II.4 SEQUENCES AND SERIES

Objectives: After the completion of this section the student

- should recall the definitions of the convergence of a sequence, and some limits;
- should recall the definition of the convergence of the infinite series;
- should be able to investigate infinite series for convergence;
- should be able to determine the interval of convergence of the power series;
- should be able to expand functions into Taylor series

Contents: 1. Sequences
2. Infinite Series
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II.4 SEQUENCES AND SERIES

1. SEQUENCES

Sequences are the ordered sets of points on the real line (real numbers), denoted by the numbers with indexes:

\[ a_k, \quad k = 0,1,2,... \]
\[ b_i, \quad i = -2,-1,0,1,2,... \]
\[ c_0,c_1,c_2,... \]

The sequences differ from the sets \( \{ a_k \} \) by two main qualities: the order of the listing is important for sequences, and the same point can appear in a sequence repeatedly. However, the braces \( \{ a_k \} \) sometimes also are used for description of the sequences.

Sequences can be treated as the functions with the discrete domain:

\[ a_k : \mathbb{N} \rightarrow \mathbb{R} \]

The distance between two consecutive points in a sequence can be measured in a natural way as the length between them

\[ |a_k - a_{k-1}| \]

Convergence

We say that a sequence \( a_k \) converges to the point \( a \) and write it

\[ \lim_{k \to \infty} a_k = a \quad \text{or} \quad a_k \rightarrow a \]

if the distance between \( a \) and the points \( a_k \) becomes arbitrarily small with increase of the index \( k \), i.e.

\[ \lim_{k \to \infty} |a - a_k| = 0 \]

In the \( \varepsilon \)-language this definition can be formulated as follows:

for any \( \varepsilon > 0 \) there exists a natural number \( K \in \mathbb{N} \) such that

\[ |a - a_k| < \varepsilon \quad \text{for all } k > K \]

If a sequence \( a_k \) does not have a limit it is called divergent.

Some Facts (Theorems):

1. Limit of the sequence is unique.

2. Convergent sequence is bounded
   (there exists \( B > 0 \) such that \( |a_k| < B \) for all \( k \)).

3. Unbounded sequence diverges.

4. Bounded non-decreasing sequence converges to its least upper bound.

5. Bounded non-increasing sequence converges to its greatest lower bound.

6. Let \( f(x) \), \( x \in \mathbb{R} \) be a continuous function such that \( f(k) = a_k \), then

\[ \lim_{x \to \infty} f(x) = a \quad \text{then} \quad \lim_{k \to \infty} a_k = a \]

Therefore, many known results and theorems derived for continuous functions can be easily extended to the sequences. For example, from \( \lim_{x \to \infty} \frac{1}{x} = 0 \) follows
that $\frac{1}{k} \to 0$. The methods such as Comparison Theorem or L’Hospital’s Rule can be used for investigation of the convergence of the sequences.

**EXAMPLES:**

1. Find the limit of the sequence $\lim_{n \to \infty} \left( l + \frac{x}{n} \right)^n$ where $x \in \mathbb{R}$.

   Associate the sequence with continuous function $f(n) = \left( l + \frac{x}{n} \right)^n$, $n > 0$.

   From the known result (Section 2.1 Example 8-9)
   $$\lim_{n \to \infty} \left( l + \frac{x}{n} \right)^n = e^x$$

   immediately follows that $\lim_{n \to \infty} \left( l + \frac{x}{n} \right)^n = e^x$.

2. Find $\lim_{n \to \infty} \frac{x^n}{n!}$ where $x \in \mathbb{R}$.

   Choose $k > |x|$, then for where $n > k + 1$

   $$\frac{x^n}{n!} < \frac{k^n}{n!} = \frac{k \cdot k \cdot \ldots \cdot k}{1 \cdot 2 \cdot \ldots \cdot (k + 1) \cdot (n - 1) \cdot n}$$

   $$= \frac{k^k}{k!} \cdot \left( \frac{k}{(k + 1) \cdot (k + 2) \cdot \ldots \cdot (n - 1) \cdot n} \right)$$

   $$\leq \frac{k^k}{k!} \cdot \frac{k}{n}$$

   $$= \frac{b}{n} \to 0 \quad \text{where} \quad b = \frac{k^k}{k!} \quad \text{is some number}$$

   Therefore, $\frac{x^n}{n!} \to 0$.

