Objectives: After the completion of this section the student should recall
- the concept of vector spaces
- the operations with vectors and matrices
- the methods of solution of linear systems of algebraic equations
- the methods of solution of the eigenvalue problem

Contents:
1. Vector Spaces
2. Linear Combination, Linear Independence, Basis
3. Vectors
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5. Linear Transformations with the Help of Matrices
6. Determinants
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1. VECTOR SPACES

A vector space \( V = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots\} \) is a set of objects called \textit{vectors}, for which two operations are defined: the operation of addition of vectors \( \mathbf{u} + \mathbf{v} \) and the operation of scalar multiplication \( k \mathbf{u} \) by the scalars from the field of real numbers \( k \in \mathbb{R} \), under the condition that they satisfy the special axioms. In the printed text, vectors are denoted by bold letters \( \mathbf{u}, \mathbf{v}, \ldots \) and in handwriting they are denoted by arrows above letters \( \vec{u}, \vec{v}, \ldots \). Scalars are denoted by lower case letters \( a, b, \ldots \).

Examples of vector spaces include the Euclidian space \( \mathbf{x} \in \mathbb{R}^n \) (column vectors of dimension \( n \)), the space of all \( m \times n \) matrices \( M_{m \times n} \), the space of all continuous functions \( f(x) \in C[a,b] \) defined on the interval \( [a,b] \), the space of all functions \( f(x) \in C^1(a,b) \) continuously differentiable on the interval \( (a,b) \).

The notion of vector spaces helps to obtain and to study the general properties of solutions of algebraic and differential equations and systems of equations.

The formal definitions of a vector space and a vector subspace are the following:

\begin{center}
\textbf{Definition 1}
\end{center}

A \textbf{linear vector space} \( V \) over the field of real numbers \( \mathbb{R} \) is a set of elements (called \textit{vectors}) with two operations:

\begin{enumerate}
\item[i)] addition of vectors:
\[ \text{if } \mathbf{u}, \mathbf{v} \in V, \text{ then } \mathbf{u} + \mathbf{v} \in V \quad \text{closure for sum} \]
\item[ii)] multiplication by a scalar:
\[ \text{if } \mathbf{u} \in V \text{ and } k \in \mathbb{R}, \text{ then } k \mathbf{u} \in V \quad \text{closure for scaling} \]
\end{enumerate}

which satisfy the following axioms:

\begin{enumerate}
\item[1)] if \( \mathbf{u}, \mathbf{v} \in V \), then \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \) \quad \text{commutative law}
\item[2)] if \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \), then \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \) \quad \text{associative law}
\item[3)] there exists a vector \( \mathbf{0} \in V \), such that \( \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \) for any \( \mathbf{u} \in V \) \quad \text{existence of zero vector}
\item[4)] for any \( \mathbf{u} \in V \) there exists a vector \( -\mathbf{u} \in V \) such that \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \) \quad \text{existence of negative vector}
\item[5)] if \( \mathbf{u} \in V \) and \( a, b \in \mathbb{R} \), then \( a(b\mathbf{u}) = (ab)\mathbf{u} \) \quad \text{associative law}
\item[6)] if \( \mathbf{u}, \mathbf{v} \in V \) and \( k \in \mathbb{R} \), then \( k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v} \) \quad \text{distributive law}
\item[7)] if \( \mathbf{u} \in V \) and \( a, b \in \mathbb{R} \), then \( (a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \) \quad \text{distributive law}
\item[8)] if \( \mathbf{u} \in V \), then \( 1 \mathbf{u} = \mathbf{u} \)
\end{enumerate}

\begin{center}
\textbf{Definition 2}
\end{center}

Let \( V \) be a vector space, then any subset \( U \subset V \) which is a vector space itself with the operations induced by \( V \) is called to be a \textbf{subspace} of \( V \).
1) Conditions i) and ii) are the closure axioms. They mean that the sum of any two vectors from the space is a vector which also belongs to this space, and any vector multiplied by any scalar is a vector which also belongs to the space. Therefore, the defined operations can always be performed without going outside of the vector space, because vector space $V$ contains all required vectors.

2) Axioms 1-8) are the vector axioms; the set of objects which satisfies these axioms is called the vector set, and its elements are called vectors.

3) We defined a vector space over the field of real numbers $\mathbb{R}$, therefore it can be referred to as a real vector space. Similarly, a complex vector space over the field of complex numbers $\mathbb{C}$ also can be defined.

4) The zero vector $\mathbf{0} \in V$ is unique. It means that if there exists a vector $\mathbf{0} \in V$ with the same property $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{u}$ then $\mathbf{0} = \mathbf{0}$

5) To show that some set $V$ is a vector space, we need to demonstrate that all ten axioms i-ii) and 1-8) are satisfied.

6) To show that some subset $U \subseteq V$ of the vector space $V$ is a subspace, we need to check only the closure axioms i) and ii), because elements of the subset of the vector space are already the vectors.

To prove the opposite, that $U \subseteq V$ is not a subspace, very often it is sufficient to show that the zero vector $\mathbf{0} \in V$ does not belong to the subset $\mathbf{0} \notin U$, and, therefore, it cannot be a subspace.

7) The negative element $-\mathbf{u} \in V$ is unique. It means that if there exists a vector $-\mathbf{w} \in V$ with the same property $\mathbf{u} + (-\mathbf{w}) = \mathbf{0}$ then $-\mathbf{u} = -\mathbf{w}$

8) $\theta \mathbf{u} = \mathbf{0}$ for any $\mathbf{u} \in V$

9) $(\theta)\mathbf{u} = -\mathbf{u}$ for any $\mathbf{u} \in V$

10) $a\mathbf{0} = \mathbf{0}$ for any $a \in \mathbb{R}$

11) $a\mathbf{u} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{u} = \mathbf{0}$

12) Vector space can be defined over an arbitrary field $\mathbb{K}$ (scalars) for which operation of multiplication of a vector by a scalar is defined:

$$\text{vector } k\mathbf{u} \text{ is defined for all } \mathbf{u} \in V \text{ and } k \in \mathbb{K}$$

If $\mathbb{K}$ is $\mathbb{R}$, then $V$ is referred to as a real vector space.
If $\mathbb{K}$ is $\mathbb{C}$, then $V$ is referred to as a complex vector space.
EXAMPLES OF VECTOR SPACES

Space of real numbers $\mathbb{R}$

1. The field of all real numbers $\mathbb{R}$ with usual algebraic operations of addition of real numbers and multiplication by a real number is a vector space. The field axioms (see Chapter I) and algebraic rules provide satisfaction of the vector space axioms. The vector space $\mathbb{R}$ is a particular case of a more general $n$-dimensional Euclidian vector space $\mathbb{R}^n$.

