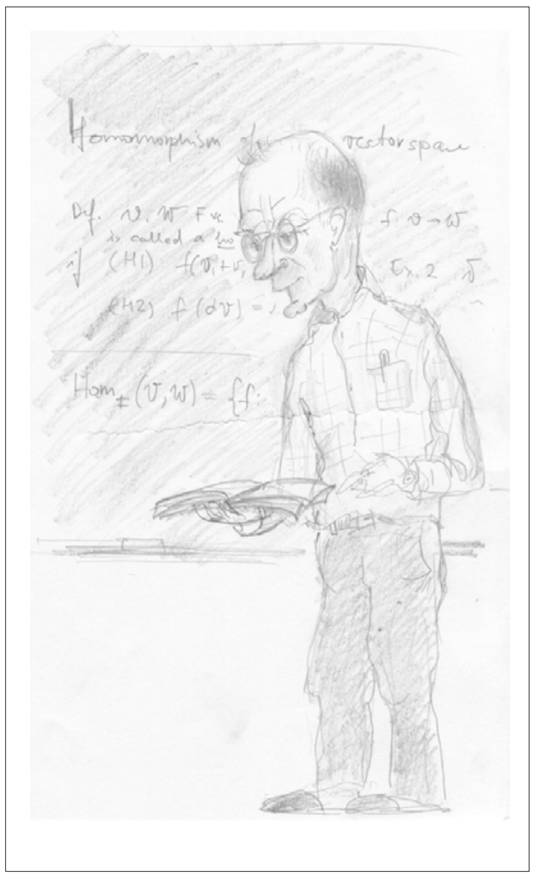
Chapter 7

INTEGRAL EQUATIONS



Chapter 7 Integral Equations

Future Topics:

7.1 Normed Vector Spaces

We will start with some definitions and results from the theory of normed vector spaces which will be needed in this chapter.

1. Euclidian vector space $\mathbb{R}^n$

The n-dimensional Euclidian vector space consists of all points

$$\mathbb{R}^n = \left\{ x = (x_1, x_2, ..., x_n) \middle| x_k \in \mathbb{R} \right\}$$

for which the following operations are defined:

Scalar product $\quad (x, y) = x_1 y_1 + x_2 y_2 + ... + x_n y_n \qquad x, y \in \mathbb{R}^n$

Norm $\quad |x| = \sqrt{(x,x)} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

Distance $\quad \rho(x, y) = |x - y|$

Convergence $\quad \lim\limits_{k \to \infty} x_k = x$ if $\lim\limits_{k \to \infty} |x - x_k| = 0$

2. Vector space $C(\bar{G})$

Vector space $C(\bar{G})$ consists of all real valued continuous functions defined on the closed domain $\bar{G} \subset \mathbb{R}^n$:

$$C(\bar{G}) = \left\{ f(x) : \bar{D} \subset \mathbb{R}^n \to \mathbb{R} \middle| continuous \right\}$$

Norm $\quad \|f\|_C = \max\limits_{x \in \bar{G}} |f(x)|$

Convergence $\quad \lim\limits_{k \to \infty} f_k = f$ if $\lim\limits_{k \to \infty} \|f - f_k\|_C = 0$

3. Vector space $L_2(G)$

The space of functions integrable according to Lebesgue (see Section 3.1)

$$L_2(G) = \left\{ f(x) : G \subset \mathbb{R}^n \to \mathbb{C} \middle| \int_G |f(x)|^2 dx < \infty \right\}$$

Inner product $\quad (f, g) = \int_G f(x) \bar{g}(x) dx$

Norm $\quad \|f\|_2 = \sqrt{(f, f)} = \int_G |f(x)|^2 dx$

The following property follows from the definition of the Lebesgue integral

$$\left| \int_G f(x) dx \right| \le \int_G |f(x)| dx$$

4. Cauchy-Bunyakovsky Inequality

$$\boxed{|(f, g)| \le \|f\|_2 \cdot \|g\|_2} \qquad \text{for all } f, g \in L_2(G)$$

Proof:

If $f, g \in L_2(G)$, then functions $|f|$, $|g|$ and any combination $\alpha|f| + \beta|g|$ are also integrable and therefore belong to $L_2(G)$.
Consider

$|f| + \lambda|g| \in L_2(G)$, $\lambda \in R$ for which we have

$$0 \le \int_G \left(|f| + \lambda|g|\right)^2 dx = \int_G |f|^2 dx + 2\lambda \int_G |fg| dx + \lambda^2 \int_G |g|^2 dx$$

