The Laplace Equation: 09 – DDRR (Dirichlet-Dirichlet-Robin-Robin)

The Laplace Equation:

\[ \nabla^2 u = 0 \]

\[ u(x) = \sum_{i=1}^{\infty} a_i \sinh(\lambda_i y) \cos(\lambda_i x - c_i y) \]

\[ u(x) = \sum_{i=1}^{\infty} b_i \sin(\lambda_i y) \cos(\lambda_i x + c_i y) \]

Supplemental problems:

\[ \nabla^2 u_i = 0 \]

Solution of supplemental problems:

\[ a_i = \frac{1}{L} \int_0^L \frac{\cos(\lambda_i x)}{\sinh(\lambda_i L)} \sinh\left(\frac{\pi \lambda_i y}{L}\right) \sinh\left(\frac{n \pi y}{L}\right) dy \]

\[ b_i = \frac{1}{L} \int_0^L \frac{\cos(\lambda_i x)}{\sin(\lambda_i L)} \sin\left(\frac{\pi \lambda_i y}{L}\right) \sinh\left(\frac{n \pi y}{L}\right) dy \]

Solution of BVP problem:

\[ u(x, y) = \sum_{i=1}^{\infty} c_i \left[ \cosh\left(\frac{n \pi x}{M}\right) \sinh\left(\frac{n \pi \lambda_i y}{M}\right) \sinh\left(\frac{n \pi y}{L}\right) \right] \]

\[ u(x, y) = \sum_{i=1}^{\infty} d_i \left[ \cos\left(\frac{n \pi x}{M}\right) \sinh\left(\frac{n \pi \lambda_i y}{M}\right) \sin\left(\frac{n \pi y}{L}\right) \right] \]

\[ u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y) \]
POISSON’S EQUATION: \(01 – \text{DDDD}\)

\[
\begin{align*}
M & \quad u = x(L - x) = f_2(x) \\
0 & \quad \nabla^2 u = F(x, y) \\
L & \quad \left[ \frac{\partial u}{\partial x} + H_3 u \right]_{x=0} = f_4(y) \\
& \quad \left[ \frac{\partial u}{\partial x} + H_3 u \right]_{x=L} = f_4(y) \\
& \quad u = x(L - x) = f_1(x)
\end{align*}
\]

Supplemental problems:

\[
\begin{align*}
f_2(x) & \\
f_4(y) & \quad \nabla^2 u = 0 \\
f_3(y) & \quad \nabla^2 u = F \\
f_1(x) & \quad \nabla^2 u = 0
\end{align*}
\]

Solution of supplemental problems:

Solution of Laplace’s homogeneous equation with non-homogeneous b.c.’s:

\[
u_3(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)
\]

Solution of Poisson’s equation with homogeneous boundary conditions

\[
u_6 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} (\lambda_n \cos \lambda_n x + H_3 \sin \lambda_n x) \sin \left( \frac{m\pi}{M} y \right)
\]

\[
A_{nm} = -\int_0^M \int_0^L F(x, y)(\lambda_n \cos \lambda_n x + H_3 \sin \lambda_n x) \sin \left( \frac{m\pi}{M} y \right) dxdy
\]

\[
= \frac{M}{2} \left( \lambda_n^2 + H_3^2 \right) \left[ L + \frac{H_3^2 + H_4^2}{2} \right] + \frac{H_3}{2} \left( \lambda_n^2 + \frac{m^2\pi^2}{M^2} \right)
\]

Where \( \lambda_n \) from positive roots of:

\[
(H_3, H_4 - \lambda^2) \sin \lambda L + (H_3 + H_4) \lambda \cos \lambda L = 0
\]

Solution of BVP for Poisson’s Equation (superposition principle):

\[
u(x, y) = u_3(x, y) + u_6(x, y)
\]