Euclidean space $\mathbb{R}^n$

2. The elements of the Euclidian vector space $\mathbb{R}^n$ are $n$-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \quad x_i \in \mathbb{R}$$

which can be treated as the coordinates of the points in $n$-dimensional geometrical space. The operations are defined in the following way:

$$\mathbf{x} + \mathbf{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$k\mathbf{x} = (kx_1, kx_2, \ldots, kx_n)$$

The zero vector is

$$\mathbf{0} = (0, 0, \ldots, 0)$$

and the negative vector is

$$-\mathbf{x} = (-1)\mathbf{x} = (-x_1, -x_2, \ldots, -x_n)$$

It is easy to verify the satisfaction of the vector space axioms for $\mathbb{R}^n$.

Euclidean space $\mathbb{R}^3$

is a particular case of $\mathbb{R}^n$:

$$\mathbf{x} = (x_1, x_2, x_3) \quad x_i \in \mathbb{R}$$

It is a physical 3-dimentional Euclidian space which is used in mathematical modeling physical and engineering problems.
Column vectors

The vector space \( \mathbb{R}^n \) is the set of all column vectors

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

for which operations are defined in the similar way:

\[
x + y = \begin{bmatrix}
x_1 + y_1 \\
x_2 + y_2 \\
\vdots \\
x_n + y_n
\end{bmatrix}, \quad kx = \begin{bmatrix}
kx_1 \\
kx_2 \\
\vdots \\
kx_n
\end{bmatrix}
\]

The column vectors are just a different form of writing the real \( n \)-tuples. In their turn, the column vectors are a particular case of the \( m \times n \) real matrices.

Space of \( m \times n \) matrices \( \mathbb{R}_{mn} \)

3. The set of all real matrices with \( m \) rows and \( n \) columns

\[
A_{mn} = (a_{ij})_{i=1,...,m}^{j=1,...,n} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \quad a_{ij} \in \mathbb{R}
\]

with operations

\[
A + B = \begin{pmatrix}
a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\
a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
\end{pmatrix}
\]

\[
kA = \begin{pmatrix}
ka_{11} & ka_{12} & \cdots & ka_{1n} \\
ka_{21} & ka_{22} & \cdots & ka_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
ka_{m1} & ka_{m2} & \cdots & ka_{mn}
\end{pmatrix}
\]

is a vector space with zero vector

\[
0_{mn} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]
Space of continuous functions  $C[a,b]$

4. The set of all real-valued *continuous* functions

$$C[a,b] = \{ f(x) : [a,b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a,b] \}$$

The operations addition of vectors and multiplication by a scalar are defined as the point-wise operations with functions

$$f + g = f(x) + g(x) \quad x \in [a,b] \quad \text{for} \quad f, g \in C[a,b]$$

$$kf = kf(x) \quad x \in [a,b] \quad \text{for} \quad f \in C[a,b], \ k \in \mathbb{R}$$

Obviously, a sum of two continuous functions on $[a,b]$ is a continuous function on $[a,b]$, and a continuous function multiplied by a scalar is still continuous. Therefore, the closure axioms i-ii) of a vector space are satisfied.

The zero vector $0 \in C[a,b]$ is a function identically equal to zero on the interval $[a,b]$.

Other vector spaces of continuous functions are:

$$C(a,b), C(-\infty, \infty)$$

Space of differentiable functions  $C^1(a,b)$

5. The set of all real valued functions continuously *differentiable* on the open interval $(a,b)$

$$C^1(a,b) = \{ f(x) : (a,b) \rightarrow \mathbb{R} \mid f' \text{ is continuous on } (a,b) \}$$

with operations identical to operations in the space $C[a,b]$.

Space of polynomials  $P_n(x)$

6. The set of all *polynomials* with degree not greater than $n$:

$$P_n(x) = \left\{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \right\}$$

is a vector space with operations identical to operations with functions.

Zero subspace

7. The simplest but important example of a vector subspace of any space $V$ over the field of real numbers is its subspace consisting of one element $\{0\} \subset V$, where vector $0 \in V$ is a zero vector of the vector space $V$:

$$0_v = \{0\}$$

It also is called the *null-subspace* of $V$. 

$$V$$

$$\{0\}$$
8. Consider the $1^{st}$ order linear ordinary differential equation

$$y' - y = 0$$

The set of all solutions of this equation (provided that they exist) is a subset of the vector space $C^1(\mathbb{R})$ (The set of all solutions is not empty – because it obviously possess the trivial solution $y(x) = 0$):

$$V = \left\{ y(x) \in C^1(\mathbb{R}) \mid y' - y = 0 \right\} \subset C^1(\mathbb{R})$$

We can verify that the set $V$ is a vector space. For this, according to comment (6), we have to verify only the conditions i-ii) of the Definition 1:

i) Assume that $y_1, y_2 \in V$ are solutions, then show that the sum of them $y_1 + y_2$ is also a solution. Indeed,

$$(y_1 + y_2)' - (y_1 + y_2) = y_1' - y_1 + y_2' - y_2 = 0$$

ii) Assume that $y_1 \in V$ is a solution, then show that the scalar multiple of it $ky_1, k \in \mathbb{R}$ is also a solution. Indeed,

$$(ky_1)' - ky_1 = ky_1' - ky_1 = k(y_1' - y_1) = k \cdot 0 = 0$$

Therefore the set of all solutions $V \subset C^1(\mathbb{R})$ is a vector space.

We investigated the structure and the properties of the set of all solutions of the differential equation and showed that they constitute the vector space even without solving the equation. In fact, the general solution of this equation is given by

$$y(x) = ce^x, \ c \in \mathbb{R}$$

They are infinitely many times differentiable functions, and therefore, they belong to the space $y(x) \in C^\infty(\mathbb{R}) \subset C^1(\mathbb{R})$.