The right hand side is a quadratic function of λ. Because this function is non-negative, its discrimenant is non-positive

$$4\left[\int_G |fg| dx\right]^2 - 4\left(\int_G |f|^2 dx\right)\left(\int_G |g|^2 dx\right) \le 0$$

$$\left[\int_G |fg| dx\right]^2 \le \left(\int_G |f|^2 dx\right)\left(\int_G |g|^2 dx\right)$$

$$\left|(f,g)\right|^2 \le \|f\|_2^2 \cdot \|g\|_2^2$$

from which the claimed inequality yields

$$\left|(f,g)\right| \le \|f\|_2 \cdot \|g\|_2$$

because $\left|(f,g)\right| = \left|\int_G f\bar{g} dx\right| \le \int_G |f\bar{g}| dx \le \int_G |f||g| dx$. ■

5. Minkowski Inequality (3rd property of the norm)

$$\boxed{\|f+g\|_2 \le \|f\|_2 + \|g\|_2} \qquad \text{for all } f,g \in L_2(G)$$

Proof:

Consider
$$\begin{aligned}
\|f+g\|_2^2 &= (f+g, f+g) \\
&= (f,f) + (f,g) + (g,f) + (g,g) \\
&\le \|f\|_2^2 + |(f,g)| + |(g,f)| + \|g\|_2^2 \\
&\le \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 \qquad \text{from C-B inequality} \\
&= \left(\|f\|_2 + \|g\|_2\right)^2
\end{aligned}$$

Then extraction of the square root yields the claimed result. ■

Note that the Minkowski inequality reduces to equality only if functions f and g are equal up to the scalar multiple, $f = \alpha g$, $\alpha \in R$ (why?).

7.2 Linear Operators

Let M and N be two linear normed vectors spaces with norms $\|\cdot\|_M$ and $\|\cdot\|_N$, correspondingly. We define an operator L as a map (function) from the vector space M to the vector space N:

$$L : M \to N$$

Introduce the following definitions concerning the operators in the vector spaces:

Operator $L : M \to N$ is **linear** if $L(\alpha f + \beta g) = \alpha L f + \beta L g$ for all $f, g \in M$ and all $\alpha, \beta \in R$

Operator $L : M \to N$ is **continuous** if from $f_k \to f$ in M follows $Lf_k \to Lf$ in N (the image of the convergent sequence in M is a convergent sequence in N)

Operator $L : M \to N$ is **bounded** if there exists $c > 0$ such that $\|Lf\|_N \le c\|f\|_M$ for all $f \in M$

The **norm** of operator $L : M \to N$ can be defined as the greatest lower bound of such constant c

$$\|L\| = \sup_{f \neq 0} \frac{\|Lf\|_N}{\|f\|_M}$$

Theorem 7.1 If operator $L : M \to N$ is bounded then it is continuous

Proof:

Let operator $L : M \to N$ be bounded, then according to the definition there exists $c > 0$ such that $\|Lf\|_N \le c\|f\|_M$.

Let $f_k \to f$ in M. That means that $\lim_{k\to\infty} \|f_k - f\|_M = 0$. From the definition of the limit it follows that for any $\varepsilon > 0$ there exists $K \in N$ such that $\|f_k - f\|_M < \varepsilon$ for all $k \ge K$.

To prove the theorem, show now that $Lf_k \to Lf$ in N or that $\lim_{k\to\infty} \|Lf_k - Lf\|_N = 0$. We have to show that for any $E > 0$ there exists $K_E \in N$ such that $\|Lf_k - Lf\|_N < E$ for all $k \ge K_E$.

Choose $\varepsilon = \dfrac{E}{c}$, then

$$\|Lf_k - Lf\|_N = \|L(f_k - f)\|_N \le c\|f_k - f\|_M < c \cdot \frac{E}{c} = E \quad \text{for all } k \ge K . \qquad \blacksquare$$

7.3 Integral Operator

Consider an operator called an **integral operator** given by the equation

$$Kf = \int_G K(x,y)f(y)dy \qquad x \in G \subset R^n$$

Obviously, that integral operator is linear.

Function $K(x,y)$ is called a **kernel** of the integral operator. We will consider kernels $K(x,y) \in L_2(G \times G)$, therefore

$$\int_G \int_G |K(x,y)|^2 \, dxdy < \infty$$

In a case of $G \subset R$, the domain $G = (a,b)$, where a,b can be finite or infinite.

Theorem 7.2 Let K be the integral operator with a kernel $K(x,y)$ continuous in $[a,b]\times[a,b]$. Then operator K is bounded, and, therefore, continuous. Moreover:

1) $K : L_2(a,b) \to C[a,b]$ $\qquad \|Kf\|_C \le M\sqrt{b-a}\|f\|_2$ $\qquad$ for $f \in L_2(a,b)$

2) $K : L_2(a,b) \to L_2(a,b)$ $\qquad \|Kf\|_2 \le M(b-a)\|f\|_2$ $\qquad$ for $f \in L_2(a,b)$

3) $K : C[a,b] \to C[a,b]$ $\qquad \|Kf\|_C \le M(b-a)\|f\|_C$ $\qquad$ for $f \in C[a,b]$

Proof:

Since $K(x,y)$ is continuous in the closed domain $[a,b]\times[a,b]$, there exists $M > 0$ such that $M = \max_{x,y\in[a,b]} |K(x,y)|$.