   Remark: L’Hospital’s Rule can be used repeatedly for $\lim_{n \to \infty} \frac{x^n}{\Gamma(n)}$, where the Gamma function $\Gamma(n)$ is the continuous differentiable extension of the factorial $n!$ for positive real numbers.

3. **Remarkable Limits:**

   1. $\frac{1}{n^a} \to 0 \quad a > 0$

   2. $x^a \to 0 \quad |x| < 1$

   3. $\frac{1}{x^a} \to 1 \quad x > 0$

   4. $\frac{1}{n^a} \to 1$

   5. $\left( l + \frac{x}{n} \right)^n \to e^x \quad x \in \mathbb{R}$

   6. $\frac{x^n}{n!} \to 0 \quad x \in \mathbb{R}$

   7. $\frac{\ln n}{n} \to 0$
Ramification: Cauchy Sequence

$a_n$ is called a **Cauchy sequence** (convergent in itself) if

$$\lim_{n,m \to \infty} |a_n - a_m| = 0$$

or in other words:

for any $\varepsilon > 0$ there exists a natural number $K \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \text{ for all } m, n > K$$

Any convergent sequence of the real numbers is a Cauchy sequence.

Any Cauchy sequence of the real numbers is convergent.

A metric set is called complete if all its Cauchy sequences are convergent. Therefore, the set of real numbers $\mathbb{R}$ is complete.

2. INFINITE SERIES

**Sigma Summation:**

**Sigma Summation** is a compact convenient way to define the sum of the numbered elements of the sets or sequences:

$$a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k$$

$$b_m + b_{m+1} + \ldots + b_M = \sum_{i=m}^{M} b_i$$

The symbol of summation $\sum$ can be treated as an operator applied to the finite sequences $a_k$

$$\sum: \mathbb{R}^n \to \mathbb{R}$$

This operator is linear:

$$\sum_{k=m}^{n} \alpha a_k = \alpha \sum_{k=m}^{n} a_k$$

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

The linearity defines the rules for operations with sigma summation. The index of summation is a “dummy” index – it means that any other letter can be used for the same result.

**Partial Sums:**

Let $a_k$, $k = 0, 1, 2, \ldots$ be a sequence. Define the **partial sums** of $a_k$ as the sum of its first consecutive elements:

$$s_0 = a_0$$

$$s_1 = a_0 + a_1$$

$$s_2 = a_0 + a_1 + a_2$$

$$\vdots$$

$$s_n = a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k \quad \text{(} n^{th} \text{ partial sum)}$$

$$s_n = s_{n-1} + a_n \quad n = 1, 2, \ldots \quad \text{(recursive definition)}$$

The set of partial sums forms a sequence

$$s_n, \quad n = 0, 1, 2, \ldots$$
Infinite Series

Define formally the **infinite series** as the sum of all elements of some infinite sequence and write it as:

\[ \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \ldots \]

with \( \infty \) in the upper limit of summation. “Formally”, because, technically, the summation of the infinite number of terms cannot be performed. Therefore, the result of summation will be defined in terms of convergence of the infinite series for which the convergence of the sequence of partial sums will be used.

Convergence

We say that the infinite series \( \sum_{k=0}^{\infty} a_k \) **converges** to a number \( L \in \mathbb{R} \) and write

\[ \sum_{k=0}^{\infty} a_k = L \]

if the sequence of partial sums of \( a_k \) **converges** to \( L \)

\[ s_n \to L \]

Divergence

If the sequence of partial sums \( s_n \) **diverges** then we say that the infinite series

\[ \sum_{k=0}^{\infty} a_k \] **diverges**.