**Intersection of vector spaces**

Let $U_i \subset V$ be the subspaces of the vector space $V$, then

$$\bigcap_{i=1}^n U_i = \left\{ u \in V \mid u \in U_i, \ i=1,...,n \right\}$$

is a subspace of $V$ (Exercise 2, p.201).
2. LINEAR COMBINATION, LINEAR INDEPENDENCE, BASIS

We need a convenient way of constructing and describing vector spaces. In the plane any point can be uniquely defined by a pair of its coordinates \((x, y)\). In vector spaces the role of coordinates is played by the coefficients in the linear combination representation of the vectors with the help of the basis vectors. For example, the set of monoms \(\{1, x, x^2\}\) is a basis of the vector space of all quadratic polynomials \(P_2(x)\) where each polynomial is represented uniquely as a linear combination of basis vectors \(a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2\) with coefficients \(a_0, a_1, a_2 \in \mathbb{R}\). Therefore, the polynomials are defined just by these coefficients.

Let \(V\) be a vector space. Recall some definitions and facts:

**Linear Combination**

Let \(u_1, u_2, \ldots, u_n \in V\). Then a vector \(c_1u_1 + c_2u_2 + \ldots + c_nu_n\) is called a **linear combination** of vectors \(u_1, u_2, \ldots, u_n\) with coefficients \(c_1, c_2, \ldots, c_n \in \mathbb{R}\).

It is obvious that any linear combination of vectors from \(V\) also is a vector of \(V\):

\[
c_1u_1 + c_2u_2 + \ldots + c_nu_n \in V
\]

**Span**

The set of all linear combinations of vectors \(u_1, u_2, \ldots, u_n \in V\) is called a **span** of vectors \(u_1, u_2, \ldots, u_n\):

\[
\text{span}\{u_1, u_2, \ldots, u_n\} \equiv \{c_1u_1 + c_2u_2 + \ldots + c_nu_n \mid c_i \in \mathbb{R}\}
\]

In this definition coefficients \(c_i\) attain all real values generating infinitely many vectors. For example, in the Euclidian space \(\mathbb{R}^3\) the span

\[
\text{span}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ c \\ 0 \end{bmatrix} \quad c \in \mathbb{R}
\]

generates a line in the \(xy\)-plane.

And the span

\[
\text{span}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ c_1 \\ c_2 \end{bmatrix} \quad c_1, c_2 \in \mathbb{R}
\]

generates the \(xy\)-plane.

The \(\text{span}\{\sin(\lambda x), \cos(\lambda x)\}\) of two vectors \(\sin(\lambda x), \cos(\lambda x) \in C^\infty(\mathbb{R})\) generates the general solution of the linear ODE \(y'' + \lambda^2 y = 0\), \(\lambda \in \mathbb{R}\).

The \(\text{span}\{e^{-mx}, e^{mx}\}\) of two vectors \(e^{-mx}, e^{mx} \in C^\infty(\mathbb{R})\) generates the general solution of the linear ODE \(y'' - m^2 y = 0\), \(m \in \mathbb{R}\). The same general solution also can be generated by the \(\text{span}\{\sinh(mx), \cosh(mx)\}\).

Because each of these linear combinations is a vector of space \(V\), the span is a subset of vector space \(V\):

\[
\text{span}\{u_1, u_2, \ldots, u_n\} \subset V
\]

We can verify that the span is a vector space itself, therefore \(\text{span}\{u_1, u_2, \ldots, u_n\}\) is a subspace of \(V\).
The question arises, if it is possible to specify some set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) such that their span generates the whole vector space \( V \)? And if yes, then what is the minimal set of vectors with this property? Then such a set can be used as a basis for vector space \( V \).

**Linear Independence**

The set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) is called *linearly independent* if their linear combination is equal to the zero vector with only all zero coefficients:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0} \quad \Rightarrow \quad \text{all } c_i = 0
\]

If a set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) is not linearly independent, then it is called *linearly dependent*. Formally, it can be stated as follows:

The set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) is called *linearly dependent* if there exist coefficients \( c_1, c_2, \ldots, c_n \in \mathbb{R} \) not all equal to zero such that the linear combination with them is equal to the zero vector:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0}
\]

Consider the following useful facts about linear independence of vectors:

1. If in the set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) one of vectors \( \mathbf{v}_k \neq \mathbf{0} \) can be represented as a linear combination of the other vectors

\[
\mathbf{v}_k = c_1 \mathbf{v}_1 + \cdots + c_{k-1} \mathbf{v}_{k-1} + c_{k+1} \mathbf{v}_{k+1} + \cdots + c_n \mathbf{v}_n
\]

then the set \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V \) is linearly dependent.

Indeed, rewrite the equation as

\[
c_1 \mathbf{v}_1 + \cdots + c_{k-1} \mathbf{v}_{k-1} + \cdots + c_n \mathbf{v}_n = \mathbf{0}
\]

where at least one coefficient is non-zero.

2. If \( \mathbf{0} \in \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) then the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is linearly dependent.

**Linear Independence in \( \mathbb{R}^m \)**

Let \( \mathbf{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \mathbf{u}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m \)

Then the set \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) is linearly dependent if there exist coefficients \( c_1, c_2, \ldots, c_n \in \mathbb{R} \) not all equal to zero such that

\[
c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n = \mathbf{0}
\]

In the component form this vector equation can be written as

\[
c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

It means that the homogeneous system of algebraic equations for \( c_1, c_2, \ldots, c_n \)

\[
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

or \( Ac = 0 \) has non-trivial solutions.

If the number of vectors \( n > m \) then the set \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) is linearly dependent.

If \( n = m \) then the set \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) is linearly independent only if \( \det A = 0 \).
Linear independence of functions

Basis of vector space \( V \)

Let \( V \) be a vector space. Then the set of vectors \( u_1, u_2, \ldots, u_n \in V \) is called a basis for the vector space \( V \) if

i) the set \( u_1, u_2, \ldots, u_n \) is linearly independent

ii) \( \text{span}\{u_1, u_2, \ldots, u_n\} = V \)

Any vector \( v \in V \) is uniquely represented as a linear combination of the basis vectors:

\[ v = c_1u_1 + c_2u_2 + \ldots + c_nu_n \]

That means that the span of \( u_1, u_2, \ldots, u_n \) generates the space \( V \).

**Theorem 1**

Every vector space possesses a basis.