1) Let $f \in L_2(a,b)$. Then because function $K(x,y)$ is continuous in $[a,b]\times[a,b]$, the function $(Kf)(x)$ is continuous in $[a,b]$, and, therefore $K : L_2(a,b) \to C[a,b]$. Consider

$$\|Kf\|_C = \max_{x\in[a,b]} |(Kf)(x)| = \max_{x\in[a,b]} \left| \int_a^b K(x,y)f(y)dy \right| = \max_{x\in[a,b]} |(K(x,y), f(y))|$$

$$\le \max_{x\in[a,b]} \|K\|_2 \|f\|_2 \qquad \text{(from Cauchy-Bunyakowski inequality)}$$

$$\le \|f\|_2 \max_{x\in[a,b]} \left[\int_a^b |K(x,y)|^2 \, dy \right]^{1/2}$$

$$\le \|f\|_2 \max_{x\in[a,b]} \left[\int_a^b M^2 \, dy \right]^{1/2}$$

$$= M\sqrt{b-a}\|f\|_2$$

2)

$$\|Kf\|_2 = \left((Kf)(x),(Kf)(x)\right)^{1/2} = \left[\int_a^b |(Kf)(x)|^2\,dx\right]^{1/2}$$

$$= \left[\int_a^b \left|\int_a^b K(x,y)f(y)dy\right|^2 dx\right]^{1/2}$$

$$\le \left[\int_a^b \|K\|_2^2 \|f\|_2^2\,dx\right]^{1/2}$$

$$= \|f\|_2 \left[\int_a^b \left(\int_a^b |K(x,y)|^2\,dy\right)dx\right]^{1/2}$$

$$\le \|f\|_2 \left[\int_a^b \left(\int_a^b M^2\,dy\right)dx\right]^{1/2}$$

$$\le M\|f\|_2 \left[\int_a^b \left(\int_a^b dy\right)dx\right]^{1/2}$$

$$= M\|f\|_2 (b-a)$$

3) Exercise ■

7.4 Integral Equations

Integral equations are equations in which the unknown function is under the integral sign. The typical integral equations for unknown function $u(x)$, $x \in G \subset R^n$ (in this chapter , we consider $x \in (a,b) \subset R$) include integral term in the form of integral operator with the kernel $K(x,y)$

$$Ku = \int_G K(x,y)u(y)dy$$

The main types of integral equations are the following:

I Fredholm integral equation

1) Fredholm's integral equation of the 1st kind:

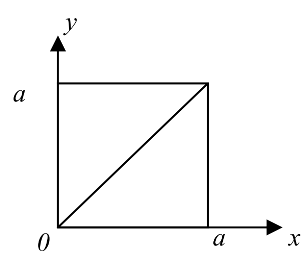
$\int_G K(x,y)u(y)dy = f(x)$	$Ku = f$	non-homogeneous eqn
$\int_G K(x,y)u(y)dy = 0$	$Ku = 0$	homogeneous eqn

2) Fredholm's integral equation of the 2nd kind: $\lambda \in C$ is a parameter

$u(x) = \lambda \int_G K(x,y)u(y)dy + f(x)$	$u = \lambda Ku + f$	non-homogeneous eqn
$u(x) = \lambda \int_G K(x,y)u(y)dy$	$u = \lambda Ku$	homogeneous eqn

II Volterra integral equation

Let $G = (0,a) \subset R$.
$K(x,y)$ is called a Volterra kernel if $K(x,y) = 0$ for $0 < x < y < a$



1) Volterra's integral equation of the 1st kind:

$$\int_0^x K(x,y)u(y)dy = f(x)$$

2) Volterra's integral equation of the 2nd kind:

$$u(x) = \lambda \int_0^x K(x,y)u(y)dy + f(x)$$

III Integro-Differential Equation includes an unknown function under the integral sign and also any derivative of the unknown function. For example:

$$\frac{du}{dx} = u(x) + \int_G K(x,y)u(y)dy + f(x)$$

An important representation of the integro-differential equation is a Radiative Transfer Equation describing energy transport in the absorbing, emitting and scattering media (analogous equations appear in the theory of neutron transport).

Some other types of integral equations will be considered in the Section 8.2.4.

Solution of integral equation is any function $u(x)$ satisfying this equation:

$u = \lambda K u + f$ non-homogeneous equation

$u = \lambda K u$ homogeneous equation

The value of the parameter $\lambda \in C$ for which the homogeneous integral equation has a non-trivial solution $u \in L_2$ which is called an **eigenvalue** of the kernel $K(x, y)$, and the corresponding solution is called an **eigenfunction** of this kernel.