Some questions can appear regarding the convergence of the infinite series. What if the order of terms in the infinite series is changed? Is it still convergent? And if it is convergent, does it converge to the same result? For example, consider

\[ a_1 + a_2 + a_3 + a_4 + \ldots \]

\[ a_2 + a_1 + a_3 + a_4 + \ldots \]

It looks that there is summation of the same terms. For summation of the finite number of terms results should be identical, but for the infinite series, in general, it is not true. Important case when the change of the order of summation in the infinite series does not change the result is the case of the absolutely convergent series.

Absolute convergence

We say that the infinite series \( \sum_{k=0}^{\infty} a_k \) **absolutely converges** if the infinite series of its absolute values \( \sum_{k=0}^{\infty} |a_k| \) converges.

Absolute convergence is especially important for the power series. It will allow the term by term operations such as differentiation and integration of the power series.

Some Facts (Theorems):

1. \( \sum_{k=0}^{\infty} a_k = L \) \quad \Rightarrow \quad \sum_{k=0}^{\infty} \alpha a_k = \alpha L \\
2. \( \sum_{k=0}^{\infty} a_k = L, \sum_{k=0}^{\infty} b_k = M \) \quad \Rightarrow \quad \sum_{k=0}^{\infty} (a_k + b_k) = L + M \\
3. \( \sum_{k=0}^{\infty} a_k \) converges \quad \Rightarrow \quad \sum_{k=m}^{\infty} a_k \) converges \\
4. \( \sum_{k=0}^{\infty} a_k = L \) \quad \Rightarrow \quad a_k \to 0 \\
5. \( a_k \not\to 0 \) \quad \Rightarrow \quad \sum_{k=0}^{\infty} a_k \) diverges \quad (basic negative test)
Tests for Convergence:

1. **Geometric series:**
   \[ \sum_{k=0}^{\infty} x_k = l + x + x^2 + \ldots \]
   \[ = \begin{cases} \frac{l}{1-x} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases} \]

2. **p-series:**
   \[ \sum_{k=1}^{\infty} \frac{l}{k^p} = l + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \]
   \[ = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases} \]

   Particular case: **harmonic series** \((p = 1)\)
   \[ \sum_{k=1}^{\infty} \frac{l}{k} = l + \frac{1}{2} + \frac{1}{3} + \ldots \text{ diverges} \]

3. **Telescoping series:**
   The series \( \sum_{k=0}^{\infty} a_k \) is telescoping if its elements can be defined as the difference \( a_k = f(k) - f(k + l) \) of values of some function and its \( n^{th} \) partial sum collapses to \( \sum_{k=0}^{n} a_k = f(0) - f(n + l) \):
   \[ \sum_{k=0}^{\infty} a_k = \left[ f(0) - f(1) \right] + \left[ f(1) - f(2) \right] + \left[ f(2) - f(3) \right] + \ldots + \left[ f(n-l) - f(n) \right] \]
   \[ = f(0) - f(n + l) \]

   Then for telescoping series:
   \[ \sum_{i=0}^{\infty} a_k = \begin{cases} f(0) - c & \text{if } f(n) \to c \\ \text{diverges} & \text{if } f(n) \to \pm\infty \end{cases} \]

4. **Alternating series:** if \( a_k \geq 0 \) for all \( k \) then the alternating series
   \[ \sum_{k=0}^{\infty} (-l)^k a_k \] converges if \( a_k \to 0 \) and \( a_{k+l} < a_k \) (Leibnitz)

   There is the estimation of the remainder of the \( n^{th} \) partial sum of the convergent alternating series:
   \[ \text{if } \sum_{k=0}^{\infty} (-l)^k a_k = L \text{ then } |L - s_n| < a_{n+1} \]

5. **Absolute convergence test:** if infinite series of absolute values of its terms is convergent then the original infinite series also is convergent, in other words
   \[ \text{if } \sum_{k=0}^{\infty} |a_k| \text{ converges then } \sum_{k=0}^{\infty} a_k \text{ converges} \]