A basis for the vector space is not unique, in general there can be many different bases, but the number of vectors in the different bases for the vector space is the same:

if the set \( u_1, u_2, \ldots, u_n \) is a basis for the vector space \( V \), then any other basis for \( V \) also consists of \( n \) vectors.

The number of vectors in the basis is called the dimension of the vector space \( V \).

Not all vector spaces have a finite dimension. For example, the space of continuous functions \( C[a,b] \) has an infinite dimension.

**Examples:**

1. The set of column vectors

\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

is a standard basis for \( \mathbb{R}^n \).

In particular, the standard basis for \( \mathbb{R}^3 \) is denoted by

\[ i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

2. The set of monoms \( \{1, x, x^2, \ldots, x^n\} \) is a standard basis for the vector space of polynomials of order \( n \) \( P_n(x) \):

\[ P_n(x) = \{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, \quad a_i \in \mathbb{R}\} \]
Functions defined on vector spaces

Let \( U \) and \( V \) be two vector spaces. Then the function \( f : U \rightarrow V \) defined between these vector spaces is called a **linear map** or a **homomorphism** if for all \( u, w \in U \) and \( c \in \mathbb{R} \) it satisfies the following conditions:

i) \( f(u + w) = f(u) + f(w) \)

ii) \( f(cu) = cf(u) \)

For example, a linear map can be defined by a matrix as a linear transformation of Euclidean vector space:

Let \( A \in \mathbb{R}^{m \times n} \) be a \( m \times n \) real matrix, then the transformation defined by

\[ v = Au \quad \text{for all} \quad u \in \mathbb{R}^n \]

is a linear map from \( \mathbb{R}^n \) to the vector space \( \mathbb{R}^m \). This linear map includes particular transformations such as dilation and rotation of the vector space.

A linear map in the vector spaces of functions can be defined by the familiar operations such as definite and indefinite integration and differentiation.

Let \( f : U \rightarrow V \) be a linear map (homomorphism). Then

**Image (Range)**

\( \text{im}(f) = \{ f(u) | u \in U \} \) is called the **image** (or **range**) of \( f \).

It can be shown that \( \text{im}(f) \subset V \) is a subspace of \( V \).

**Rank**

\( \text{rank}(f) = \dim(\text{im}(f)) \) is called the **rank** of \( f \).

Rank is a dimension of the image of \( f \).

**Kernel**

\( \ker(f) = \{ u \in U | f(u) = 0 \} \) is called the **kernel** of \( f \).

Kernel is a subset of all vectors in space \( U \) which are mapped into the zero vector of the space \( V \). It can be shown that \( \ker(f) \subset U \) is a subspace of \( U \).

**Theorem 2**

\[ \text{rank}(f) = \dim U - \dim \ker(f) \]

The linear maps help to obtain and to study the properties of solutions of algebraic and differential equations.
3. VECTORS

Recall the definition of the column-vectors in $\mathbb{R}^n$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$x_i \in \mathbb{R}$

for which operations of vector addition and multiplication by a scalar are defined by the operations with their components:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix}$$

$$k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}$$

The set of all column-vectors $\mathbb{R}^n$ is a vector space of dimension $n$ with the standard basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

4. MATRICES

Recall the vector space of all real matrices with $m$ rows and $n$ columns $\mathbb{R}_{mn}$

$$\mathbf{A}_{m,n} = (a_{ij})_{i=1,\ldots,m; j=1,\ldots,n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} \in \mathbb{R}$

The set of elements with equal indices $\{a_{kk}\}$ is called the main diagonal. If the number of rows is equal to the number of columns, $m = n$, then the matrix is called a square matrix.

A matrix can be interpreted as a combination of its column-vectors:

$$\mathbf{A}_{mn} = (\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n)$$

with column-vectors $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $j = 1,2,\ldots,n$

or their row vectors:

$$\mathbf{A}_{mn} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix}$$

with row-vectors $\mathbf{A}_i = (a_{i1} \ a_{i2} \ \cdots \ a_{in})$ $i = 1,2,\ldots,m$

Column-vectors $\mathbf{x} \in \mathbb{R}_n$ can be treated as the $n \times l$ matrices.

Equality $\mathbf{A} = \mathbf{B}$

Equality of matrices: two matrices of the same size $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{mn}$ are equal if and only if all the corresponding matrix elements are equal:

$$\mathbf{A} = \mathbf{B} \quad \text{if} \quad (a_{ij})_{i=1,\ldots,m; j=1,\ldots,n} = (b_{ij})_{i=1,\ldots,m; j=1,\ldots,n}$$
Multiplication by a scalar $kA$

Let $A \in \mathbb{R}^{mn}$, $k \in \mathbb{R}$, then

$$kA = \begin{pmatrix}
ka_{11} & ka_{12} & \cdots & ka_{1n} \\
ka_{21} & ka_{22} & \cdots & ka_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
ka_{m1} & ka_{m2} & \cdots & ka_{mn}
\end{pmatrix}$$

Addition $A + B$

Addition of two matrices of the same size $A, B \in \mathbb{R}^{mn} \rightarrow A + B \in \mathbb{R}^{mn}$

$$A + B = \begin{pmatrix}
a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\
a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
\end{pmatrix}$$

Transpose $A^T$

Transpose of a matrix $A \in \mathbb{R}^{mn} \rightarrow A^T \in \mathbb{R}^{nm}$

$$A^T = (a_{ij})_{i=1,...,m}^{j=1,...,n} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} = (a_{ji})_{j=1,...,n}^{i=1,...,m} = \begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{pmatrix}$$

This operation interchanges the components $a_{ij} \leftrightarrow a_{ji}$.

