)=\frac{1}{M}\int_0^M[g(x,y)-u_s(x,y)]dy$$

$$B_m(x)=\frac{2}{M}\int_0^M[g(x,y)-u_s(x,y)]cos\frac{m\pi}{M}ydy\quad m=1,2,...$$

on the other hand by definition they are

$$B_0(x)=\frac{1}{M}\int_0^M[g(x,y)-u_s(x,y)]dy\qquad=\sum_{n=1}^{\infty}A_{0n}\,sin\left(\frac{n\pi}{L}x\right)$$

$$B_m(x)=\frac{2}{M}\int_0^M[g(x,y)-u_s(x,y)]cos\frac{m\pi}{M}ydy\quad=\sum_{n=1}^{\infty}A_{mn}\,sin\left(\frac{n\pi}{L}x\right)$$

in which A_{mn} are coefficients of a sine Fourier series expansion. Therefore,

$$A_{0n}\qquad=\frac{2}{L}\int_0^L\left\{\frac{1}{M}\int_0^M[g(x,y)-u_s(x,y)]dy\right\}sin\frac{n\pi}{L}xdx$$
$$=\frac{2}{LM}\int_0^L\int_0^M[g(x,y)-u_s(x,y)]sin\left(\frac{n\pi}{L}x\right)dydx$$

$$A_{mn}\qquad=\frac{2}{L}\int_0^L\left\{\frac{2}{M}\int_0^M[g(x,y)-u_s(x,y)]cos\frac{m\pi}{M}ydy\right\}sin\frac{n\pi}{L}dx$$
$$=\frac{4}{LM}\int_0^L\int_0^M[g(x,y)-u_s(x,y)]cos\left(\frac{m\pi}{M}y\right)sin\left(\frac{n\pi}{L}y\right)dydx$$

3. Solution of IBVP

Now, when coefficients of a double series expansion for transient solution are determined, we can recall the equation for the change of dependent variable and write a solution for the original IBVP:

$$u(x,y,t) = U(x,y,t) + u_s(x,y)$$

$$\boxed{\begin{aligned} u(x,y,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, sin\left(\frac{n\pi}{L}x\right) cos\left(\frac{m\pi}{M}y\right) e^{-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t} \\ &\quad + a_0 L + \sum_{m=1}^{\infty}\left(a_m\, sinh\frac{m\pi}{M}L\right) cos\frac{m\pi}{M}y \end{aligned}}$$

where coefficients of expansions are

$$A_{0n} = \frac{2}{LM}\int_0^L\int_0^M [g(x,y) - u_s(x,y)]\, sin\left(\frac{n\pi}{L}x\right) dydx$$

$$A_{mn} = \frac{4}{LM}\int_0^L\int_0^M [g(x,y) - u_s(x,y)]\, cos\left(\frac{m\pi}{M}y\right) sin\left(\frac{n\pi}{L}y\right) dydx$$

$$a_0 = \frac{1}{LM}\int_0^M f(y)dy$$

$$a_m = \frac{2}{M\, sinh\frac{m\pi}{M}L}\int_0^M f(y)\, cos\left(\frac{m\pi}{M}y\right) dy$$

4. Example

Maple solution: *heat5dn-2.mws*

$L = 2$
$M = 4$
$\alpha = 0.5$

$f(y) = 1$ fixed temperature at the boundary

$g(x,y) = x(x-L) + y(y-M)$ parabolic initial temperature distribution

019 **4.6.2.4) 4. Example** *(heat5dn-2.mws)* **2-D Heat Equation** **DD-NN**

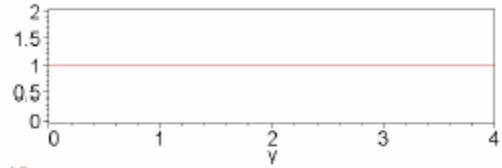
```
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> L:=2;M:=4;alpha:=0.5;
```

$$L := 2$$
$$M := 4$$
$$\alpha := 0.5$$

```
> f(y):=1;
```

$$f(y) := 1$$

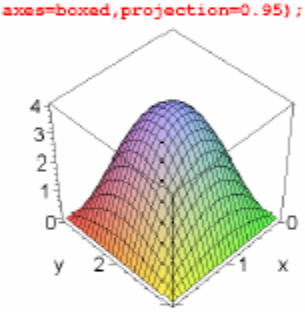
```
> plot(f(y),y=0..M,axes=boxed);
```




```
> g(x,y):=x*(x-L)*y*(y-M);
```

$$g(x,y) := x(x-2)y(y-4)$$

```
> plot3d(g(x,y),x=0..L,y=0..M,axes=boxed,projection=0.95);
```



Steady State Solution:

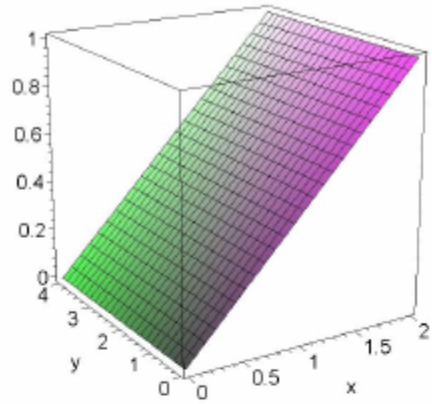
```
> a[0]:=int(f(y),y=0..M)/L/M;
```

$$a_0 := \frac{1}{2}$$

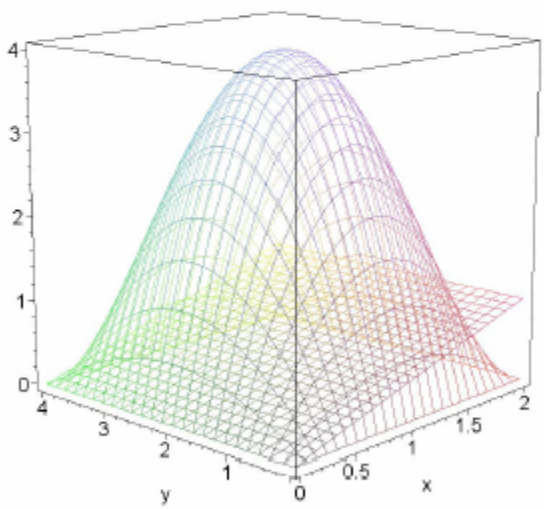
```
> a[m]:=2*int(f(y)*cos(m*Pi*y/M),y=0..M)/sinh(m*Pi*L/M)/M;
```

$$a_m := \frac{2\sin(m\pi)}{m\pi\sinh\left(\frac{m\pi}{2}\right)}$$

```
> us[m](x,y):=a[m]*sinh(m*Pi*x/M)*cos(m*Pi*y/M):
> us(x,y):=a[0]*x+sum(us[m](x,y),m=1..2):
> plot3d(us(x,y),x=0..L,y=0..M,axes=boxed,projection=0.92);
```




```
> plot3d({us(x,y),g(x,y)},x=0..L,y=0..M,axes=boxed,style=wireframe,projection=0.92);
```



Supplemental Problem:

```
> w(x,y):=g(x,y)-us(x,y);
```

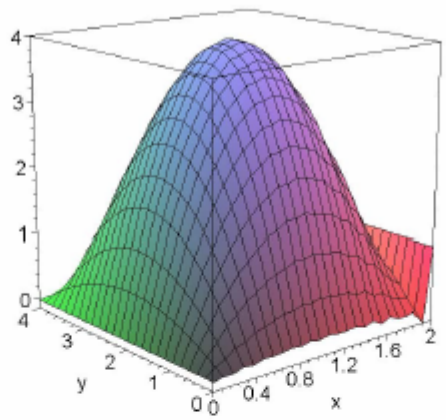
$$w(x,y) := x(x-2)y(y-4) - \frac{x}{2}$$

```
> A[n,0]:=2*int(int(w(x,y),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:
> A[n,m]:=4*int(int(w(x,y)*cos(m*Pi*y/M),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:
> U[n,0](x,y,t):=A[n,0]*sin(n*Pi*x/L)*exp(-n^2/L^2*Pi^2*t/alpha^2):
> U[n,m](x,y,t):=A[n,m]*sin(n*Pi*x/L)*cos(m*Pi*y/M)*exp(-(m^2/M^2+n^2/L^2)*Pi^2*t/alpha^2):
> U(x,y,t):=sum(U[n,0](x,y,t),n=1..20)+sum(sum(U[n,m](x,y,t),m=1..25),n=1..20):
```