This test emphasizes the importance of investigation of infinite series for absolute convergence. There are several very efficient tests for the absolute convergence of infinite series:
Absolute convergence tests: Let \( \sum_{k=0}^{\infty} a_k \) be an infinite series with \( a_k \geq 0 \) for all \( k \). Then

1. **Integral Test:**
   
   If \( \int_{\infty}^{x} f(x) \, dx \) converges then \( \sum_{k=1}^{\infty} f(k) \) converges

2. **Ratio Test:**

   If \( \frac{a_{k+1}}{a_k} \to \lambda \) then
   
   \[
   \begin{cases}
   \sum_{k=0}^{\infty} a_k \text{ converges if } \lambda < 1 \\
   \text{no conclusion if } \lambda = 1 \\
   \sum_{k=0}^{\infty} a_k \text{ diverges if } \lambda > 1
   \end{cases}
   \]

3. **Root Test:**

   If \( \left( a_k \right)^{\frac{1}{k}} \to \rho \) then
   
   \[
   \begin{cases}
   \sum_{k=0}^{\infty} a_k \text{ converges if } \rho < 1 \\
   \text{no conclusion if } \rho = 1 \\
   \sum_{k=0}^{\infty} a_k \text{ diverges if } \rho > 1
   \end{cases}
   \]

4. **Basic Comparison Test:**

   a) Let \( c_k \geq a_k \) for all \( k \), then

   If \( \sum_{k=0}^{\infty} c_k \) converges then \( \sum_{k=0}^{\infty} a_k \) converges

   b) Let \( 0 \leq d_k \leq a_k \) for all \( k \), then

   If \( \sum_{k=0}^{\infty} d_k \) diverges then \( \sum_{k=0}^{\infty} a_k \) diverges

5. **Limit Comparison Test:**

   Let \( L > 0 \) and \( b_k > 0 \) for all \( k \). If \( \frac{a_k}{b_k} \to L \), then either

   \[
   \begin{cases}
   \sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k \text{ converge or } \\
   \sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k \text{ diverge}
   \end{cases}
   \]
Examples:

1. Harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges because improper integral \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges.

2. Find the values of \( x \) for which \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-1)^n \) is absolutely convergent.

   Identify \( a_n = (-1)^n \frac{(x-1)^n}{n} \), then consider \( \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{|x-1|^n}{n} \).

   Use the ratio Test: \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{|x-1|^{k+1}}{k+1} \frac{n}{|x-1|^k} \)

   \(= \lim_{k \to \infty} \frac{n}{k+1} |x-1| \)

   \(= \lim_{k \to \infty} \left( 1 - \frac{1}{n+1} \right) |x-1| = |x-1| \)

   Therefore, the infinite series of absolute values is convergent when \( |x-1| < 1 \)

   The solution of this inequality is \( 0 < x < 2 \).

   For \( x < 0 \) and \( x > 2 \) infinite series is divergent.

   The question is not resolved for the case \( |x-1| = 1 \) which consists of:

   \( x = 0 \) then \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{|x-1|^n}{n} = \sum_{k=1}^{\infty} \frac{1}{n} \) diverges

   \( x = 2 \) then \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{|x-1|^n}{n} = \sum_{k=1}^{\infty} \frac{2}{n} \) diverges

   Answer: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-1)^n \) absolutely converges for \( 0 < x < 2 \).

Remarkable Series:

3. Although even if there exists a positive answer on the convergence of the infinite series, the actual value to which it converges in many cases is not known. Here, we present some results for which the sum of infinite series can be determined (note that in each case the convergence of the infinite series can be easily established with the appropriate convergence test):