Properties of transpose:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(A^{-1})^T = (A^T)^{-1}$ for invertible matrices

Symmetric Matrix

The square matrix $A$ is called symmetric if $A^T = A$.
The square matrix $A$ is called skew-symmetric if $A^T = -A$.
Every square matrix can be separated into a sum of a symmetric matrix and a skew-symmetric matrix:

$$A = A_s + A_a = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Matrix multiplication $AB$

Matrix Multiplication: the product of two matrices $AB$ is defined if the number of the columns of the matrix $A$ is equal to the number of rows of the matrix $B$:

For $A \in \mathbb{R}^{mn}, B \in \mathbb{R}^{np} \rightarrow AB \in \mathbb{R}^{mp}$

Let $A = A_{mn} = (a_{ij})_{i=1,...,m}^{j=1,...,n} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}$

Let $B = B_{np} = (b_{ij})_{i=1,...,m}^{j=1,...,n} = \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1p} \\
b_{21} & b_{22} & \cdots & b_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mp}
\end{pmatrix}$

$$AB = (ab_{ij})_{i=1,...,m}^{j=1,...,p} = \begin{pmatrix}
a_1b_1 + a_2b_2 + \cdots + a_nb_n \\
a_1b_{11} + a_2b_{12} + \cdots + a_nb_{1p} \\
\vdots & \vdots & \ddots & \vdots \\
a_1b_{11} + a_2b_{12} + \cdots + a_nb_{1p}
\end{pmatrix}$$
The product $AB$ yields an $m \times p$ matrix which can be defined in one of the following ways:

$$AB = \begin{pmatrix}
\sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp} \\
\sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k}b_{kp} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp}
\end{pmatrix}$$

$$(AB)_{ij} = a_{i1}b_{j1} + a_{i2}b_{j2} + \cdots + a_{in}b_{jn}$$

The element in the intersection of the $i^{th}$ row and the $j^{th}$ column $(AB)_{ij}$ of the matrix $AB$ is obtained by a scalar product of the $i^{th}$ row of the matrix $A$ with the $j^{th}$ column of the matrix $B$.

Properties of matrix multiplication:

Let $A \in \mathbb{R}_{mn}$, $B \in \mathbb{R}_{np}$, $B_1 \in \mathbb{R}_{np}$, $B_2 \in \mathbb{R}_{np}$, $C \in \mathbb{R}_{pr}$, $m,n,p,r \in \mathbb{N}$

1. $cAB = A(cB) = c(AB)$
2. $(AB)C = A(BC)$ \hspace{1cm} associative law
3. $A(B_1 + B_2) = AB_1 + AB_2$ \hspace{1cm} distributive law
4. $(B_1 + B_2)C = B_1C + B_2C$ \hspace{1cm} distributive law
5. $AB = 0 \iff A = 0$ or $B = 0$
6. $AB_1 = AB_2 \iff B_1 = B_2$
7. $(AB)^T = B^TA^T$ \hspace{1cm} transpose of a product
8. $A0 = 0$ \hspace{1cm} zero matrix
9. $AI = A$ \hspace{1cm} unit matrix
10. $AB \neq BA$ \hspace{1cm} do not commute
Example:

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{pmatrix}
= 
\begin{pmatrix}
11 & 14 & 17 & 20 \\
23 & 30 & 37 & 44 \\
35 & 46 & 57 & 68
\end{pmatrix}
\]

**Particular case:**
product of a matrix and a vector \( A_{mxn} x_{nxl} = b_{nxl} \)

**System of algebraic equations:**
System of \( m \) algebraic equations for \( n \) unknowns \( x_1, x_2, ..., x_n \):

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix}
\]

\[
a_{ij}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\]

Let \( x, y \in \mathbb{R}^n \) be vectors. Two types of vector product can be defined

**Inner product** of column-vectors (dot product) is defined as

\[
x \cdot y \equiv x^T y = 
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
= x_1y_1 + x_2y_2 + \cdots + x_ny_n
\]

The result is a number, \( x \cdot y \in \mathbb{R} \).

**Outer-product** (dyadic product) of column-vectors is defined as

\[
x y^T = 
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\begin{bmatrix}
y_1 & y_2 & \cdots & y_n
\end{bmatrix}
= 
\begin{bmatrix}
x_1y_1 & x_1y_2 & \cdots & x_1y_n \\
x_2y_1 & x_2y_2 & \cdots & x_2y_n \\
\vdots & \vdots & \ddots & \vdots \\
x_ny_1 & x_ny_2 & \cdots & x_ny_n
\end{bmatrix}
\]

The result is a matrix, \( xy^T \in \mathbb{R}_{mn} \).
5. LINEAR TRANSFORMATIONS WITH THE HELP OF MATRICES

Let \( A_{mn} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^{mn} \) where \( a_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{mi} \end{pmatrix} \in \mathbb{R}^m \), \( i = 1, 2, \ldots, n \).

Matrix multiplication of vectors generates a linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^m : x \rightarrow Ax \).

A vector \( x \in \mathbb{R}^n \) is transformed to a vector \( Ax \in \mathbb{R}^m \).

The resulting vector can be written as

\[
Ax = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} x_i a_i \in \mathbb{R}^m, \quad x_i \in \mathbb{R}
\]

Therefore, the image of a linear map is a set of all linear combinations of the columns of the matrix \( A \) which by the definition is a span. Therefore, the image is a subspace of the vector space \( \mathbb{R}^m \); let us call it the column space of the matrix \( A \) and denote it in the following way:

\[
\text{column space} \quad \text{col} A = \text{span}\{a_1, a_2, \ldots, a_n\}, \quad \text{col} A \subset \mathbb{R}^m
\]

**rank**

\[
\text{rank} A = \text{dim} \text{col} A
\]

Recall the definition of the kernel of a linear map denoted by

\[
\ker A = \left\{ x \in \mathbb{R}^n \mid Ax = 0 \right\}, \quad \ker A \subset \mathbb{R}^n
\]

Therefore, the kernel of a linear transformation defined with the help of the matrix \( A \) is a set of all solutions of a homogeneous system of linear algebraic equations \( Ax = 0 \).

The column space of the matrix \( A \) is a set of all vectors \( b \in \mathbb{R}^m \) which are images of some vectors \( x \in \mathbb{R}^n \), i.e., for which there exists a solution of the system of non-homogeneous system of linear algebraic equations \( Ax = b \).
Theorem 3

\[ \dim(\ker A) + \rank A = n \quad \text{and} \quad \rank A = n - \dim(\ker A) < m \]

Consider transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) given by \( A \mathbf{x} \in \mathbb{R}^m \).

This important theorem of linear algebra allows one to make some conclusions about the solutions of the system of linear algebraic equations:

If \( \mathbf{b} \in \text{col} A \) then there exists \( \mathbf{x} \in \mathbb{R}^n \) such that \( A\mathbf{x} = \mathbf{b} \).

If \( \mathbf{c} \notin \text{col} A \) then there is no \( \mathbf{x} \in \mathbb{R}^n \) such that \( A\mathbf{x} = \mathbf{c} \).