Eigenvalue problem We will distinguish eigenvalue problems for the integral kernel (integral equation):

$$u = \lambda K u$$

and for the integral operator

$$Ku = \frac{1}{\lambda} u$$

The eigenvalues of the integral operator are recipical to eigenvalues of the integral kernel, and eigenfunctions are the same in both cases.

7.5 Solution Methods for Integral Equations

1. The Method of Successive Approximations for Fredholm's Integral Equation

For the integral equation

$$u = \lambda K u + f$$

the following iterations of the method of successive approximations are set by:

$$u_0(x) = f(x)$$

$$u_n(x) = \lambda K u_{n-1} + f \qquad n = 1,2,...$$

Lemma 7.1 $\boxed{u_n(x) = \sum_{k=0}^{n} \lambda^k K^k f}$ where $K^k = \underbrace{K(K(\cdots K))}_{k\ times}$

Proof by mathematical induction (assume that the formula for *n* is true):

$n = 0$ $\quad u_0(x) = \lambda^0 K^0 f = f(x)$ confirmed

$n = n+1$ $\quad u_{n=1}(x) = \lambda K u_n + f$ by definition

$$= \lambda K\left(\sum_{k=0}^{n} \lambda^k K^k f\right) + f \qquad \text{by assumption}$$

$$= f + \sum_{k=0}^{n} \lambda^{k+1} K^{k=1} f \qquad \text{linearity}$$

$$= f + \sum_{p=1}^{n+1} \lambda^p K^p f \qquad \text{change of index } p = k+1$$

$$= \lambda^0 K^0 f + \sum_{p=1}^{n+1} \lambda^p K^p f$$

$$= \sum_{p=0}^{n+1} \lambda^p K^p f$$

$$= \sum_{k=0}^{n+1} \lambda^k K^k f \qquad \text{change of index } p = k \quad \blacksquare$$

Neumann Series $\quad \sum_{k=0}^{\infty} \lambda^k K^k f$ is called to be the Neumann Series

Estimation of iterations $\quad \|K^n f\|_C = \|K(K^{n-1} f)\|_C$

$$\le M(b-a)\|K^{n-1} f\|_C \qquad \text{Theorem 7.2 (3)}$$

$$\le M^2 (b-a)^2 \|K^{n-2} f\|_C$$

$$\dots$$

$$\le M^n (b-a)^n \|f\|_C$$

$$\left\|\sum_{k=0}^{\infty}\lambda^k K^k f\right\|_C \le \|f\|_C \sum_{k=0}^{\infty}|\lambda|^k M^k (b-a)^k$$

$$= \|f\|_C \sum_{k=0}^{\infty}\left[|\lambda| M(b-a)\right]^k$$ geometric series

converges if $|\lambda| < \dfrac{1}{M(b-a)}$

$$= \frac{\|f\|_C}{1-|\lambda| M(b-a)}$$

Therefore, the Neumann series $\sum_{k=0}^{\infty}\lambda^k K^k f$ converges for $|\lambda| < \dfrac{1}{M(b-a)}$.

Denote the sum of the Neumann series as a function $u(x)$:

$$u(x) = \sum_{k=0}^{\infty}\lambda^k K^k f$$

Show that this function satisfies integral the equation $u = \lambda K u + f$. Consider iterations

$u_n(x) = \lambda K u_{n-1} + f$

then

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lambda K \lim_{n\to\infty} u_{n-1}(x) + f \\ &= \lambda \int_a^b K(x,y) \lim_{n\to\infty} u_{n-1}(y) dy + f \\ &= \lambda \int_a^b K(x,y) u(y) dy + f \end{aligned}$$

And, recalling estimation, $\|u(x)\|_C \le \dfrac{\|f\|_C}{1-|\lambda| M(b-a)}$

Show that this solution is unique. For that, it is enough to show that the homogeneous equation $u = \lambda K u$ has only a trivial solution. Indeed, if $u_0 = \lambda K u_0$, then $u_0 \in C[a,b]$ and , according to Theorem 6.2 3),

$\|u_0\|_C \le |\lambda| M(b-a) \|u_0\|_C$, then

$\left[1-|\lambda| M(b-a)\right]\|u_0\|_C \le 0$

Because $|\lambda| < \dfrac{1}{M(b-a)}$, $\left[1-|\lambda| M(b-a)\right] > 0$ and, therefore, $\|u_0\|_C = 0$. That yields, that $u\ (x) = 0$ for all $x \in [a,b]$. So, only the trivial solution exists for the homogeneous equation.