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>1.</td>
<td>( \sum_{k=1}^{\infty} \frac{1}{k^2} )</td>
<td>( \frac{\pi^2}{6} )</td>
</tr>
<tr>
<td>2.</td>
<td>( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} )</td>
<td>( \frac{\pi^2}{12} )</td>
</tr>
<tr>
<td>3.</td>
<td>( \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} )</td>
<td>( \frac{\pi^2}{8} )</td>
</tr>
<tr>
<td>4.</td>
<td>( \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^2} )</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>5.</td>
<td>( \sum_{k=0}^{\infty} \frac{1}{k!} )</td>
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<tr>
<td>6.</td>
<td>( \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} )</td>
<td>( -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \pi )</td>
</tr>
<tr>
<td>7.</td>
<td>( \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} )</td>
<td>( -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 )</td>
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</tbody>
</table>
3. POWER SERIES

Power Series: We define the power series as the function \( P : \mathbb{R} \to \mathbb{R} \)

\[
P(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_k, x \in \mathbb{R}
\]

which for the fixed values of variable \( x = c \) becomes an infinite series

\[
P(c) = \sum_{k=0}^{\infty} a_k c^k
\]

The power series is convergent at \( x = c \) if \( \sum_{k=0}^{\infty} a_k c^k \) converges, and the power series is divergent at \( x = c \) if \( \sum_{k=0}^{\infty} a_k c^k \) diverges.

The power series can be treated as the generalization of the polynomials to the infinite number of terms.

Interval of convergence: The convergence of the power series has the following symmetry properties:

a) If \( \sum_{k=0}^{\infty} a_k x^k \) converges at \( x = c \)

then \( \sum_{k=0}^{\infty} a_k x^k \) converges for all \( -|c| < x < |c| \)

b) If \( \sum_{k=0}^{\infty} a_k x^k \) diverges at \( x = b \)

then \( \sum_{k=0}^{\infty} a_k x^k \) diverges for all \( x < -|b| \) and \( x > |b| \)

Analysis shows that there are three possible symmetrical cases for the convergence of the power series:

1) \( \sum_{k=0}^{\infty} a_k x^k \) converges only at \( x = 0 \)

2) \( \sum_{k=0}^{\infty} a_k x^k \) converges for all \( -\infty < x < \infty \)

3) \( \sum_{k=0}^{\infty} a_k x^k \) converges for \( -R < x < R \) for some \( R > 0 \), and diverges for \( x < -R \) and \( x > R \).

The maximum open interval \((−R,R)\) in which the power series is convergent is called the interval of convergence, where \( R \) is called the radius of convergence.

If \((−R,R)\) is the interval of convergence of the power series \( \sum_{k=0}^{\infty} a_k x^k \), then \( \sum_{k=0}^{\infty} a_k x^k \) converges on \((−R,R)\) absolutely.

At the boundary points of the interval of convergence, the power series has to be investigated separately. Example 3.2 is an example of investigation of the power series for the interval of convergence. Consider also the modification of this example:
Example:

Determine the interval of convergence of the power series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k \]

Because of the symmetry property (a), it is enough to investigate the power series for absolute convergence. Use the ration test:

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{|x|^{k+1}}{k+1} = \lim_{k \to \infty} \frac{k}{k+1} = |x|
\]

Therefore, the series is convergent for \(|x| < 1\). Consider the end points of the interval of convergence \((-1, 1)\):

\[ x = -1 \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \text{ diverges (harmonic)} \]

\[ x = 1 \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (1)^k = \sum_{k=1}^{\infty} (-1)^k \text{ converges} \]

Answer: \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k \) converges for all \( x \in (-1, 1) \).

The radius of convergence \( R = 1 \).

Radius of convergence:

The ratio test and the root test can be rewritten in terms of the radius of convergence \( R \) of the power series \( \sum_{k=0}^{\infty} a_k x^k \) in the following way:

1) Ratio Test: \( R = \lim_{k \to \infty} \left| \frac{a_k}{a_{k+1}} \right| \)
2) Root Test: \( R = \lim_{k \to \infty} |a_k|^{1/k} \)

Differentiation and Integration:

Let \( \sum_{k=0}^{\infty} a_k x^k \) be the power series with the radius of convergence \( R \).