If the homogeneous system \( A\mathbf{x} = 0 \) has only the trivial solution \( \mathbf{x} = 0 \), then the \( \ker A = \{0\} \) and \( \dim(\ker A) = 0 \) and therefore \( \rank A = n \).

Then for any \( \mathbf{b} \in \text{col} A \) there exists the unique \( \mathbf{x} \in \mathbb{R}^n \) such that \( A\mathbf{x} = \mathbf{b} \).

If \( A \) is a square matrix \( (m=n) \), then it means that \( \text{col} A = \mathbb{R}^m \) is the whole space \( \mathbb{R}^m \), and therefore, for any \( \mathbf{b} \in \mathbb{R}^m \) there exists \( \mathbf{x} \in \mathbb{R}^n \) such that \( A\mathbf{x} = \mathbf{b} \).

If columns of \( A \) are linearly independent then the dimension of the vector space spanned by \( n \) linearly independent vectors is \( n \), \( \rank A = n \) and \( \dim(\ker A) = 0 \).

Therefore, there is only the trivial solution for \( A\mathbf{x} = 0 \).
6. DETERMINANTS

Determinants are defined only for square matrices $A \in \mathbb{R}^{n \times n}$ ($n \times n$ matrices). The determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is a number calculated according to special rules. The determinant is denoted in one of the following ways:

$$\det A$$

The determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is a number calculated according to special rules. The determinant is denoted in one of the following ways:

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

**Definition 1**

One possible definition of the $\det A$ is based only on its properties (it does not show actually how to calculate the determinant):

**Determinant** of the square matrix $A \in \mathbb{R}^{n \times n}$ is defined as a function $\det A : \mathbb{R}^{n \times n} \to \mathbb{R}$ which satisfies the following conditions:

1. the determinant is linear in every row

$$\det \begin{pmatrix} A_i + B_j \\ \vdots \end{pmatrix} = \det A_i + \det B_j$$

for every $i = 1, 2, \ldots, n$

$$\det cA = c \det A$$

where the dots indicate unchanged rows.

2. If matrix $A$ contains two equal rows $A_i = A_j$ then $\det A = 0$.

3. Determinant of the **unit matrix**

$$\det I = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

**Definition 2**

The other definition demonstrates the structure of the determinant and provides the method of its calculation:

**Determinant** of the square matrix $A \in \mathbb{R}^{n \times n}$ is a number equal to the sum of all possible products of $n$ matrix entries such that every term contains exactly one entry of each row and each column of the matrix weighted by 1 or $-1$ in the following way:

$$A \in \mathbb{R}^{2 \times 2} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A \in \mathbb{R}^{3 \times 3} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
Where the minor $A_{ij}$ is defined as a matrix obtained from the matrix $A$ by deleting the $i^{th}$ row and the $j^{th}$ column

the cofactor $C_{ij} = (-1)^{i+j} |A_{ij}| = (-1)^{i+j} \det A_{ij}$

Then the definition of the determinant of the matrix $A \in \mathbb{R}^{n \times n}$ can be formalized in the following expansion over the $1^{st}$ row form:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

**Theorem 4 (the cofactor expansion theorem)**

The determinant can be calculated by the cofactor expansion over any row:

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{ik} = \sum_{k=1}^{n} a_{ik} C_{ik} \quad i = 1, 2, ..., n$$

or any column:

$$\det A = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} A_{kj} = \sum_{k=1}^{n} a_{kj} C_{kj} \quad j = 1, 2, ..., n$$

In practice, for calculation of the determinants the cofactor expansion method is used only for small matrices ($2 \times 2$ or $3 \times 3$) or matrices of special form (diagonal, triangle, sparse, etc).

**Theorem 5 (the determinant of the triangle matrices)**

The determinant of the upper-triangle, the low-triangle and the diagonal matrices are equal to the product of the elements on the main diagonal:

$$\begin{vmatrix} d_{11} & * & \cdots & * \\ 0 & d_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{vmatrix} = \begin{vmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{vmatrix} = d_{11} d_{22} \cdots d_{nn}$$

**Theorem 6 (properties of the determinant)**

Let $A, B \in \mathbb{R}^{n \times n}$, then

1. $\det(cA) = c^n \det A$
2. $\det(AB) = \det A \det B$
3. $\det(A^T) = \det A$
4. $\det A = 0 \Rightarrow \text{rank} A < n$
5. $\det(A^{-1}) = \frac{1}{\det A}$ for invertible $A$
6. If two rows of $A$ are the same, then $\det A = 0$
7. If $A$ has a row(or a column) all of zeroes, then $\det A = 0$
7. MATRIX INVERSE

**Definition** Let matrix $A \in \mathbb{R}^{n \times n}$ be a **square** matrix. If there exists a matrix $C \in \mathbb{R}^{n \times n}$ such that
\[
CA = I \quad \text{left inverse}
\]
\[
AC = I \quad \text{right inverse}
\]
then the matrix $A$ is said to be **invertible**, and the matrix $C$ is called an **inverse** of $A$. It is denoted by $A^{-1}$:

**Inverse matrix** $A^{-1}$

$$A^{-1}A = AA^{-1} = I$$

**Theorem 7** If the inverse of $A \in \mathbb{R}^{n \times n}$ exists then it is **unique**

**Theorem 8** Let $A, B \in \mathbb{R}^{n \times n}$ be invertible matrices, then

a) $(A^{-1})^{-1} = A$ (A^{-1} is also invertible)

b) $(AB)^{-1} = B^{-1}A^{-1}$

c) $(A^T)^{-1} = (A^{-1})^T$

**Theorem 9** If $A \in \mathbb{R}^{n \times n}$ is invertible then $A$ is row equivalent to $I$

**Theorem 10** If $A \in \mathbb{R}^{n \times n}$ is invertible then $\det A \neq 0$

**Theorem 11** If $A \in \mathbb{R}^{n \times n}$ is not invertible then $\det A = 0$

**singular matrix** Matrix with zero determinant $\det A = 0$ is called **singular**.

**Theorem 12** If $A \in \mathbb{R}^{n \times n}$ is invertible then for any $b \in \mathbb{R}^{n}$ the linear system $Ax = b$ has a unique solution $x = A^{-1}b$