The non-homogeneous equation $u = \lambda K u + f$ can be rewritten in the form

$(I - \lambda K) u = f$

where I is an identity operator

Then solution of this equation can be treated as an inversion of the operator

$u = (I - \lambda K)^{-1} f$

Therefore, if $|\lambda| < \dfrac{1}{M(b-a)}$, then there exists an inverse operator $(I - \lambda K)^{-1}$.

The abovementioned results can be formulated in the following theorem:

Theorem 7.3 Fredholm's integral equation

$$u = \lambda K u + f$$

with $|\lambda| < \dfrac{1}{M(b-a)}$ and continuous kernel $K(x,y)$ has a unique solution $u(x) \in C[a,b]$ for any $f(x) \in C[a,b]$.
This solution is given by a convergent Neumann series

$$u(x) = \sum_{k=0}^{\infty} \lambda^k K^k f$$

and satisfies

$$\|u(x)\|_C \le \frac{\|f\|_C}{1 - |\lambda| M(b-a)} .$$

If $|\lambda| < \dfrac{1}{M(b-a)}$, then there exists an inverse operator $(I - \lambda K)^{-1}$.

Conditions of Theorem 7.3 are only just sufficient conditions; if these conditions are not satisfied, solution of the integral equation still can exists and the Neumann series can be convergent.

Example 7.1 Find the solution of the integral equation

$$u(x) = e^x + \frac{1}{e}\int_0^1 u(y)dy$$

by the method of successive approximations and in the form of the Neumann series.

Identify: $K(x,y) = 1$ $\quad f(x) = e^x$ $\quad b - a = 1$

$M = 1$ $\quad \lambda = \dfrac{1}{e}$

Check condition: $|\lambda| < \dfrac{1}{M(b-a)}$ $\quad \dfrac{1}{e} < \dfrac{1}{1 \cdot 1} < 1$

1) iterations:

$$u_0(x) = e^x$$

$$u_1(x) = e^x + \frac{1}{e}\int_0^1 u_0(y)dy = e^x + \frac{1}{e}\int_0^1 e^x dy = e^x + \frac{1}{e}\left[e^x\right]_0^1 = e^x + 1 - \frac{1}{e}$$

$$u_2(x) = e^x + \frac{1}{e}\int_0^1 u_1(y)dy = e^x + \frac{1}{e}\int_0^1\left(e^x + 1 - \frac{1}{e}\right)dy = e^x + 1 - \frac{1}{e^2}$$

…

$$u_n(x) = e^x + \frac{1}{e}\int_0^1 u_{n-1}(y)dy = e^x + 1 - \frac{1}{e^n}$$

Then solution of the integral equation is a limit of iterations

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(e^x + 1 - \frac{1}{e^n}\right) = e^x + 1$$

This result can be validated by a direct substitution into the given integral equation.

2) Neumann series:

$$u(x) = \sum_{k=0}^{\infty} \lambda^k K^k f \qquad = f(x) + \lambda^1 K^1 f + \lambda^2 K^2 f + \cdots$$

$$\begin{aligned}
f(x) &= e^x \\
Kf &= \int_0^1 e^x dy \qquad = e - 1 \\
K^2 f &= \int_0^1 (e-1) dy \quad = e - 1 \\
&\dots \\
K^n f &= e - 1
\end{aligned}$$

Then the Neumann series is

$$\begin{aligned}
u(x) &= e^x + \frac{1}{e}(e-1) + \frac{1}{e^2}(e-1) + \cdots + \frac{1}{e^n}(e-1) + \cdots \\
&= e^x - (e-1) + (e-1)\sum_{n=0}^{\infty} \frac{1}{e^n} \\
&= e^x - e + 1 + \frac{(e-1)}{1 - \frac{1}{e}} \\
&= e^x - e + 1 + e \\
&= e^x + 1
\end{aligned}$$

So, the Neumann series approach produces the same solution.

2. The Method of Successive Substitutions for Fredholm's Integral Equation (the Resolvent Method)

Iterated kernel

Let integral operator K has a continuous kernel $K(x,y)$, then define:

Repeated operator $\quad K^n = K\left(K^{n-1}\right) = \left(K^{n-1}\right)K \qquad n = 2,3,\ldots$

It has a has a kernel $\quad K_n(x,y) = \int_G K(x,y')K_{n-1}(y',y)dy'$

Indeed,

$$(Kf)(x) = \int_G K(x,y)f(y)dy$$

$$\begin{aligned}(K^2 f)(x) &= [K(Kf)](x) \\ &= \int_G K(x,y')\left[\int_G K(y',y)f(y)dy\right]dy' \\ &= \int_G\left[\int_G K(x,y')K(y',y)dy'\right]f(y)dy\end{aligned}$$

$\ldots$

Kernel

$$\begin{aligned}K_n(x,y) &= \int_G K(x,y')K_{n-1}(y',y)dy' \\ &= \int_G K_{n-1}(x,y')K(y',y)dy'\end{aligned}$$

is called an **iterated kernel**. Kernels $K_n(x,y)$ are continuous, and if domain $G=(a,b)$, then

$$\left|K_n(x,y)\right| \le M^n(b-a)^{n-1}$$

Resolvent

Function defined by the infinite series

$$R(x,y,\lambda) = \sum_{k=0}^{\infty} \lambda^k K_{k+1}(x,y)$$

is called a **resolvent**.