Define \( P(x) = \sum_{k=0}^{\infty} a_k x^k \) for all \( x \in (-R, R) \). Then

1. \( P(x) \) is differentiable on \((-R, R)\) and its derivative \( P'(x) \) is defined by the power series with the same radius of convergence obtained by the term by term differentiation of the power series \( P(x) \):

\[ P'(x) = \sum_{k=0}^{\infty} \frac{d}{dx} (a_k x^k) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad x \in (-R, R) \]

Moreover, the function \( P(x) \) has derivatives of all orders.

2. \( P(x) \) is integrable on \((-R, R)\) and its indefinite integral is defined by the power series with the same radius of
convergence obtained by the term by term integration of the power series $P(x)$:

$$\int P(x)\,dx = \sum_{k=0}^{\infty} \int a_k x^k \,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + c \quad x \in (-R,R)$$

**Identity Theorems:**

Let $R_1$ be the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$ and let $R_2$ be the radius of convergence of $\sum_{k=0}^{\infty} b_k x^k$.

Denote $r = \min\{R_1,R_2\}$. Then

1. If $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k$ for all $x \in (-r,r)$, then $a_k = b_k$ for all $k \geq 0$

2. If $\sum_{k=0}^{\infty} a_k x^k = 0$ for all $x \in (-r,r)$, then $a_k = 0$ for all $k \geq 0$

**Algebraic Operations:**

1. $\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ (sum of two power series)

2. $\alpha \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \alpha a_k x^k \quad \alpha \in \mathbb{R}$

3. $\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j b_{k-j} \right) x^k$ (Cauchy product)

   for all $x \in (-r,r)$.

**Power Series in $(x-x_0)^k$:**

The similar definitions and results can be formulated for the power series in terms of the powers of $(x-x_0)$:

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

which is centered now at the point $x = x_0$. It can be reduced to the considered power series by the change of variable $\xi = x-x_0$. It is called the expansion in the power series about the point $x = x_0$.

**Analytic Function**

Real valued function $f(x)$ is called **analytic** at the point $x = x_0$ if it can be represented by the power series in some neighborhood of $x = x_0$:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

Real valued function $f(x)$ is called **analytic** on the open interval $I$ if it is analytic at each point $x_0 \in I$.

Power series will be an important tool for the convenient representation of the analytic functions. The truncated power series is used for evaluation of transcendental functions by computers and calculators.

Many solutions of the differential equations in engineering can be obtained only in the form of the power series. The functions defined by these power-series solutions are called the special functions. Among them are the Bessel functions, the spherical functions, and many others.
4. TAYLOR SERIES

The analytical functions which are represented by the power series are infinitely many times differentiable in its interval of convergence. They can be represented by the power series about any other point from its interval of convergence.

Theorem (Taylor’s Series)
Let \( f(x) \) be an analytic function defined by the power series
\[
f(x) = \sum_{k=0}^{\infty} a_k x^k
\]
which converges in the interval \((-R, R)\), (can be \( R = \infty \)).

If \( x_0 \in (-R, R) \), then the function \( f(x) \) can be expanded in the power series about the point \( x = x_0 \)
\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k
\]
which converges in the interval \((x_0 - |R - x_0|, x_0 + |R - x_0|)\).

This expansion is called the Taylor Series. If the point of expansion is \( x_0 = 0 \), then it is called the Maclaurin Series.

Now the definition of the functions, its evaluation and investigation of its properties becomes very simple, because the Taylor series consists just of simple arithmetic operations. What do we mean by that? Recall how the function \( f(x) = \sin x \) was defined: in the right triangle by the ratio \( \frac{b}{a} \). It is not very practical to draw the right triangle with one angle equal to \( x \) and to measure its sides each time when you need the value of \( \sin x \). Now we can define it by the Taylor series
\[
\sin x = x - \frac{x^3}{3!} + ...
\]
The only problem left is that in practice we can calculate only the finite number of terms in this expansion. The accuracy of the truncation of the Taylor series is estimated in the following theorem about the approximation of the functions by the Taylor polynomial. In this theorem, the function \( f(x) \) is not necessarily analytic, it is enough to be just \( n + 1 \) times differentiable.