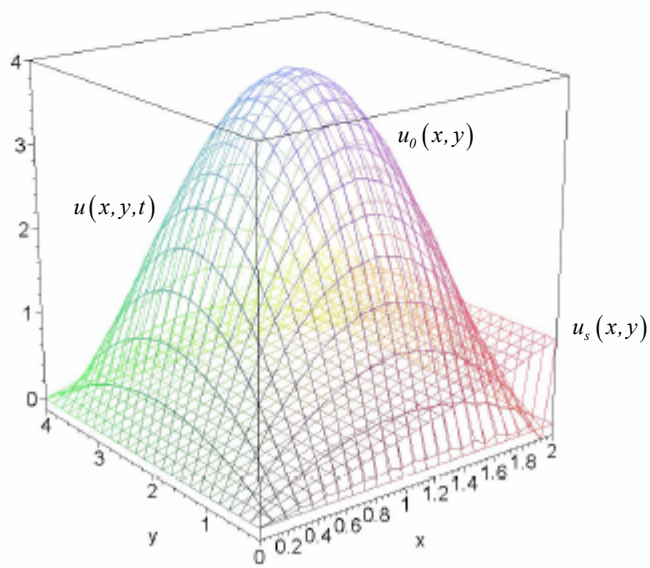
```
> U(x,y,0):=subs(t=0,U(x,y,t)):
```

Solution of IBVP:

```
> u(x,y,t):=us(x,y)+U(x,y,t):
> u(x,y,0):=subs(t=0,u(x,y,t)):
> animate3d(u(x,y,t),x=0..L,y=0..M,t=0..0.3,frames=100,axes=boxed,projection=0.92);
```

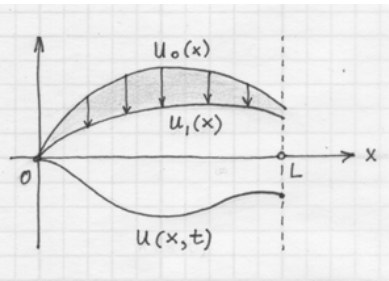



```
> animate3d({us(x,y),g(x,y),u(x,y,0),u(x,y,t)},x=0..L,y=0..M,t=0..0.3,frames=100,axes=boxed
  ,style=wireframe,projection=0.92);
```



4.6.4 Wave Equation

4.6.4 1) homogeneous equation with homogeneous boundary conditions



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

Initial conditions: $u(x,0) = u_0(x)$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x)$$

Boundary conditions: $u(0,t) = 0$, $\quad t > 0$ (Dirichlet)

$$k_2 \frac{\partial u(L,t)}{\partial x} + h_2 u(x,t) = 0, \quad t > 0 \quad \text{(Robin)}$$

Denote $\quad H_2 = \dfrac{h_2}{k_2}$

1. Separation of variables

we assume that the function $u(x,t)$ can be represented as a product of two functions each of a single variable

$$u(x,y) = X(x)T(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \qquad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t) \qquad \text{substitute into equation}$$

$a^2 X''(x)T(t) = X(x)T''(t)$ After separation of variables, one gets

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = \mu \qquad \text{with a separation constant } \mu$$

That yields two ordinary differential equations:

$X'' - \mu X = 0$ and $T'' - \mu a^2 T = 0$

2. Sturm-Liouville problem

$$X'' - \mu X = 0$$

boundary conditions:

$\underline{x = 0}$ $\quad X(0)T(t) = 0 \quad \Rightarrow X(0) = 0$

$\underline{x = L}$ $\quad X'(L)T(t) + H_2 X(L)T(t) = 0 \quad \Rightarrow X'(L) + H_2 X(L) = 0$

This Sturm-Liouville problem has the following solution with $\mu_n = -\lambda_n^2$:

eigenvalues $\quad \lambda_n$ are positive roots of equation $\lambda \cos \lambda L + H_2 \sin \lambda_n L = 0$

eigenfunctions $\quad X_n(x) = \sin \lambda_n x$

Then solutions of the second differential equation $T'' + \lambda_n^2 a^2 T = 0$ are

$$T_n(t) = c_1 \cos \lambda_n at + c_2 \sin \lambda_n at$$

basic solutions:

$$\boxed{u_n(x,t) = X_n T_n = \sin \lambda_n x (c_1 \cos \lambda_n at + c_2 \sin \lambda_n at)}$$

We are looking for a solution in the vector space with the basis $\{u_n(x,t)\}$:

$$u(x,t)=\sum_{n=1}^{\infty} a_n u_n(x,t)=\sum_{n=1}^{\infty} a_n \sin\lambda_n x(c_1 \cos\lambda_n at + c_2 \sin\lambda_n at)$$

$$u(x,t)=\sum_{n=1}^{\infty} \sin\lambda_n x(a_n c_1 \cos\lambda_n at + a_n c_2 \sin\lambda_n at)$$

$$u(x,t)=\sum_{n=1}^{\infty} \sin\lambda_n x(b_n \cos\lambda_n at + d_n \sin\lambda_n at)$$

initial conditions:

$\underline{t=0}$ $\quad u(x,0)=\sum_{n=1}^{\infty} b_n \sin\lambda_n ax = u_0(x)$

which is a generalized Fourier series expansion of the function $f(x)$ over the interval $(0,L)$ with coefficients

$$b_n=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

The derivative with respect to t of the assumed solution is

$$\frac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}\lambda_n a \sin\lambda_n x(-b_n \sin\lambda_n at + d_2 \cos\lambda_n at)$$

Then the second initial condition yields

$\underline{t=0}$ $\quad \frac{\partial u(x,0)}{\partial t}=\sum_{n=1}^{\infty} d_n\lambda_n a \sin\lambda_n x = u_1(x)$

Again, it can be treated as a Fourier series with coefficients

$$d_n\lambda_n a=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

$$d_n=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\lambda_n a\left(\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}\right)}$$

Then the solution of the initial-boundary value problem is:

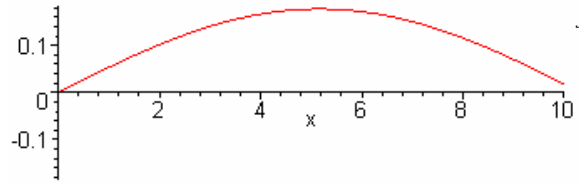
3. Solution:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\sin\lambda_n x\{b_n \cos\lambda_n at + d_n \sin\lambda_n at\} \\ &= \sum_{n=1}^{\infty}\frac{\sin\lambda_n x}{\left(\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}\right)}\left\{\left[\int_0^L u_0(x)\sin\lambda_n x dx\right]\cos\lambda_n at+\left[\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\lambda_n a}\right]\sin\lambda_n at\right\} \end{aligned}$$

4. Normal modes of string vibration

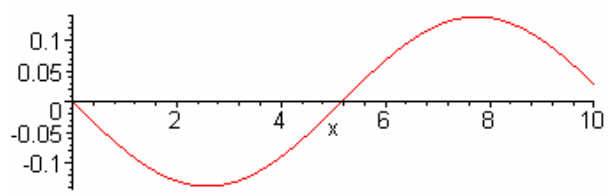
Solution of the Wave Equation is obtained as a sum of terms of the form $u_n(x,t) = X_n T_n = \sin \lambda_n x (c_1 \cos \lambda_n at + c_2 \sin \lambda_n at)$ which we called the basic solutions, but as the contributors to the vibration of string, these functions are known as **normal modes**. In our example, for $n = 1,2,3,4,\ldots$ they have the following shape (see Maple file for animation):

```
> m1:=subs(n=1,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m1},x=0..L,t=0..9);
```



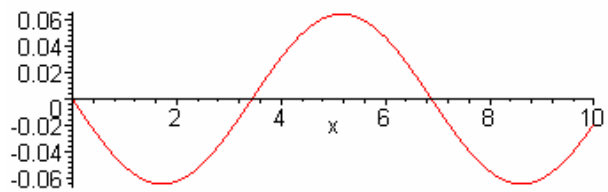
fundamental mode

```
> m2:=subs(n=2,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m2},x=0..L,t=0..9);
```



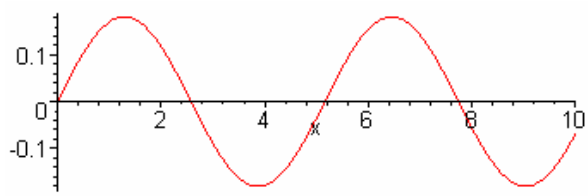
1st overtone

```
> m3:=subs(n=3,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m3},x=0..L,t=0..9);
```



2nd overtone

```
> m4:=subs(n=4,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m4},x=0..L,t=0..9);
```



3rd overtone

The first of these normal modes is called the **fundamental mode**, others are called the **first overtone**, the **second overtone**, and so on. The **frequency** of oscillation of the normal mode is increased with its number and is determined by the corresponding eigenvalue λ_n and coefficient a which has a physical sense of the speed of propagation of the waves (speed of sound). There are fixed points in the vibration of overtones.