Theorem 6.4 Solution of integral equation $u = \lambda K u + f$ with continuous kernel $K(x,y)$ is unique in $C[a,b]$ for $|\lambda| < \dfrac{1}{M(b-a)}$, and for any $f \in C[a,b]$ is given by

$$u(x) = f(x) + \lambda\int_a^b R(x,y,\lambda)f(y)dy$$

i.e. there exists inverse operator

$$(I-\lambda K)^{-1} = I + \lambda R\,, \qquad |\lambda| < \frac{1}{M(b-a)}$$

Example 6.2 Find solution of integral equation

$$u(x)=\frac{23}{6}x+\frac{1}{8}\int_0^1 xyu(y)dy$$

by the resolvent method .

Identify: $K(x,y)=xy$ $f(x)=\frac{23}{6}x$ $b-a=1$

$M=1$ $\lambda=\frac{1}{8}$

Check condition: $|\lambda|<\frac{1}{M(b-a)}$ $\frac{1}{8}<\frac{1}{1\cdot 1}<1$

Iterated kernels:

$$K_1(x,y) = xy$$

$$K_2(x,y) = \int_0^1 K(x,y')K(y',y)dy' = \int_0^1 xy'y'ydy' = xy\left[\frac{y'^3}{3}\right]_0^1 = \frac{xy}{3}$$

$$K_3(x,y) = \int_0^1 K_2(x,y')K(y',y)dy' = \int_0^1 \frac{xy'}{3}y'ydy' = \frac{xy}{3}\left[\frac{y'^3}{3}\right]_0^1 = \frac{xy}{3^2}$$

…

$$K_n(x,y) = \frac{xy}{3^{n-1}}$$

Resolvent:

$$R(x,y,\lambda) = \sum_{k=0}^{\infty}\lambda^k K_{k+1}(x,y)$$

$$= xy+\frac{1}{8}\frac{xy}{3}+\frac{1}{8^2}\frac{xy}{3^2}+\frac{1}{8^3}\frac{xy}{3^3}+\ldots+\frac{1}{8^n}\frac{xy}{3^n}+\ldots$$

$$= xy\left[1+\frac{1}{8}\frac{1}{3}+\frac{1}{8^2}\frac{1}{3^2}+\frac{1}{8^3}\frac{1}{3^3}+\ldots+\frac{1}{8^n}\frac{1}{3^n}+\ldots\right]$$

$$= xy\frac{1}{1-\frac{1}{24}}$$

$$= \frac{24}{23}xy$$

Solution:

$$u(x) = f(x)+\lambda\int_a^b R(x,y,\lambda)f(y)dy$$

$$= \frac{23}{6}x+\frac{1}{8}\int_0^1 \frac{24}{23}xy\frac{23}{6}ydy$$

$$= \frac{23}{6}x+\frac{1}{2}x\int_0^1 y^2dy$$

$$= \frac{23}{6}x+\frac{1}{2}x\left[\frac{y^3}{3}\right]_0^1$$

$$= 4x$$

3. The Method of Successive Approximations for the Volterra Integral Equation of the 2^{nd} kind

Consider the Volterra integral equation of the 2^{nd} kind (note that equation of the 1^{st} kind can be reduced to the 2^{nd} kind by diffrerention)

$$u(x) = \lambda \int_0^x K(x,y) u(y)\, dy + f(x)$$

where $K(x,y)$ is a continuous Volterra kernel.

The method of successive approximation is defined by the following iterations:

$$u_0(x) \quad = f(x)$$

$$u_n(x) \quad = \sum_{k=0}^{n} \lambda^n K^k f \qquad = \lambda K u_{n-1} + f$$

Theorem 6.5 The Volterra integral equation of the 2^{nd} kind

$$u(x) = \lambda \int_0^x K(x,y) u(y)\, dy + f(x)$$

with continuous Volterra kernel $K(x,y)$ and with any $\lambda \in R$

has a unique solution $u(x) \in C[0,a]$ for any $f(x) \in C[0,a]$. This solution is given by a uniformly convergent Neumann series

$$u(x) = \sum_{k=0}^{\infty} \lambda^n \left(K^k f\right)(x)$$

and its norm satisfies

$$\|u(x)\|_C \le \|f\|_C e^{|\lambda| Ma}$$

Example 6.3 Find solution of integral equation

$$u(x) = 1 + \int_0^x u(y) dy$$

by the method of successive approximations.