Theorem (Taylor’s Polynomial)
Let \( f(x) \) be \( n + 1 \) times continuously differentiable in \((a, b)\), and let \( x_0 \in (a, b) \). Then for any \( x \in (a, b) \)
\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x)
\]
where
\[
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_{n+1}(x)
\]
where the remainder: 1) \( R_{n+1}(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(s)(x-s)^n \, ds \) or

2) (Lagrange formula) There exists a point \( c \) between \( x \) and \( x_0 \) such that

\[
R_{n+1}(x) = f^{(n+1)}(c) \frac{(x-a)^{n+1}}{(n+1)!}
\]

Then the maximum possible error can be estimated as

\[
|R_{n+1}(x)| \leq \left[ \max_{s \in (x,x_0)} f^{(n+1)}(s) \right] \frac{(x-x_0)^{n+1}}{(n+1)!}
\]

**Examples:**

1. **(Definition of the exponential function \( e^x \))**

   Defined the real-valued function \( e^x : \mathbb{R} \to \mathbb{R} \) by the power series:

   \[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}
\]

   a) Find the radius of convergence

   \[
   R = \lim_{k \to \infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k \to \infty} \frac{x^k}{k!} \frac{(k+1)!}{x^{k+1}} = \lim_{k \to \infty} \frac{(k+1)}{|x|} = \infty \quad \text{for any } x
   \]

   Therefore, the exponential function is defined for any \( x \in \mathbb{R} \)

   b) Find the derivative of the exponential function:

   \[
   \frac{d}{dx}(e^x) = \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right)' = \sum_{k=0}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n!} = e^x
   \]

   c) Derive \( e^{x+y} = e^x e^y \) (use Cauchy product)

2. **(Table of Taylor series expansions)\[
\begin{array}{ccc}
\text{Interval of convergence:} & & \\
\hline
1. & \frac{l}{1-x} & \sum_{k=0}^{\infty} x^k & -1 < x < 1 \\
2. & \frac{l}{1+x} & \sum_{k=0}^{\infty} (-1)^k x^k & -1 < x < 1 \\
3. & \frac{l}{x} & \sum_{k=0}^{\infty} (-1)^k (x-1)^k & 0 < x < 2 \\
4. & \sin x & \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} & -\infty < x < \infty \\
5. & \cos x & \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} & -\infty < x < \infty \\
6. & \ln(1+x) & \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} & -1 < x < 1 \\
7. & e^x & \sum_{k=0}^{\infty} \frac{x^k}{k!} & -\infty < x < \infty
\end{array}
\]
5. REVIEW QUESTIONS:

1. What is a sequence?
2. What is the difference between a set and a sequence?
3. What is the convergent sequence?
4. How the limit theorems for functions are related to sequences?
5. What remarkable limits of sequences do you recall?
6. What is infinite series?
7. In what sense can we say the sum of the infinite series?
8. What are partial sums?
9. What is convergent infinite series?
10. What is absolute convergence? Why it is important?
11. What are the properties of the convergent infinite series?
12. What convergence tests do you recall?
13. What tests for absolute convergence do you recall?
14. What remarkable infinite series do you recall?
15. What is a power series?
16. What is convergence of power series?
17. What is interval of convergence?
18. What are the properties of interval of convergence?
19. What is the radius of convergence? How it can be determined?
20. When a power series is absolutely convergent?
21. What operations can be performed with the power series and how?
22. What is a power series in terms of \( x - x_0 \) ?
23. What is an analytic function?
24. How an analytic function can be defined?
25. What is the Taylor series?
26. What is the Maclaurin series?
27. What functions can be expanded into Taylor series?
28. What is the Taylor polynomial?
29. What functions can be approximated by the Taylor polynomial?
30. What is a remainder? How it can be estimated?
31. What remarkable Taylor series do you recall?
EXERCISES:

1) Write the first six terms of the sequence:
   a) \( a_k = k3^k \)
   b) \( a_k = (-1)^k \frac{x^k}{k!} \)
   c) \( a_k = \frac{1}{k} + \frac{1}{k^2} \)
   d) \( a_k = \cos \frac{\pi}{2} \)