**Proof:**
a) $x = A^{-1}b$ is a solution: b) Let $u \in \mathbb{R}^n$ be another solution

$$A(A^{-1}b) = b \quad \text{Au} = b$$

$$\left(AA^{-1}\right)b = b \quad A^{-1}(Au) = A^{-1}b$$

$$Ib = b \quad (A^{-1}A)u = x$$

$$b = b \quad u = x$$

**Inversion by row reduction:** The algorithm for finding $A$ (for row reduction procedure see below):

1) start with augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$

2) by Gaussian elimination reduce to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$

**Inverse of $2 \times 2$ matrix:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**Cofactor formula for inverse:** Define the adjoint of matrix $A \in \mathbb{R}^{n \times n}$ as the matrix which entries are cofactors:

$$\text{adj} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}, \quad C_{ij} = (-1)^{i+j} \det A_{ij}$$

then the inverse of matrix $A$ is defined by

$$A^{-1} = \frac{\text{adj} A}{\det A}$$
**Example:** Let \( \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \)

1) **Row reduction (inversion of a unit matrix):**

\[
\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{5}{3} & 0 & 1 \end{bmatrix}
\]

\( \begin{array}{c}
R_1/2 \\
R_2 - 5R_1 \rightarrow R_2
\end{array} \)

\[
\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{bmatrix}
\]

\( R_1 - R_2 \rightarrow R_1 \)

\[
\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{bmatrix}
\]

\( R_1 - R_2 \rightarrow R_1 \)

\( \mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \)

2) \( \text{det} \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = 2 \cdot 3 - 1 \cdot 5 = 1 \)

\[
\mathbf{A}^{-1} = \frac{1}{\text{det} \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}
\]

3) **Cofactor formula for inverse:**

\( A_{11} = \begin{bmatrix} 3 \end{bmatrix} \) \( \text{det} A_{11} = 3 \Rightarrow C_{11} = (-1)^{1+1} 3 = 3 \)

\( A_{12} = \begin{bmatrix} 5 \end{bmatrix} \) \( \text{det} A_{12} = 5 \Rightarrow C_{12} = (-1)^{1+2} 5 = -5 \)

\( A_{21} = \begin{bmatrix} 1 \end{bmatrix} \) \( \text{det} A_{21} = 1 \Rightarrow C_{21} = (-1)^{2+1} 1 = -1 \)

\( A_{22} = \begin{bmatrix} 2 \end{bmatrix} \) \( \text{det} A_{22} = 2 \Rightarrow C_{22} = (-1)^{2+2} 2 = 2 \)

\[
\text{adj} \mathbf{A} = \frac{1}{\text{det} \mathbf{A}} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}
\]
8. **ROW REDUCTION:** The following *elementary operations with matrix rows*, their symbolic designation and affiliated with them elementary matrices are introduced as:

### Interchange

**Operation:**

\[ R_i \leftrightarrow R_j \]

**Symbolic designation:**

\[ E_i = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \end{bmatrix} \]

**Affiliated elementary matrix:**

\[ \begin{vmatrix} E_i \end{vmatrix} = -1 \]

**Example:**

\[ E_i \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \]

### Scaling

**Operation:**

\[ kR_i \]

**Symbolic designation:**

\[ E_2 = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \end{bmatrix} \]

**Affiliated elementary matrix:**

\[ \begin{vmatrix} E_2 \end{vmatrix} = k \]

**Example:**

\[ E_2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \]

### Replacement

**Operation:**

\[ R_i + kR_j \rightarrow R_i \]

**Symbolic designation:**

\[ E_3 = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \end{bmatrix} \]

**Affiliated elementary matrix:**

\[ \begin{vmatrix} E_3 \end{vmatrix} = 1 \]

**Example:**

\[ E_3 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \]

With the help of the elementary matrices, the elementary row operations can be performed by matrix multiplication (with the corresponding acting on the determinant for square matrices):

**Operation interchange**

\[ E_i A \]

**Affiliated determinant:**

\[ \begin{vmatrix} E_i \end{vmatrix} = -1 \]

**Example:**

\[ \begin{vmatrix} E_i A \end{vmatrix} = -\begin{vmatrix} A \end{vmatrix} \]

**Operation scaling**

\[ E_2 A \]

**Affiliated determinant:**

\[ \begin{vmatrix} E_2 \end{vmatrix} = k \]

**Example:**

\[ \begin{vmatrix} E_2 A \end{vmatrix} = k \begin{vmatrix} A \end{vmatrix} \]

**Operation replacement**

\[ E_3 A \]

**Affiliated determinant:**

\[ \begin{vmatrix} E_3 \end{vmatrix} = 1 \]

**Example:**

\[ \begin{vmatrix} E_3 A \end{vmatrix} = \begin{vmatrix} A \end{vmatrix} \]

Two matrices \( B \) and \( A \) are called *row equivalent* if one of them is obtained from the other by a sequence of elementary operations

\[ B = E_{k_1} \ldots E_{k_i} A \]

Row equivalent matrices have the same *kernel*. **Row Reduced Echelon Form:**

Row Reduced Echelon Form (RREF) is a matrix which entries satisfy the following conditions:

1. All non-zero rows are above any row with all zeroes.
2. Each leading entry (first non-zero element) of a row is to the right of the leading entry of the row above.
3. All entries in the column below the leading entry are zero.
4. All leading entries are 1.
5. Leading entry (pivot) is the only non-zero entry in the column.

**Theorem 13**

Every matrix \( A \) is row equivalent to the unique RREF \( U \).
Row Reduced Algorithm – Gaussian Elimination

Step 1

\[
\begin{pmatrix}
3 & -3 & 4 & 0 & 2 \\
1 & -1 & 2 & 3 & 1 \\
-2 & 2 & -2 & 3 & 1 \\
1 & -1 & -2 & 5 & 1
\end{pmatrix}
\]

Find the first pivot column (leftmost non-zero column)

Step 2

\[
\begin{pmatrix}
1 & -1 & 2 & 3 & 1 \\
3 & -3 & 4 & 0 & 2 \\
-2 & 2 & -2 & 3 & 1 \\
1 & -1 & -2 & 5 & 1
\end{pmatrix}
\]

Select any non-zero entry in the pivot column as a pivot.