The whole motion of the string is a superposition of vibration of all overtones with different amplitude. The involvement of different modes in the vibration of string is determined by initial conditions. If for representation of the initial shape of the string at rest, different modes are required, then all of them will be present in the undamped vibration of the string. But if the initial shape of the string is exactly one of the overtones, then only this mode will be present the string vibration. This phenomenon is called **standing waves**. Standing waves do not propagate, only shrink and swell in the same shape.

020 **4.6.4 1) Example** *(wave-1-DR.mws)* **WAVE EQUATION** X(0)=0, X'(L)+HX(L)=0 (Dirichlet-Robin)

```
> restart;
> L:=10;H[2]:=3;a:=2;
```

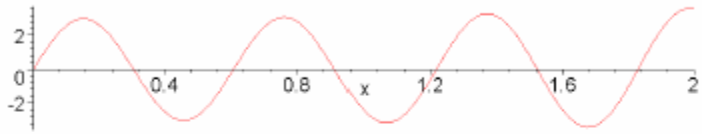
$$L := 10$$
$$H_2 := 3$$
$$a := 2$$

equation:

```
> w(x):=x*cos(x*L)+H[2]*sin(x*L);
```

$$w(x) := x\cos(10\,x) + 3\sin(10\,x)$$

```
> plot(w(x),x=0..2);
```



Eigenvalues:

```
> lambda:=array(1..100);
```

$$\lambda := \mathrm{array}(1\,..\,100, [\ ])$$

```
> n:=1: for m from 1 to 100 do z:=fsolve(w(x)=0,x=m/5..(m+1)/5): if type(z,float) then
  lambda[n]:=z: n:=n+1 fi od:
> for i to 5 do lambda[i] od;
```

$$0.3040584751$$
$$0.6083126850$$
$$0.9129369619$$
$$1.218068300$$
$$1.523798943$$

```
> N:=n-1;
```

$$N := 64$$

```
> n:='n':i:='i':
```

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*x);
```

$$X_n := \sin(\lambda_n\,x)$$

Squared norm:

```
> NX2[n]:=int(X[n]^2,x=0..L);
```

$$NX2_n := -\frac{1}{2}\,\frac{\cos(10\,\lambda_n)\sin(10\,\lambda_n) - 10\,\lambda_n}{\lambda_n}$$

Initial conditions:

sinusoidal:

```
> u0(x):=sin(x/L);u1(x):=sin(2*x/L);
```

$$u0(x) := \sin\left(\frac{x}{10}\right)$$
$$u1(x) := \sin\left(\frac{x}{5}\right)$$

impulse:

```
> u0(x):=(Heaviside(x-6)-Heaviside(x-7));u1(x):=0;
```

$$u0(x) := \mathrm{Heaviside}(x-6) - \mathrm{Heaviside}(x-7)$$
$$u1(x) := 0$$

parabolic:

```
> u0(x):=x*(L-x);u1(x):=1;
```

$$u0(x) := x\,(10 - x)$$
$$u1(x) := 1$$

Fourier coefficients:

```
> b[n]:=int(u0(x)*X[n],x=0..L)/NX2[n];
```

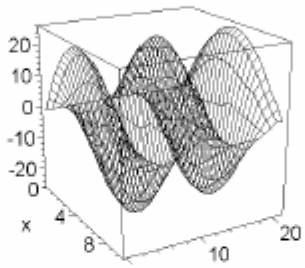
$$b_n := \frac{4\,(5\,\lambda_n \sin(10\,\lambda_n) + \cos(10\,\lambda_n) - 1)}{\lambda_n^{\,2}\,(\cos(10\,\lambda_n)\sin(10\,\lambda_n) - 10\,\lambda_n)}$$

```
> d[n]:=int(u1(x)*X[n],x=0..L)/NX2[n]/lambda[n]/a;
```

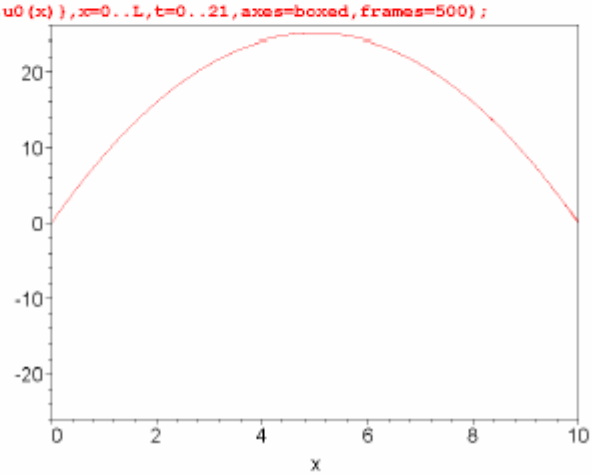
$$d_n := \frac{\cos(10\,\lambda_n) - 1}{\lambda_n\,(\cos(10\,\lambda_n)\sin(10\,\lambda_n) - 10\,\lambda_n)}$$

Solution:

```
> u(x,t):=sum(X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t)),n=1..N):
> plot3d((u(x,t)),x=0..L,t=0..21,axes=boxed,style=wireframe,projection=0.9,color=black);
```




```
> with(plots):
Warning, the name changecoords has been redefined
> animate((u(x,t),u0(x)),x=0..L,t=0..21,axes=boxed,frames=500);
```



impulse:

```
> u0(x):=(Heaviside(x-6)-Heaviside(x-7));u1(x):=0;
```

$$u0(x) := \text{Heaviside}(x-6) - \text{Heaviside}(x-7)$$

$$u1(x) := 0$$

Fourier coefficients:

```
> b[n]:=int(u0(x)*X[n],x=0..L)/NX2[n];
```

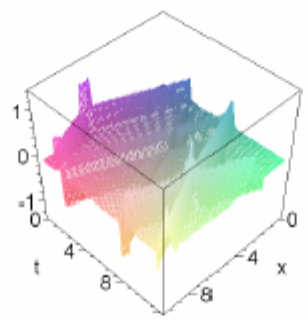
$$b_n := -2\left(\frac{2\,(32\cos(\lambda_n)^6 - 48\cos(\lambda_n)^4 + 18\cos(\lambda_n)^2 - 1 - 64\cos(\lambda_n)^7 + 112\cos(\lambda_n)^5 - 56\cos(\lambda_n)^3 + 7\cos(\lambda_n))}{\lambda_n} + \frac{-32\cos(\lambda_n)^6 + 48\cos(\lambda_n)^4 - 18\cos(\lambda_n)^2 + 1 + 64\cos(\lambda_n)^7 - 112\cos(\lambda_n)^5 + 56\cos(\lambda_n)^3 - 7\cos(\lambda_n)}{\lambda_n}\right)\lambda_n \Big/ (\cos(10\,\lambda_n)\sin(10\,\lambda_n) - 10\,\lambda_n)$$

```
> d[n]:=int(u1(x)*X[n],x=0..L)/NX2[n]/lambda[n]/a;
```

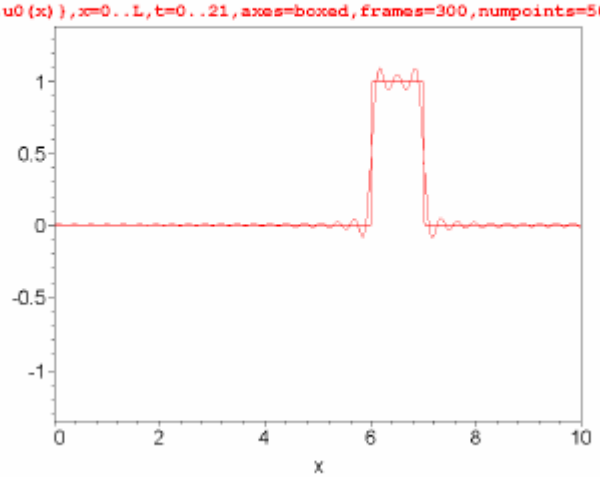
$$d_n := 0$$

Solution:

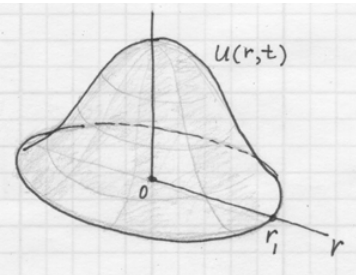
```
> u(x,t):=sum(X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t)),n=1..N):
> plot3d((u(x,t)),x=0..L,t=0..12,axes=boxed,projection=0.9,style=wireframe,grid=[150,150]);
```