Identify: $K(x,y) = 1 \qquad f(x) = 1$

$M = 1 \qquad \lambda = 1$

$$K^0 f \quad = f(x) \quad = 1$$

$$K^1 f \quad = \int_0^x K(x,y)\left(K^0 f\right)(y) dy \quad = \int_0^x 1 \cdot 1 dy \quad = [y]_0^x \quad = x$$

$$K^2 f \quad = \int_0^x K(x,y)\left(K^1 f\right)(y) dy \quad = \int_0^x 1 \cdot y dy \quad = \left[\frac{y^2}{2}\right]_0^x \quad = \frac{x^2}{2}$$

$$K^3 f \quad = \int_0^x K(x,y)\left(K^2 f\right)(y) dy \quad = \int_0^x 1 \cdot \frac{y^2}{2} dy \quad = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^x \quad = \frac{x^3}{2 \cdot 3}$$

…

$$K^n f \quad = \frac{x^n}{n!}$$

Solution: $$u(x) \quad = \sum_{k=0}^{\infty} \lambda^n \left(K^k f\right)(x) \quad = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad = e^x$$

7.6 Connection between integral equations and initial and boundary value problems

1. Reduction of IVP to the Volterra integral equation

Example 7.4 Reduce IVP

$$u' - 3x^2 u = 0 \qquad u(0) = 1$$

to the Volterra integral equation.

Integrate the differential equation from 0 to x:

$$\int_0^x \left(u' - 3x^2 u\right) dx = 0$$

$$\int_0^x (u') dx - \int_0^x \left(3x^2 u\right) dx = 0$$

$$u(x) - u(0) - 3\int_0^x x^2 u dx = 0$$ use the initial condition $u(0) = 1$

$$u(x) = 1 + 3\int_0^x x^2 u dx$$ is a Volterra equation with $K(x, y) = x^2$

2. Reduction of the Volterra integral equation to IVP

Recall the Liebnitz rule for differentiation of expressions with integrals:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x, y) dy \quad = \int_{a(x)}^{b(x)} \frac{g(x, y)}{\partial x} dy + g[x, b(x)]\frac{db(x)}{dx} - g[x, a(x)]\frac{da(x)}{dx}$$

In particularly,

$$\frac{d}{dx}\int_0^x g(y) dy \quad = g(x)$$

$$\frac{d}{dx}\int_0^x g(x, y) dy \quad = \int_0^x \frac{g(x, y)}{\partial x} dy + g(x, x)$$

Reduction of the Volterra integral equation to IVP is performed by consecutive differentiation of the integral equation with respect to variable x and substitution $x = 0$ for setting of the initial conditions.

Example 7.5 Reduce the Volterra integral equation

$$u(x) = x^3 + \int_0^x (x - y)^2 u(y) dy$$

initial value problem.

substitute $x = 0$ to get initial condition

$$u(x) = x^3 + \int_0^x (x - y)^2 u(y) dy \qquad u(0) = 0^3 + \int_0^0 (x - y)^2 u(y) dy \qquad u(0) = 0$$

$$u'(x) = 3x^2 + \int_0^x 2(x-y)u(y)dy \qquad u'(0) = 30^2 + \int_0^0 2(x-y)u(y)dy \quad u'(0) = 0$$

$$u''(x) = 3x^2 + 2\int_0^x u(y)dy \qquad u''(0) = 30^2 + 2\int_0^x u(y)dy \qquad u''(0) = 0$$

$$u'''(x) = 6x + 2u(x)$$

Therefore, the integral equation is reduced to IVP for 3rd order ODE:

$$u'''(x) = 6x + 2u(x) \qquad \begin{aligned} u(0) &= 0 \\ u'(0) &= 0 \\ u''(0) &= 0 \end{aligned}$$

3. Reduction of BVP to the Fredholm integral equation

Recall repeated integration formulas:

$$\int_0^x \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1) dt_1 dt_2 \cdots dt_{n-1} dt_n = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt$$

Example 7.6 Reduce the boundary value problem

$$y''(x) + y(x) = x \qquad x \in (0, \pi)$$
$$y(0) = 1$$
$$y(\pi) = \pi - 1$$

to the Fredholm integral equation.