2) Find the limit of the following sequences:
   a) \( a_k = \frac{2k^2+k+2}{3k^2-5k+1} \)
   b) \( a_k = \frac{\ln(k^2)}{k} \)
   c) \( a_k = \cos \frac{\pi}{k} \)
   d) \( a_k = \frac{k!}{k} \)

3) Find (evaluate, to what number this series converges):
   a) \( \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \)
   b) \( \sum_{k=0}^{\infty} \frac{2^k}{3^k} \)

4) Investigate for convergence (absolute convergence) the following infinite series:
   a) \( \sum_{k=0}^{\infty} \frac{1}{k^3+1} \)
   b) \( \sum_{k=0}^{\infty} \frac{1}{4k+10} \)
   c) \( \sum_{k=0}^{\infty} \frac{k^2}{k^4+2k+1} \)
   d) \( \sum_{k=0}^{\infty} \frac{1}{\ln(k+1)} \)
   e) \( \sum_{k=0}^{\infty} \sin \left( \frac{\pi}{k} \right) \)
   f) \( \sum_{k=0}^{\infty} \frac{4}{k(\ln k)} \)
   g) \( \sum_{k=0}^{\infty} \frac{3^k}{k^3} \)
   h) \( \sum_{k=0}^{\infty} \frac{k}{2^k} \)
   i) \( \sum_{k=0}^{\infty} \frac{k^k}{k!} \)
   j) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \)
   k) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \)

5) Find the interval of convergence of the following power series:
   a) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{k} x^k \)
   b) \( \sum_{k=0}^{\infty} \frac{1}{k^2} x^k \)
   c) \( \sum_{k=0}^{\infty} \frac{k}{5^k} x^k \)
   d) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2} (x+3)^k \)

6) Find the Taylor series expansions about \( x_0 = 0 \) of the following functions:
   a) \( \frac{1}{(1-x)^2} \)
   b) \( \sqrt{x+1} \)
   c) \( \sin(x^2) \)
   d) \( \sec^2 x \)
6. SEQUENCES AND SERIES WITH MAPLE:

Sequences:

> restart;

finite sequence:

> s[n]:={seq(1/n,n=1..10)};

\[ s_n := \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\} \]

limit of the sequence:

> limit(n/(n+2),n=infinity);

1

Series (create partial sum and evaluate the limit of partial sums):

> s[n]:=sum(1/3^k,k=0..n);

\[ s_n := \sum_{k=0}^{n} \frac{1}{3^k} \]

> limit(s[n],n=infinity);

\[ \frac{3}{2} \]

Ratio Test:

> restart;
> a[k]:=k!/k^k;

\[ a_k := \frac{k!}{k^k} \]

> a[k+1]:=subs(k=k+1,a[k]);

\[ a_{k+1} := \frac{(k+1)!}{(k+1)^{(k+1)}} \]

> simplify(a[k+1]/a[k]);

\[ (k+1)^{(k-k)} k^k \]

> limit(a[k+1]/a[k],k=infinity);

\[ e^{(-1)} \]

Expansion in Power Series:

> restart;
> f:=sin(x);

\[ f := \sin(x) \]

> ps:=series(f,x=0,9);

\[ ps := x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + O(x^9) \]

> p(x):=convert(ps,polynom);

\[ p(x) := x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 \]

> plot({f,p(x)},x=-6..6,y=-4..4);
Expansion in Power Series:

\[ f(x) = e^{-x} \]

\[ ps := taylor(f, x=1, 3) \]

\[ p(x) := convert(ps, polynom) \]

\[ plot([f, p(x)], x=-2..4, y=-1..3) \]

Expansion in Taylor Series:

\[ f(x) = x \ln(x) \]

\[ ps := taylor(f, x=1, 3) \]

\[ p(x) := convert(ps, polynom) \]

\[ plot([f, p(x)], x=-2..4, y=-1..3) \]