```
> with(plots):
> animate((u(x,t),u0(x)),x=0..L,t=0..21,axes=boxed,frames=300,numpoints=500);
```



4.6.4 *2) Wave Equation in polar coordinates with angular symmetry*



$$\boxed{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}} \qquad u(r,t),\ \ r\in[0,r_1],\ t>0$$

Initial conditions: $u(r,0)=u_0(r)$

$$\frac{\partial u(r,0)}{\partial t}=u_1(r)$$

Boundary condition: $u(r_1,t)=0 \quad t>0$ (Dirichlet)

1. Separation of variables

Assume

$u(r,y)=R(r)T(t)$

then

$$\frac{\partial u}{\partial r}=R'(r)T(t),\quad \frac{\partial^2 u}{\partial r^2}=R''(r)T(t),\ \frac{\partial^2 u}{\partial t^2}=R(r)T''(t).$$

Substitute into the equation

$$R''(r)T(t)+\frac{1}{r}R'(r)T=\frac{1}{a^2}R(r)T''(t)$$

After separation of variables (division by $R(t)T(t)$), we receive

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=\frac{1}{a^2}\frac{T''}{T}=\mu \quad \text{with a separation constant } \mu$$

it yields two ordinary differential equations:

$$\boxed{R''+\frac{1}{r}R'-\mu R=0}$$

$$\boxed{T''-\mu a^2 T=0}$$

boundary condition:

$\underline{r=r_1}$ $\qquad u(r_1,t)=R(r_1)T(t)=0 \quad\Rightarrow\quad R(r_1)=0$

2. Solution of Sturm-Liouville problem

Consider first the boundary value problem from which we expect to obtain eigenvalues and eigenfunctions for construction of a functional vector space. Consider the equation for $R(r)$ for which we have a homogeneous boundary condition:

$$R''+\frac{1}{r}R'-\mu R=0 \qquad R(r_1)=0$$

Multiplication by r^2 yields

$$r^2R''+rR'-\mu r^2R=0$$

Solution of this equation depends on the form of the separation constant

$$R(r)=\begin{cases} d_1I_0(\lambda r)+d_2K_0(\lambda r) & \mu>0 \quad \mu=\lambda^2 \\ d_1\ln r+d_2 & \mu=0 \quad \mu=0 \\ d_1J_0(\lambda r)+d_2Y_0(\lambda r) & \mu<0 \quad \mu=-\lambda^2 \end{cases}$$

See, how these solutions were obtained and how the boundary condition can be satisfied:

1) $\boxed{\mu>0}$ Denote $\mu=\lambda^2$. Then the equation becomes

$$r^2R''+rR'-\left(\lambda^2r^2+0^2\right)R=0$$

This is a modified Bessel equation of integer order 0 for the independent variable $x = \lambda r$. General solution of this equation is given by

$$R(r) = d_1 I_0(\lambda r) + d_2 K_0(\lambda r)$$

At the interior point of the membrane $r = 0$, the solution has to be finite, therefore, we require the coefficient d_2 before function $K_0(\lambda r)$ (which is unbounded at $r = 0$) to be equal to zero, $d_2 = 0$.

Consider the boundary condition:

$$\underline{r = r_1} \qquad R(0) = d_1 I_0(0) = d_1 = 0$$

Therefore, the case $\mu > 0$ leads to a trivial solution.

2) $\boxed{\mu = 0}$ Equation becomes

$$R'' + \frac{1}{r} R' = 0$$

The order of equation can be reduced by a change of independent variable $\widetilde{R} = R'$. The equation for $\widetilde{R}$ is a first order linear differential equation

$$\widetilde{R}' + \frac{1}{r} \widetilde{R} = 0$$

solution of which is obtained with the help of an integrating factor

$$\widetilde{R}(r) = d_1 \left(e^{\int \frac{1}{r} dr} \right)^{-1} = d_1 \left(e^{\ln r} \right)^{-1} = d_1 (r)^{-1} = \frac{d_1}{r}$$

Then the solution of the original equation is

$$R(r) = \int \widetilde{R}(r) dr = \int \frac{d_1}{r} dr = d_1 \ln r + d_2$$

To have a finite solution at point $r = 0$, we must put $d_1 = 0$.

Then the boundary condition leads to $d_2 = 0$, and we end up with a trivial solution.

3) $\boxed{\mu < 0}$ Denote $\mu = -\lambda^2$. Then the equation becomes

$$r^2 R'' + r R' + \left(\lambda^2 r^2 - 0^2 \right) R = 0$$

which we can identify as a Bessel equation of order 0. The general solution of Bessel equation of integer order is given by

$$R(r) = d_1 J_0(\lambda r) + d_2 Y_0(\lambda r)$$

where $J_0(\lambda r)$ and $Y_0(\lambda r)$ are, correspondingly, Bessel's functions of the 1st and the 2nd kind of zero order.

Before considering the boundary condition, we can make one observation. The deflection of membrane anywhere in the domain $r \in [0, r_1]$ is assumed to be finite (moreover, the wave equation is derived in the assumption of small deflection). The Bessel function of the 2nd kind $Y_0(\lambda r)$ approaches $-\infty$ when r goes to 0. In our case, $r = 0$ is the interior point of the membrane. Therefore, for function $R(r)$ to be bounded at $r = 0$, the coefficient d_2 should be equal to 0.

Then solution of the Bessel equation becomes

$$R(r) = d_1 J_0(\lambda r)$$

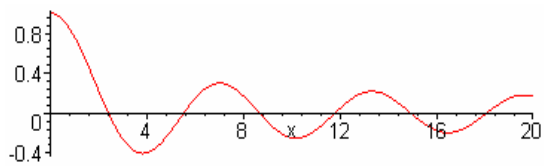
The boundary condition implies

$$R(r_1) = d_1 J_0(\lambda r_1) = 0$$

If we want a non-trivial solution, then $d_1 \neq 0$, and we receive

$$\boxed{J_0(\lambda r_1) = 0}$$

The figure shows the graph of the function $w(\lambda) = J_0(\lambda r_1)$ with $r_1 = 1$



The roots of the equation $\lambda_1, \lambda_2, \ldots$ are the values of the parameter λ for which the boundary value problem has a non-trivial solution (they are called **eigenvalues**).
The corresponding solutions (**eigenfunctions**) are

$$\boxed{R_n(r) = J_0(\lambda_n r)}$$

To determine the weight function, transform the Bessel equation to the self-adjoint form of the Sturm-Liouville problem. Find the integrating factor

$$m(r) = \frac{1}{r^2} e^{\int_0^r \frac{r}{r^2} dr} = \frac{1}{r}$$

and reduce the equation to the self-adjoint form

$$-\frac{1}{r}\left[rR'\right]' = \lambda^2 R \qquad \left[rR'\right]' + \lambda^2 rR = 0 \qquad p(r) = r$$

Then, according to the Sturm-Liouville theorem, the set of functions
$R_n = \{J_0(\lambda_n r)\} \quad n = 1,2,\ldots$
is a complete set of functions orthogonal with respect to weight r over interval $[0, r_2]$, e.g.

$$\int_0^{r_2} rR_n(r)R_m(r)dr = 0 \text{ when } n \neq m$$

<u>solution for *T*</u>

The result of a negative separation constant $\mu = -\lambda^2$ agrees with a physical sense of solution for $T(t)$:

$$T'' - \mu a^2 T = 0 \qquad T(t) = \begin{cases} c_1 e^{\sqrt{\mu} at} + c_2 e^{-\sqrt{\mu} at} & \mu > 0 \\ c_1 + c_2 t & \mu = 0 \\ c_1 \cos\sqrt{-\mu} at + c_2 \sin\sqrt{-\mu} at & \mu < 0 \end{cases}$$

We expect a periodic solution for an undamped vibration of membrane. There should be no constant terms either because boundary conditions and the equation are homogeneous. Therefore, only the case of a negative separation constant may be accepted for our problem, $\mu = -\lambda_n^2$ (or positive eigenvalues). Then solutions $T_n(t)$ with determined eigenvalues are

$$T_n(t) = c_{1,n} \cos \lambda_n at + c_{2,n} \sin \lambda_n at$$

<u>3. Basic solutions</u>

For basis functions we take the solutions of the wave equation satisfying boundary conditions

$$\boxed{u_n(r,t) = J_0(\lambda_n r)(c_{1,n} \cos \lambda_n at + c_{2,n} \sin \lambda_n at)}$$

We are looking for solution of the given i.b.v.p. in the vector space spanned by this basis:

$$u(r,t)=\sum_{n=1}^{\infty}a_nJ_0(\lambda_n r)(c_{1,n}\cos\lambda_n at+c_{2,n}\sin\lambda_n at)$$
$$=\sum_{n=1}^{\infty}J_0(\lambda_n r)(a_nc_{1,n}\cos\lambda_n at+a_nc_{2,n}\sin\lambda_n at)$$
$$=\sum_{n=1}^{\infty}J_0(\lambda_n r)(b_{1,n}\cos\lambda_n at+b_{2,n}\sin\lambda_n at)$$

We will choose the values of coefficients in such a way that initial conditions are satisfied.

4. Initial conditions

Consider the first initial condition

$$u(r,0)=\sum_{n=1}^{\infty}b_{1,n}J_0(\lambda_n r)=u_0(r)$$

then coefficients for the generalized Fourier series are defined as

$$b_{1,n}=\frac{\int_0^{r_1}ru_0(r)J_0(\lambda_n r)\,dr}{\int_0^{r_1}rJ_0^2(\lambda_n r)\,dr}\qquad \boxed{b_{1,n}=\frac{\int_0^{r_1}ru_0(r)J_0(\lambda_n r)\,dr}{\frac{r_1^2}{2}J_1^2(\lambda_n r_1)}}$$

The second condition for the derivative with respect to time

$$\frac{\partial u(r,t)}{\partial t}=\sum_{n=1}^{\infty}J_0(\lambda_n r)(-b_{1,n}\lambda_n a\sin\lambda_n at+b_{2,n}\lambda_n a\cos\lambda_n at)$$

becomes

$$\frac{\partial u(r,0)}{\partial t}=\sum_{n=1}^{\infty}b_{2,n}\lambda_n aJ_0(\lambda_n r)=u_1(r)$$

Then coefficients in this generalized Fourier expansion are

$$b_{2,n}\lambda_n a=\frac{\int_0^{r_1}ru_1(r)J_0(\lambda_n r)\,dr}{\int_0^{r_1}rJ_0^2(\lambda_n r)\,dr}\quad\Rightarrow\quad \boxed{b_{2,n}=\frac{\int_0^{r_1}ru_1(r)J_0(\lambda_n r)\,dr}{\lambda_n a\frac{r_1^2}{2}J_1^2(\lambda_n r_1)}}$$

Then solution of the initial-boundary value problem for the wave equation is

5. Solution

$$u(r,t)=\sum_{n=1}^{\infty}J_0(\lambda_n r)(b_{1,n}\cos\lambda_n at+b_{2,n}\sin\lambda_n at)$$

$$\boxed{u(r,t)=\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\frac{r_1^2}{2}J_1^2(\lambda_n r_1)}\left\{\left[\int_0^{r_1}ru_0(r)J_0(\lambda_n r)\,dr\right]\cos\lambda_n at+\left[\frac{1}{a\lambda_n}\int_0^{r_1}ru_1(r)J_0(\lambda_n r)\,dr\right]\sin\lambda_n at\right\}}$$

021 *(mem1-2.mws)* **4.6.4 2) Example** Vibrating of circular membrane (example2 for wave2-1.doc) - **Standing Waves**