Set $\qquad y''(x) = u(x)$

integrate $\qquad \int_0^x y''(t)dt = \int_0^x u(t)dt$

$$y'(x) - y'(0) = \int_0^x u(t)dt$$

integrate $\qquad \int_0^x [y'(t_2) - y'(0)]dt_2 = \int_0^x \left[\int_0^{t_2} u(t_1)dt_1 \right] dt_2$

$$y(x) - y(0) - y'(0)x = \int_0^x \left[\int_0^{t_2} u(t_1)dt_1 \right] dt_2$$

$$y(x) - y(0) - y'(0)x = \int_0^x (x-t)u(t)dt \qquad \text{repeated integration}$$

Use the first boundary condition

$$y(x) = 1 + y'(0)x + \int_0^x (x-t)u(t)dt$$

In this expression, $y'(0)$ is not known. Substitute $x = \pi$ and apply the second boundary condition

$$y(\pi) = 1 + y'(0)\pi + \int_0^\pi (\pi - t)u(t)dt$$

$$\pi - 1 = 1 + y'(0)\pi + \int_0^\pi (\pi - t)u(t)dt$$

Solve for $y'(0)$

$$y'(0) = 1 - \frac{2}{\pi} - \frac{1}{\pi}\int_0^\pi (\pi - t)u(t)dt$$

Then

$$\begin{aligned} y(x) &= 1 + \left[1 - \frac{2}{\pi} - \frac{1}{\pi}\int_0^\pi (\pi - t)u(t)dt\right]x + \int_0^x (x - t)u(t)dt \\ &= 1 + x - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^\pi (\pi - t)u(t)dt + \int_0^x (x - t)u(t)dt \end{aligned}$$

Now substitute this expression for $y(x)$ and $y''(x) = u(x)$ into the original differential equation

$$u + 1 + x - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^\pi (\pi - t)u(t)dt + \int_0^x (x - t)u(t)dt = x$$

$$u + 1 - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^\pi (\pi - t)u(t)dt + \int_0^x (x - t)u(t)dt = 0$$

$$u = -1 + \frac{2}{\pi}x + \frac{x}{\pi}\int_0^\pi (\pi - t)u(t)dt - \int_0^x (x - t)u(t)dt$$

$$u = -1 + \frac{2}{\pi}x + \frac{x}{\pi}\int_0^x (\pi - t)u(t)dt + \frac{x}{\pi}\int_x^\pi (\pi - t)u(t)dt - \int_0^x (x - t)u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \left[\frac{x}{\pi}\int_0^x (\pi - t)u(t)dt - \int_0^x (x - t)u(t)dt\right] + \frac{x}{\pi}\int_x^\pi (\pi - t)u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \left[\int_0^x \frac{x}{\pi}(\pi - t)u(t)dt - \int_0^x (x - t)u(t)dt\right] + \int_x^\pi \frac{x(\pi - t)}{\pi}u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \int_0^x \left[\frac{x}{\pi}(\pi - t) - (x - t)\right]u(t)dt + \int_x^\pi \frac{x(\pi - t)}{\pi}u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \int_0^x \frac{t(\pi - x)}{\pi}u(t)dt + \int_x^\pi \frac{x(\pi - t)}{\pi}u(t)dt$$

It yields a Fredholm integral equation

$$u = \frac{2}{\pi}x - 1 + \int_0^\pi K(x,t)u(t)\,dt$$

with a kernel

$$K(x,t) = \begin{cases} \dfrac{t(\pi - x)}{\pi} & 0 \le t \le x \\ \dfrac{x(\pi - t)}{\pi} & x \le t \le \pi \end{cases}$$

Exercises

1. Prove part 3) of the Theorem 6.2.

2. Classify each of the following integral equations as Fredholm or Volterra integral equation, linear or non-linear, homogeneous or non-homogeneous, identify the parameter λ and the kernel $K(x,y)$:

a) $u(x) = x + \int_0^1 xyu(y)dy$

b) $u(x) = 1 + x^2 + \int_0^x (x-y)u(y)dy$

c) $u(x) = e^x + \int_0^x yu^2(y)dy$

d) $u(x) = \int_0^1 (x-y)^2 u(y)dy$

e) $u(x) = 1 + \frac{x}{4}\int_0^1 \frac{1}{x+y}\frac{1}{u(y)}dy$

3. Reduce the following integral equation to an initial value problem

$$u(x) = x + \int_0^x (y-x)u(y)dy$$

4. Find the equivalent Volterra integral equation to the following initial value problem

$$y''(x) + y(x) = \cos x \qquad y(0) = 0 \qquad y'(0) = 1$$

5. Derive the equivalent Fredholm integral equation for the following boundary value problem

$$y'' + y = x \quad x \in (0,1) \qquad y(0) = 1 \qquad y(1) = 0$$

6. Solve the following integral equations by using the successive approximation method and the resolvent method:

a) $u(x) = x + \lambda\int_0^1 xyu(y)dy$

b) $u(x) = x + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos xu(y)dy$

7. Solve the following integral equation by using the successive approximation method

$$u(x) = 1 - \int_0^x (y-x)u(y)dy$$

8. Solve the following integral equations:

a) $u(x) = \sin 2t + \int_0^t u(t-s)\sin(s)ds$

b) $u(x) = t^2 + \int_0^t u'(t-s)e^{-as}ds \qquad u(0) = 0$