REVIEW OF THERMAL WAVES, 3D TRANSIENT HEAT CONDUCTION IN A LAYERED SAMPLE, AND ANISOTROPIC MEDIA

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OVERVIEW

Detailed knowledge of nuclear fuels and its properties can help improve the performance, prediction, and safety of nuclear energy production. During the burnup process, the composition of nuclear fuels changes, along with the thermal conductivity. As a result, heat flows through it differently over its lifetime. Calculating these properties can provide a detailed model of the material. Knowing these spatially resolved properties will allow for better predictive modeling of the fuel or material when heated, especially with the next generation of fuels being developed [1]. A specific instance of this is Post-Irradiation Examination (PIE) of used fuel. In radiated environments materials can swell or experience non thermal creep. Knowing the properties of the material at a small scale instead of in bulk will help understand these behaviors better.

Recent advancements in non-contact thermometry with quantum dot photoluminescence and improved accuracy of the reconstructed temperature with neural networks make it possible to accurately measure the thermal response of a heated material. But current measurement methods are costly, and lack the spatial resolution necessary to model the strong radial temperature variations in a nuclear fuel rod. The math discussed in this paper relates a project to make an infrared laser diode provides periodic heating to the material to the periodic thermal energy from the laser at the surface. The photodiode in the player will capture the emitted light from the quantum dots, which can be related to temperature change.

![Figure 1 - Example of thermal wave generated with heating laser and resulting phase delayed thermal wave](image)

This review paper will discuss the concepts of thermal waves and how to solve for a sample’s material properties in a n-layered transient 3d model. The idea is to have a known temperature input and then determine the thermal properties based on the heating response of the material. Thermal wave models relate the difference in the amplitude and phase of these waves (Figure 1) to determine the thermal properties of the material. There is also a discussion included on how to solve for thermal properties when they are not homogeneous in the control volume being analyzed which come from [2].

INTRODUCTION TO THERMAL WAVES

As an introduction to thermal waves we will first look at an isotropic homogeneous semi-infinite medium [3] that is periodically heated by a laser power source with the mathematical representation of

\[
\frac{Q_o}{2} \left[ 1 + \cos(\omega t) \right] \tag{1}
\]

with \(Q_o\) representing the source intensity. The transient heat equation in one direction will be used to solve for the temperature profile in this scenario.

\[
\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0 \tag{2}
\]

The boundary condition will equate the conduction into the material to the periodic thermal energy from the laser at the surface.

\[
-k \frac{\partial T}{\partial x} = \frac{Q_o}{2} \left[ 1 + \cos(\omega t) \right] = \frac{Q_o}{2} \left[ 1 + \cos(\omega t) \right] = \text{Re} \left[ \frac{Q_o}{2} \left[ 1 + \exp(j\omega t) \right] \right] \tag{3}
\]

Within the real part of the heating it divides into \(\frac{Q_o}{2}\) and \(Q_o \exp(j\omega)\). Only the transient part will be analyzed in the following solution. To solve the transient 1-D heat equation we will assume the periodic component solution is of this form:

\[
T(x, t) = \text{Re} \left( T(x) \exp(\omega t) \right) \tag{4}
\]

The general solution to the problem becomes

\[
T(x) = A \exp(-\alpha x) + B \exp(\sigma x) \tag{5}
\]

The sample in this case is represented as a semi-infinite media. Assuming the \(T(x)\) is finite as \(x\) goes to infinite, \(B\) is zero. The boundary condition in (3) represents the heat from the laser being equal to the conduction into the material at the surface and
can be used to solve for the constant $A$ in (5). Solving the equation out it becomes

$$T(x, t) = \frac{Q_0}{2\sqrt{\rho c k \omega \mu}} \exp \left( -\frac{x}{\mu} \right) \exp \left( i \omega t - \frac{\pi}{4} \right)$$

(6)

$$\mu = \frac{2\alpha}{\sqrt{\omega \mu}}$$

(7)

The thermal diffusion length of the periodic wave represents how far into the material the thermal change penetrates, and is a function of the thermal diffusivity and frequency of the heating laser. This simple model will not work to calculate the phase delay of thermal waves, because with our method of temperature sensing we can only measure the temperature at the surface of the material when $x = 0$. In this case the terms with $x$ disappear and the phase delay at the surface is a constant $\pi/4$. To find the properties of a sample, the phase delay needs to be known as a function of frequency. One way to model this more accurately is to treat the laser heating as a volumetric generation in the heat diffusion equation and treat the quantum dots on the surface as a layer over the material being analyze, turning the model into a 2-layer system, each with different properties. This approach is developed and used in [4] and will be discussed here.