```
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> a:=0.05;
```

$$a := 0.05$$

```
> r1:=1;
```

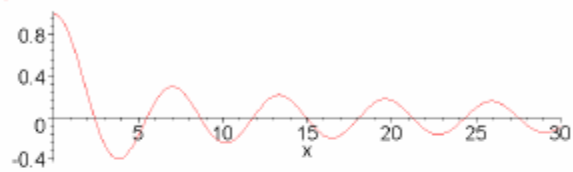
$$r1 := 1$$

Eigenvalues:

```
> w(x) :=BesselJ(0,x*r1);
```

$$w(x) := \mathrm{BesselJ}(0, x)$$

```
> plot(w(x),x=0..30);
```




```
> lambda:=array(1..50);
```

$$\lambda := \mathrm{array}(1 .. 50, [\ ])$$

```
> n:=1: for m from 1 to 300 do z:=fsolve(w(x)=0,x=m/10..(m+1)/10); if type (z,float) then
  lambda[n]:=z: n:=n+1 fi od:
> for i to 5 do lambda[i] od;
```

$$2.404825558$$
$$5.520078110$$
$$8.653727913$$
$$11.79153444$$
$$14.93091771$$

```
> N:=n-1;
```

$$N := 9$$

```
> n:='n':i:='i':
```

Initial conditions:

```
> f(r) :=BesselJ(0,lambda[4]*r);
```

$$f(r) := \mathrm{BesselJ}(0, 11.79153444\, r)$$

```
> g(r) :=0;
```

$$g(r) := 0$$

```
> n:=4;
```

$$n := 4$$

Fourier coefficients:

```
> b[n]:=int(r*f(r)*BesselJ(0,lambda[n]*r),r=0..r1)/int(r*BesselJ(0,lambda[n]*r)^2,r=0..r1);
```

$$b_4 := 1.000000000$$

```
> d[n]:=int(r*g(r)*BesselJ(0,lambda[n]*r),r=0..r1)/int(r*BesselJ(0,lambda[n]*r)^2,r=0..r1)/
  a/lambda[n];
```

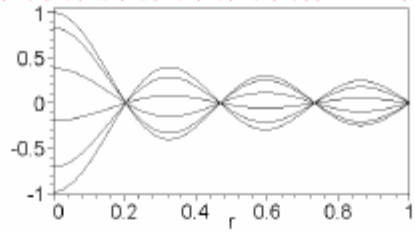
$$d_4 := 0.$$

```
> u[n](r,t) :=BesselJ(0,lambda[n]*r)*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t));
```

$$u_4(r, t) := 1.000000000\ \mathrm{BesselJ}(0, 11.79153444\, r) \cos(0.5895767220\, t)$$

```
> u(r,t) :=u[n](r,t):
> u(r,0) :=subs(t=0,u(r,t)):
> u(r,1) :=subs(t=1,u(r,t)):
> u(r,2) :=subs(t=2,u(r,t)):
> u(r,3) :=subs(t=3,u(r,t)):
> u(r,4) :=subs(t=4,u(r,t)):
> u(r,5) :=subs(t=5,u(r,t)):
```

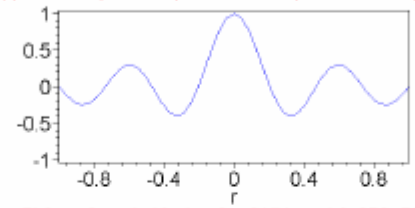
```
> plot([f(r),u(r,0),u(r,1),u(r,2),u(r,3),u(r,4),u(r,5)],r=0..r1,color=black,axes=boxed);
```



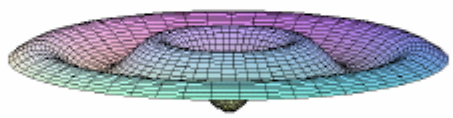

```
> un(-r,t)=u(r,t):
> T:=2*Pi/a/lambda[1];
```

$$T := 16.63322309\,\pi$$

```
> animate([un(-r,t),u(r,t)],r=-1..1,t=0..T,frames=100,color=blue,axes=boxed);
```

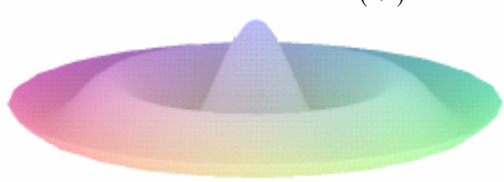



```
> cylinderplot([r,theta,u(r,5)],r=0..r1,theta=0..2*Pi,grid=[50,50]);
```



```
> animate3d([r,theta,u(r,t)],r=0..r1,theta=0..2*Pi,t=0..2*T,coords=cylindrical,style=patchn
ogrid,frames=100);
```

$u(r,t)$



4.6.5 PDE in spherical coordinates

Consider a BVP generated by separation of variables in a PDE in spherical coordinates. We will only see what the Sturm-Liouville problems are in this case. Most of them we solve in Chapter 5 Special Functions.

1. Laplace's Equation

Recall the general form of Laplace's Equation in spherical coordinates for the function $u(r,\phi,\theta)$, $r \in D$:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{1}$$

or with differential operators written in self-adjoint form:

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2}$$

separation of variables

Assume

$$u(r,\phi,\theta) = R(r)\Phi(\phi)\Theta(\theta) \tag{3}$$

Substitute into equation (1)

$$R''\Phi\Theta + \frac{2}{r}R'\Phi\Theta + \frac{1}{r^2\sin^2\theta}R\Phi''\Theta + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}R\Phi\Theta' + \frac{1}{r^2}R\Phi\Theta'' = 0$$

Multiply the equation by $\dfrac{r^2}{R\Phi\Theta}$

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} + \frac{\Theta''}{\Theta} = 0$$

Consider the axisymmetric case ($\dfrac{\partial}{\partial\phi} = 0$):

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} + \frac{\Theta''}{\Theta} = 0$$

Separate variables and set both sides of the equation equal to the same constant

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = -\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} - \frac{\Theta''}{\Theta} = \mu$$

<u>It yields two equations:</u>

1) $r^2\dfrac{R''}{R} + 2r\dfrac{R'}{R} = \mu$

which can be rewritten in the form

$$r^2R'' + 2rR' - \mu R = 0 \qquad \text{(Euler-Cauchy equation)}$$

or in the self-adjoint Sturm-Liouville form

$$-\frac{1}{1}\left(r^2R'\right)' = (-\mu)R \tag{4}$$

Solutions of this equation are sought in the form $R = r^m$ (see Section 2.3.6).

2) $\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \mu\Theta = 0$

Use change of independent variable $x = \cos\theta$, then

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx}$$

$$\Theta'' = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}\left(-\sin\theta\frac{d\Theta}{dx}\right) = -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{d\theta}\left(\frac{d\Theta}{dx}\right)$$

$$= -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{dx}\left(\frac{d\Theta}{dx}\right)\frac{dx}{d\theta} = -\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2}$$

Substitute into equation

$$-\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2} - \frac{\cos\theta}{\sin\theta}\sin\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\sin^2\theta\frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \mu\Theta = 0$$

or in self-adjoint Sturm-Liouville form:

$$-\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Theta}{dx}\right] = \mu\Theta \tag{5}$$

This equation is called Legendre's differential equation. It happens that its solution is bounded only if the separation constant is a non-negative integer of the form

$$\mu = n\left(n+1\right) \qquad n = 0,1,2,...$$

Its solution consists of Legendre polynomials $P_n\left(x\right)$ (see Section 5.7).

2. Heat Equation

Consider the axisymmetric heat equation for $u\left(r,t\right)$, $r \in D$, $t > 0$ in spherical coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = a^2\frac{\partial u}{\partial t} \tag{6}$$

separation of variables

Assume

$$u\left(r,t\right) = R\left(r\right)T\left(t\right)$$

Substitute into equation (6)

$$R''T + \frac{2}{r}R'T = a^2RT'$$

divide by RT and separate variables

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} = a^2\frac{T'}{T} = \mu$$

It yields two ordinary differential equations. Equation for R is

$$r^2R'' + 2rR' - \mu r^2R = 0 \tag{7}$$

which is a spherical Bessel equation of zero order (see equation (25) in Sec. 5.6 with $n = 0$).

It has a self-adjoint Sturm-Liouville form

$$-\frac{1}{r^2}\left(r^2R'\right)' = \left(-\mu\right)R$$

Its solutions are given by spherical Bessel functions

$$j_0\left(r\right) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}\left(r\right)}{\sqrt{r}}$$

$$y_0\left(r\right) = \sqrt{\frac{\pi}{2}}\frac{Y_{1/2}\left(r\right)}{\sqrt{r}}$$

4.6.6 Singular Sturm-Liouville Problem

We studied a regular Sturm-Liouville Problem in which the ordinary differential equation is set in the finite interval and both boundary conditions do not vanish. In a singular Sturm-Liouville problem not all of these conditions hold. Usually, the interval is not finite, and one or both boundary conditions are missing. Instead of boundary conditions, when the solution may not exist at the boundaries, the eigenfunctions should satisfy some limiting conditions. One of such requirements can be the following:
Let y_1 and y_2 be eigenfunctions corresponding to two distinct eigenvalues λ_1 and λ_2, correspondingly. Then they have to satisfy the following condition:

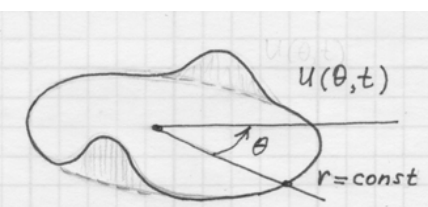
$$\lim_{x\to x_2^-} p(x)[y_1(x)y_2'(x)-y_2(x)y_1'(x)] = \lim_{x\to x_1^+} p(x)[y_1(x)y_2'(x)-y_2(x)y_1'(x)]$$

In the other cases the absence of boundary conditions is because of the periodical or cycled domain, when we demand that the solution should be continuous and smooth

$y(x_1)=y(x_2)$ and $y'(x_1)=y'(x_2)$

In this case, it is still possible to have the orthogonal set of solutions $\{y_n(x)\}$ on $[x_1,x_2]$.
We will not study the formal approach to solution of such problems, but rather discuss the practical examples of its application.
Here, we consider an interesting example of a singular SLP in a cycled domain with no boundary conditions. Physical demonstration of this example can be seen on the ceiling of the hall of the Eyring Science Building.

Example 1 Consider vibration of a thin closed ring string of radius r described in polar coordinates by deflection over the plane $z=0$ $u(\theta,t)$, $\theta\in[0,2\pi]$, $t>0$



The Wave Equation reduces to

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = a^2\frac{\partial^2 u}{\partial t^2} \qquad r=const$$

with initial conditions

$$u(\theta,0)=u_0(\theta)$$

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)$$

There are no boundaries for a closed string, but rather a physical condition for a continuous and smooth string:

$$u(0,t)=u(2\pi,t) \quad t>0$$

$$\frac{\partial u}{\partial\theta}(0,t)=\frac{\partial u}{\partial\theta}(2\pi,t) \quad t>0$$

separation of variables

Assume $\quad u(\theta,t)=\Theta(\theta)T(t)$

Substitute into equation $\quad \frac{1}{r^2}\Theta''T = a^2\Theta T''$

Separate variables $\quad \frac{\Theta''}{\Theta} = a^2r^2\frac{T''}{T} = \mu \qquad \mu$ is a separation constant

Consider

$$\frac{\Theta''}{\Theta}=\mu$$

$$\Theta''-\mu\Theta=0$$

We already have experience with solution of this special equation for regular Sturm-Liouville Problems and know that in all cases except the case of both boundary conditions of Neumann type, only a negative separation constant ,

$\mu = -\lambda^2$, generates eigenvalues and eigenfunctions. General solution in this case is

$$\Theta(\theta) = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$$

This solution suits our problem because it is periodic. The values of λ which satisfy periodicity on the interval $\theta \in [0,2\pi]$, are

$$\lambda_n = \frac{2n\pi}{2\pi} = n$$

Therefore, solutions are

$$\Theta_n(\theta) = c_{1,n} \cos n\theta + c_{2,n} \sin n\theta$$

Obviously, that for all $n = 0,1,2,...$ 2π is a period for this solution and for its derivative

$$\Theta_n'(\theta) = -c_{1,n} n \sin n\theta + c_{2,n} n \cos n\theta$$

With these values of the separation constant, $\mu_n = -\lambda_n^2 = -n^2$, $n = 0,1,2,...$ consider the equation for $T(t)$:

$$a^2 r^2 \frac{T''}{T} = -n^2$$

$$T'' + \frac{n^2}{a^2 r^2} T = 0$$

which also has a periodic (in t) general solution

$$T_n(t) = c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t$$

Then periodic solution of the wave equation can be constructed in the form of an infinite series:

$$u(\theta,t) = \Theta(\theta)T(t) \qquad = \sum_{n=0}^{\infty} \Theta_n(\theta) T_n(t)$$

$$= \sum_{n=0}^{\infty} \left(c_{1,n} \cos n\theta + c_{2,n} \sin n\theta\right)\left(c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t\right)$$

$$= \sum_{n=0}^{\infty} \left(c_{1,n} c_{3,n} \cos n\theta \cos \frac{n}{ar} t + c_{1,n} c_{4,n} \cos n\theta \sin \frac{n}{ar} t + c_{2,n} c_{3,n} \sin n\theta \cos \frac{n}{ar} t + c_{2,n} c_{4,n} \sin n\theta \sin \frac{n}{ar} t\right)$$

$$= \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta \cos \frac{n}{ar} t + b_{2,n} \cos n\theta \sin \frac{n}{ar} t + b_{3,n} \sin n\theta \cos \frac{n}{ar} t + b_{4,n} \sin n\theta \sin \frac{n}{ar} t\right)$$

where coefficients b are new arbitrary constants which can be chosen in such a way that this solution will satisfy the initial conditions.
Consider the first initial condition:

$$t = 0 \qquad u(\theta,0) = u_0(\theta) \qquad = \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right)$$

$$= b_{1,0} + \sum_{n=1}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right)$$

which can be treated as a standard Fourier series expansion of the function $u_0(\theta)$ on the interval $[0,2\pi]$. Therefore, the coefficients of this expansion are

$$b_{1,0} = \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta) d\theta$$

$$b_{1,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \cos n\theta d\theta$$

$$b_{3,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \sin n\theta d\theta$$

For the second initial condition, differentiate the solution first with respect to t

$$\frac{\partial u}{\partial t}(\theta,t)=\sum_{n=0}^{\infty}\left(-b_{1,n}\frac{n}{ar}\cos n\theta\sin\frac{n}{ar}t+b_{2,n}\frac{n}{ar}\cos n\theta\cos\frac{n}{ar}t-b_{3,n}\frac{n}{ar}\sin n\theta\sin\frac{n}{ar}t+b_{4,n}\frac{n}{ar}\sin n\theta\cos\frac{n}{ar}t\right)$$

then apply the second initial condition

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)\qquad=\sum_{n=0}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

$$=b_{2,n}\cdot 0+\sum_{n=1}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

Where the coefficients are determined as

$$b_{2,0}\cdot 0=\frac{1}{2\pi}\int_0^{2\pi}u_1(\theta)d\theta$$

$$b_{2,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\cos n\theta d\theta$$

$$b_{4,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\sin n\theta d\theta$$

Coefficient $b_{2,0}$ can be any constant, it will not influence the initial speed of the string, but not to influence the initial shape of the string it has to be chosen equal to zero (otherwise, initially the string will shifted by $b_{2,0}$ and will not be centered over the plane $z=0$):

$b_{2,0}=0$

Therefore, solution of the problem is given by the infinite series

$$\boxed{u(\theta,t)=b_{1,0}+\sum_{n=1}^{\infty}\left(b_{1,n}\cos n\theta\cos\frac{n}{ar}t+b_{2,n}\cos n\theta\sin\frac{n}{ar}t+b_{3,n}\sin n\theta\cos\frac{n}{ar}t+b_{4,n}\sin n\theta\sin\frac{n}{ar}t\right)}$$

where coefficients are determined according to abovementioned formulas.

Consider particular cases (Maple examples):

1) isolated wave;

$u_0(\theta)$

$u(\theta,t)$

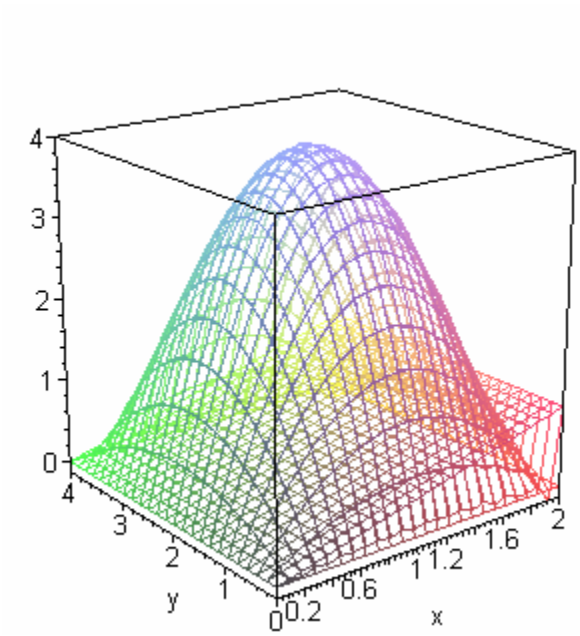
2) standing waves

$u(\theta,t)$

Chapter 4

Partial Differential Equations

EXERCISES



1. Let $D \subset \mathbb{R}^3$ be a domain (open connected set), and let S be the boundary of D ($S = \bar{D} \setminus D$).
Show that if $\mathbf{r} \in S$ is a point of the boundary of D,
then any ball $B(\mathbf{r}, R)$ with $R \in \mathbb{R}, R > 0$ includes points both from D and $\mathbb{R}^3 \setminus D$, i.e.
$B(\mathbf{r}, R) \cap D \neq \varnothing$ and $B(\mathbf{r}, R) \cap (\mathbb{R}^3 \setminus D) \neq \varnothing$.

Remark: this property is usually used as the more general definition:
If $A \subset \mathbb{R}^n$ is an arbitrary subset of $\mathbb{R}^n$ (not necessarily domain), then
$x \in \mathbb{R}^n$ is called *a boundary point* of A if for any radius $R > 0$:
$B(x, R) \cap A \neq \varnothing$ and $B(x, R) \cap (\mathbb{R}^n \setminus A) \neq \varnothing$
The set $\partial A = \{x \in \mathbb{R}^n \mid x \text{ is boundary point of } A\}$ is called the boundary of A in $\mathbb{R}^n$.

Examples:
a) $\partial(0,1] = \{0,1\}$
b) $\partial\{a\} = \{a\}$ (the boundary of an insulated point is the point itself)
c) $\partial\mathbb{Q} = \mathbb{R}$
d) $\partial\mathbb{Z} = \mathbb{Z}$
e) $\partial\varnothing = \varnothing$
f) $\partial\mathbb{R}^n = \varnothing$
g) $\partial\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} = \left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} \cup \{0\}$

2. Transform the Heat Equation in Cartesian coordinates (8) to cylindrical coordinates (9), using the conversion formulas between coordinate systems: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

3. Set up a mathematical model (choose an appropriate coordinate system and dimension of the problem, write the governing equation and corresponding initial and boundary conditions) for the following engineering models (do not solve the problem):

a) A very thin long wire dissipates energy in the massive layer of the stagnant media with the rate per unit length q $\left[\frac{W}{m}\right]$. The media has a thermal conductivity k $\left[\frac{W}{m \cdot K}\right]$. Determine the stationary temperature distribution in the media.

b) In the massive layer of homogeneous material (with thermal properties k, ρ, c_p) which was initially at the uniform temperature T_0, a localized heat source spontaneously started to dissipate energy with the rate q $[W]$. Determine the development of the temperature field in the material.

c) A very long tree trunk of radius R in the forest is exposed to the surrounding air (average wind speed is v $\left[\frac{m}{s}\right]$), but the dense crown prevents the direct sun radiation of the trunk. Set up the mathematical model describing the temperature distribution in the tree trunk during the day. Conductivity in the tree depends on direction: it is much higher along the tree than in the radial direction.

d) A wide reservoir of water of L meters deep is exposed to the solar irradiation G_0 $\left[\frac{W}{m^2}\right]$ incident at the angle θ. Penetration of the solar radiative flux along the path s is described by the Lambert-Beer Law $G(x) = G_0 \cos\theta e^{-\kappa s}$, where κ $\left[\frac{1}{m}\right]$ is the gray absorption coefficient of water.

Then the solar energy dissipated in water (radiative dissipation source or the divergence of Radiative flux) is determined by $Q(s) = -\frac{dG(x)}{dx}\left[\frac{W}{m^3}\right]$. Set up the mathematical model describing the equilibrium temperature field in the water layer.

e) Two opposite sides of the long column are insulated. There is an intensive condensation of the water steam on one of the other sides. The last side is exposed to the convective environment at temperature T_∞ and convective coefficient $h\left[\frac{W}{m^2 \cdot K}\right]$. Due to some chemical reaction there is production energy in the column with the volumetric rate $\dot{q}\left[\frac{W}{m^3}\right]$. Initially, column was at the uniform temperature T_0. Describe the transient temperature distribution inside of the column.

4. 1) Find solution of the Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the domain $D = (0, L) \times (0, M)$ subject to boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \text{ and } u\big|_{y=M} = f(x)$$

2) Sketch the graph of solution for $L = 2$, $M = 3$ and $f(x) = x(L - x)$

3) What is the solution for $f(x) = cos\frac{\pi x}{L}$? Sketch the graph.

5. 1) Find solution of the Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the domain $D = (0, L) \times (0, \infty)$ subject to boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \text{ and } u\big|_{y=0} = f(x)$$

2) Sketch the graph of solution for $L = 5$ and

a) $f(x) = 10$

b) $f(x) = sin\frac{\pi}{2L}x$

c) $f(x) = cos\frac{\pi}{2L}x$

6. 1) Find solution of the Poisson Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y)$$

in the domain $D = (0, L) \times (0, M)$ subject to boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = 10,\ u\big|_{y=0} = f(x),\ u\big|_{y=M} = 0$$