In this case the governing equation of the 3D model is

$$k_{ij} T_{x,ij} - \rho_c c_s^2 \partial_t T = Q_f$$

(8)

$$k_{ij} T_{s,ij} - \rho_c c_s^2 \partial_t T = Q_s$$

(9)

With the following heat source terms

$$Q_f = \frac{P \alpha_f (1 - R_f)}{\pi a_1 a_2} \exp \left( -\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \alpha_f x_3 \right)$$

$$\left( \frac{1}{2} + \frac{1}{4} \exp(i \omega t) + \exp(-i \omega t) \right)$$

(10)

$$Q_s = \frac{P \alpha_s (1 - R_f)(1 - R_s)}{\pi a_1 a_2} \exp(-\alpha_f h) \exp \left( -\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \alpha_s (x_3 - h) \right)$$

$$\left( \frac{1}{2} + \frac{1}{4} \exp(i \omega t) + \exp(-i \omega t) \right)$$

(11)

as volumetric generation from the laser where $P$ represents the power of the modulated heating laser, $a_1$ and $a_2$ are the major and minor axes of the laser spot, and $R$ is the reflectivity of the top surface. The subscripts $f$ and $s$ represent the film and the substrate respectively. The directions of the anisotropic conductivity is represented by $i$ and $j$. The surface temperature is derived in Fourier space as $\hat{T}_o(\xi_1, \xi_2)$ with $\xi_1$ and $\xi_2$ as the integral variables. After derivation, $\hat{T}_o(\xi_1, \xi_2) = A(\xi_1, \xi_2) + B(\xi_1, \xi_2) + E(\xi_1, \xi_2)$ with $A$ and $B$ from the following boundary conditions

$$A \cdot [B] = R$$

(12)

and

$$N_{11} = (i k_{f31} \xi_1 - i k_{f32} \xi_2 - \eta_f k_{f33})$$

$$N_{12} = (i k_{f31} \xi_1 - i k_{f32} \xi_2 + \eta_f k_{f33})$$

$$N_{13} = 0$$

$$N_{21} = (i k_{f31} \xi_1 - i k_{f32} \xi_2 - \eta_f k_{f33}) \exp(-\eta_f h)$$

$$N_{22} = (i k_{f31} \xi_1 - i k_{f32} \xi_2 + \eta_f k_{f33}) \exp(-\eta_f h)$$

$$N_{23} = (i k_{s31} \xi_1 + i k_{s32} \xi_2 + \eta_s k_{s33})$$

$$N_{31} = R_{th}(i k_{f31} \xi_1 + i k_{f32} \xi_2 - \eta_f k_{f33}) \exp(-\eta_f h) - \exp(-\eta_f h)$$

$$N_{32} = R_{th}(i k_{f31} \xi_1 + i k_{f32} \xi_2 - \eta_f k_{f33}) \exp(-\eta_f h) - \exp(-\eta_f h)$$

$$N_{33} = 1$$

$$R_1 = E(i k_{f31} \xi_1 + i k_{f32} \xi_2 + \alpha_f k_{f33})$$

$$R_2 = E(i k_{f31} \xi_1 + i k_{f32} \xi_2 + \alpha_f k_{f33}) \exp(-\alpha_f h) - F(i k_{s31} \xi_1 + i k_{s32} \xi_2 + \alpha_s k_{s33})$$

$$R_3 = E[1 - R_{th}(i k_{f31} \xi_1 + i k_{f32} \xi_2 + \alpha_f k_{f33})] \exp(-\alpha_f h) - F$$

(13)

The terms $E$ and $F$ are obtained from the particular solution as

$$E = \frac{p a_f (1 - R_f)}{2 \pi} \frac{\exp\left( \frac{\xi_1 a_1^2}{4} + \frac{\xi_2 a_2^2}{4} \right)}{k_{f32}^2 + 2 a_1 (i k_{f31} \xi_1 + i k_{f32} \xi_2 - 2 \xi_1 k_{f32} - k_{f33}^2 \xi_2 - 2 i \omega \rho \xi_3)}$$

$$F = \frac{p a_s (1 - R_f)(1 - R_s)}{2 \pi} \frac{\exp\left( -\frac{\xi_1 a_1^2}{4} - \frac{\xi_2 a_2^2}{4} \right)}{k_{s33}^2 + 2 a_1 (i k_{s31} \xi_1 + i k_{s32} \xi_2 - 2 \xi_1 k_{s32} + k_{s33} \xi_3 - 2 i \omega \rho \xi_3)}$$

(14)

where $\eta_f$ and $\eta_s$ are defined from the homogeneous solutions and are obtained by applying those solutions to the following diffusion equation.
\[ k \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + (k_{12} + k_{21}) \frac{\partial^2 T}{\partial x \partial y} \\
+ (k_{13} + k_{31}) \frac{\partial^2 T}{\partial x \partial z} + (k_{23} + k_{32}) \frac{\partial^2 T}{\partial y \partial z} \\
+ g(x, y, z, t) = \rho c \frac{\partial T(x, y, z, t)}{\partial t} \]  

where \( k_{12} = k_{21}, \ k_{13} = k_{31}, \ \text{and} \ k_{23} = k_{32} \) by the reciprocal relation. The information presented here may hopefully be useful to someone looking to analyze thermal waves in layered samples or for someone examining the heat diffusion equation for conduction in anisotropic materials. This overview contains relevant information for exposure to the topic, and the citations provide a much more in depth discussion of the topic if necessary for a particular application.
REFERENCES
4. Hua, Z., Hybrid Photothermal Technique for Microscale Thermal Con, in Mechanical and Aerospace Engineering. 2013, Utah State University.