2) Sketch the graph of solution for $L = 4$, $M = 2$ and

a) $f(x) = 5$ and $F(x, y) = xy(L - x)(M - y)$

b) $f(x) = x$ and $F(x, y)$ of your choice

7.

1) Solve the Laplace Equation in the cylindrical domain

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \qquad u(r,z): \qquad 0\le r<r_1,\ 0<z<z_1$$

Boundary conditions:

$$u(r_1,z)=0 \qquad 0<z<z_1$$
$$u(r,0)=0 \qquad 0\le r<r_1$$
$$u(r,z_1)=f(r) \qquad 0\le r<r_1$$

2) Display some creativity in visualization of solution for

$r_1=2$

$z_1=5$

$f(r)=(r-r_1)^2$

8.

1) Solve the Laplace Equation in the cylindrical domain

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \qquad u(r,z): \qquad 0\le r<r_1,\ 0<z<z_1$$

Boundary conditions:

$$u(r_1,z)=2 \qquad 0<z<z_1$$
$$u(r,0)=0 \qquad 0\le r<r_1$$
$$u(r,z_1)=3 \qquad 0\le r<r_1$$

2) Display some creativity in visualization of solution for

$r_1=4$

$z_1=10$

9.

1. Solve the Dirichlet problem for the Heat Equation:

$$\frac{\partial^2 u}{\partial x^2}=a^2\frac{\partial u}{\partial t} \qquad u(x,t): \quad x\in[0,L],\ t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions:

$$u(0,t)=0,\quad t>0 \qquad \text{(Dirichlet)}$$
$$u(L,t)=0,\quad t>0 \qquad \text{(Dirichlet)}$$

2. Sketch the graph of solution for $L=3$ and $a=0.1$ and initial conditions:

a) $u_0(x)=1$

b) $u_0(x)=x(L-x)$

c) $u_0(x)=\sin 2x$

3. Observe the Maximum principle for the Dirichlet problem for the Heat Equation

10.

1) Solve the Heat Equation in the cylindrical domain with angular symmetry

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial u}{\partial t} \qquad u(r,z): \qquad 0\le r<r_1,\ t>0$$

Boundary condition: $u(r_1,t)=f_1 \qquad t>0$

Initial condition $u(r,0)=u_0(r)$

2) Display some creativity in visualization of solution for
$r_1 = 0.5$
$a = 3000$
$f_1 = 70$
$u_0(r) = 25r^2 + 20$

3) Give some physical interpretation of the problem

11. Superposition Principle for Non-Homogeneous Heat Equation with Non-Homogeneous Boundary Conditions:

Heat Equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + F(x) \qquad u(x,t): \quad x \in (0,L),\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = g_0,\ t > 0$ (Dirichlet)

$$\frac{\partial u(L,t)}{\partial x} = g_L, \quad t > 0 \qquad \text{(Neumann)}$$

Supplemental problems:

a) steady state solution:

$$\frac{\partial^2 u_s}{\partial x^2} = F(x) \qquad u_s(x): \quad x \in (0,L)$$

$$u_s(0) = g_0$$
$$\frac{\partial u_s}{\partial x}(L) = f_L$$

b) transient solution:

$$\frac{\partial^2 U}{\partial x^2} = a^2 \frac{\partial U}{\partial t} \qquad U(x,t): \quad x \in (0,L),\ t > 0$$

$$U(x,0) = u_0(x) - u_s(x)$$

$$U(0,t) = 0, \quad t > 0$$
$$U(L,t) = 0 \quad t > 0$$

First supplemental problem is a BVP for ODE;
the second supplemental problem is an IBVP problem for a homogeneous Heat Equation with homogeneous boundary conditions.

Show that $u(x,t) = U(x,t) + u_s(x)$ is a solution of the non-homogeneous IBVP.
Solve the problem with $F(x) = 0, g_0 = 1, g_L = 3$ and $u_0(x) = x(4-x)$.
Sketch the graph.

12. a) Find eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$X''(x) - \mu X(x) = 0$$
$$-X'(0) + HX(0) = 0 \qquad \text{Robin}$$
$$X'(L) = 0 \qquad \text{Neumann}$$

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$$f(x) = xe^{-x}$$

Sketch the graph for $L = 2$, $H = 3$, and $n = 5$ and $n = 20$.

14. a) Find eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$X''(x) - \mu X(x) = 0$$

$X'(0) = 0$ Neumann

$X'(L) + H_2 X(L) = 0$ Robin

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$$f(x) = xe^{-x}$$

Sketch the graph for $L = 2$, $H = 3$, and $n = 5$ and $n = 20$.

15. a) Find eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$X''(x) - \mu X(x) = 0$$

$-X'(0) + H_1 X(0) = 0$ Robin

$X'(L) + H_2 X(L) = 0$ Robin

(do not try to get the solution given in the table for the Sturm-Liouville problem)

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$f(x) = xe^{-x}$ in the interval $(0, L)$

Sketch the graph for $L = 2$, $H = 3$, and $n = 5$ and $n = 20$.

16. a) Reduce the following BVP to a Sturm-Liouville problem:

$x^2 u'' + 2xu' + \mu u = 0$

$u(1) = 0$

$u(e) = 0$

and find eigenvalues and eigenfunctions.

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$f(x) = xe^{-x}$ in the interval $(1, e)$

Sketch the graph for $n = 5$ and $n = 20$.

17. a) Reduce the following BVP to a Sturm-Liouville problem:

$$x^2u'' + 3xu' + \mu u = 0$$
$$u(1) = 0$$
$$u'\left(\sqrt{e}\right) = 0$$

and find eigenvalues and eigenfunctions.

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$f(x) = xe^{-x}$ in the interval $\left(1, \sqrt{e}\right)$

Sketch the graph for $n = 5$ and $n = 20$.

18. a) Reduce the following BVP to a Sturm-Liouville problem:

$$xu'' + u' + \frac{\mu}{x}u = 0$$
$$u(1) = 0$$
$$u(e) = 0$$

and find eigenvalues and eigenfunctions.

Use the obtained set of eigenfunctions for generalized Fourier series representation of the function

$f(x) = xe^{-x}$ in the interval $(1, e)$

Sketch the graph for $n = 5$ and $n = 20$.

19. The set of functions $\{1, x, x^2, x^3, \ldots\}$ (monoms) is linearly independent in $L_2(-1,1)$.

a) Using the Gram-Schmidt orthogonalization process, construct an orthonormal basis of $L_2(-1,1)$.

b) Use the obtained orthogonal set for a generalized Fourier series representation of the function

$$f(x) = \begin{cases} -1 & x \in (-1,0) \\ 1 & x \in (0,1) \end{cases}$$

20. Prove that if the set $\{\varphi_n\}$ is orthonormal, then it is linearly independent (show it for a finite set).

21. Show that the set $\left\{ sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x \right\}$ is orthogonal in $L_2(0,L)$.

22. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ x \in (0,L),\ t > 0$$

initial condition: $u(x,0) = u_0(x)$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x)$$

boundary conditions: $u(0,t) = 0, \quad t > 0$ (Dirichlet)

$u(L,t) = 0, \quad t > 0$ (Dirichlet)

Sketch the graph of solution with $L = 2$, $a = 0.5$, and

a) $u_1(x) = -0.1$, $u_0(x) = x^2(L-x)^2$

b) $u_1(x) = 0$, $u_0(x) = sin\frac{6\pi}{L}x$
(observe the phenomena called standing waves)

23. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

initial condition:

$$u(x,0) = u_0(x)$$
$$\frac{\partial u(x,0)}{\partial t} = u_1(x)$$

boundary conditions:

$$-u'(0,t) + H_1 u(0,t) = 0 \quad , \quad t > 0 \qquad \text{Robin}$$
$$u(L,t) = 0 \quad , \quad t > 0 \qquad \text{Dirichlet}$$

Sketch the graph of solution with $L = 5$, $a = 2.0$, and

a) $u_1(x) = 0.2$, $u_0(x) = (L-x)^2$

b) $u_1(x) = 0$, $u_0(x) = X_5(x)$ (eigenfunction)

24. Set up and solve the IBVP for a 2-D Wave Equation in Cartesian coordinates.
Sketch the graph of the solution.

25. Solve the IBVP for the Heat Equation in polar coordinates with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = a^2\frac{\partial u}{\partial t} \qquad u(r,t),\ \ r \in [0,r_1),\ t > 0$$

Initial conditions: $u(r,0) = u_0(r)$

Boundary condition: $k\frac{\partial u(r_1,t)}{\partial r} + hu(r_1,t) = f_1 \qquad t > 0$

And sketch the graph of solution for
$r_1 = 2$, $a = 0.5$, $k = 0.1$, $h = 12$, $f_1 = 2$, and $u_0(r) = (r - r_1)^2$

(hint: find the first steady state solution)

26. a) Solve the IBVP:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + F(x) \qquad u(x,t),\ \ x \in (0,L),\ t > 0$$

initial condition: $u(x,0) = u_0(x)$

boundary conditions: $u(0,t) = f_1 \qquad t > 0$ (Dirichlet)

$$k\frac{\partial u(L,t)}{dx} + hu(L,t) = f_2 \qquad t > 0 \qquad \text{(Robin)}$$

b) Sketch the graph of solution with
$L = 4$, $a = 0.5$, $k = 2.0$, $u_0(x) = x(x - L/2) + 5$, $f_1 = 10$, $f_2 = 1$, $F(x) = x$

27. Find the solution for vibration of the annular membrane with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = a^2 \frac{\partial^2 u}{\partial t^2} \qquad u(r,t),\ \ r \in (r_1, r_2),\ t > 0$$

Initial conditions: $u(r,0) = u_0(r)$

$$\frac{\partial u}{\partial t}(r,0) = u_1(r)$$

Boundary condition: $u(r_1,t) = 0 \qquad t > 0$

$u(r_2,t) = 0 \qquad t > 0$

And sketch the graph of solution for
$r_1 = 1$, $r_2 = 2$ $a = 0.5$, $u_0 = (r - r_1)(r_2 - r)$, and $u_1(r) = 0$.

ALLEGORY OF GEOMETRY

(Museum of Louvre)