If necessary interchange rows to move pivot into pivot position

Step 3

\[
\begin{pmatrix}
1 & -1 & 2 & 3 & 1 \\
0 & 0 & -2 & -9 & -1 \\
0 & 0 & 2 & 9 & 3 \\
0 & 0 & -4 & 2 & 0
\end{pmatrix}
\]

Use row replacement operations to create zeroes in all positions below the pivot

Step 4

\[
\begin{pmatrix}
1 & -1 & 2 & 3 & 1 \\
0 & 0 & \frac{9}{2} & \frac{-1}{2} & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 20 & 0
\end{pmatrix}
\]

Ignore the pivot rows and repeat steps 1-3 to the remaining matrix

Step 5

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Starting from the rightmost pivot use row replacement operations to create all zeroes above pivot

(back elimination)

We obtained the RREF

Row operations:

\[R_1 \leftrightarrow R_2\]

\[R_2 - 3R_1 \rightarrow R_2\]

\[R_3 + 2R_1 \rightarrow R_3\]

\[R_4 - R_1 \rightarrow R_4\]

\[R_2 - 2R_3 \rightarrow R_2\]

\[R_2 \rightarrow \frac{R_2}{2}\]

\[R_3 \rightarrow \frac{R_3}{20}\]

\[R_4 \rightarrow \frac{R_4}{2}\]

\[R_5 \rightarrow \frac{R_5}{10}\]
9. LINEAR SYSTEMS: We introduced all necessary foundation and tools to solve and to determine the properties of a system of algebraic equations. The linear system of $m$ algebraic equations for $n$ unknowns $x_1, x_2, \ldots, x_n$ can be defined in the following form:

$$
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
$$

where the **system coefficients** $a_{ij} \in \mathbb{R}$ and the **system constants** $b_j \in \mathbb{R}$ are fixed real numbers.

System coefficients can be organized in the matrix component form or in the column-vector form:

$$
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \in \mathbb{R}^{m \times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m
$$

The vector of constants and the vector of unknowns can be written as:

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m
$$

**Augmented matrix** (matrix of coefficients with additional column of constants):

$$
A^* = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix} \in \mathbb{R}^{m \times (n+1)}, \quad A^* = (a_1 \ a_2 \ \ldots \ a_n \ | \ b)
$$

With the help of these notations, the linear system can be written in the matrix form:

**non-homogeneous system**

$$
Ax = b
$$

or in the vector form: $x_1a_1 + x_2a_2 + \cdots + x_na_n = b$

Associated with this system is the **homogeneous system** ($b = 0$):

**homogeneous system**

$$
Ax = 0
$$

**Solution** of the non-homogeneous system $Ax = b$ is any vector $x \in \mathbb{R}^n$ which satisfies this system. If the linear system has at least one solution, then it is called **consistent**, otherwise it is **inconsistent**.

The homogeneous system $Ax = 0$ has at least one solution $x = 0$ (**trivial solution**), and therefore, it is always consistent. The non-zero vector $x \neq 0$ satisfying $Ax = 0$ is called a **non-trivial solution**.

For the linear system we want to get the answer to the following questions:

- Is the system consistent?
- And if yes, then how many solutions does it have and how are they found?

The main method of solution of the linear system is Gaussian Elimination: row reduction of the matrix $A$ for a homogeneous system, and row reduction of an augmented matrix $A^*$ for a non-homogeneous system.
Row reduction of $A$ for homogeneous system $Ax = 0$. Solution space of homogeneous system $\ker A$.

If in the RREF of matrix $A$ there exists a non-pivot column, then this column corresponds to a free variable. In this case, there exists a non-trivial solution of the homogeneous system.

For example, let the RREF have the following form:

\[
\begin{bmatrix}
1 & 0 & 0 & \ast \\
0 & 1 & 0 & \ast \\
0 & 0 & 1 & \ast \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

pivot rows

\[
\begin{bmatrix}
\ast & 0 & 0 & 3 \\
0 & \ast & 0 & 4 \\
0 & 0 & \ast & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

pivot rows

Then the corresponding linear system with arbitrary values for free parameters $x_2$ and $x_3$ can be written as

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 0 \\
x_3 + 4x_4 &= 0 \\
x_4 + x_5 &= 0 \\
x_5 &= x_4
\end{align*}
\]

Solve each equation in terms of free variables

\[
\begin{align*}
x_1 &= -2x_2 - 3x_3 \\
x_2 &= x_2 \\
x_3 &= -4x_3 \\
x_4 &= -x_3 \\
x_5 &= x_4
\end{align*}
\]

and rewrite the system in column-vector form; and then expand the right hand side into component form with free variables as coefficients

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
-2x_2 - 3x_3 \\
x_2 \\
-4x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
-4 \\
0 \\
0
\end{bmatrix}x_2 + \begin{bmatrix}
-3 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}x_3
\]

For any values of free parameters $x_2, x_3 \in \mathbb{R}$, this equation provides the solution $x$ of the homogeneous system. Therefore, the solution vector is a linear combination of two vectors or a span of these two vectors.

\[
x_c = \ker A = \text{the solution space of the homogeneous system } Ax = 0
\]

(also called a \textit{complimentary solution})

\[
x_c = \ker A = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\}
\]
This solution set of the homogeneous system $Ax = 0$ is a vector space which we called the kernel of the matrix $A$. Here, we explicitly constructed the vector space $\text{ker } A$. Because it is spanned by two linearly independent vectors, it has a dimension two: $\dim(\text{ker } A) = 2$. Then according to the Theorem, the dimension of the set spanned by the column vectors of the matrix $A$ is determined as

$$\dim(\text{ker } A) + \text{rank } A = n \quad \text{rank } A = n - \dim(\text{ker } A) = 5 - 2 = 3$$

This dimension corresponds to the number of pivot columns in the RREF of $A$. If all columns in RREF are pivot columns then there are no free variables, and the homogeneous system in this case has a trivial solution $x = 0$.

Here, we demonstrated the technique of explicit construction of the solution space $\text{ker } A$ of the homogeneous system $Ax = 0$.

There are some facts about the consistency of linear systems:

**Theorem 14** If the number of unknowns $n$ is greater than the number of equations $m$, then homogeneous linear system has the non-trivial solutions:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$ with $m < n$, then there always exists $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \neq 0$ such that $Ax = 0$.

**Theorem 15** The dimension of the solution space $\text{ker } A$ of the homogeneous linear system $Ax = 0$ is equal to the number of free variables in the row reduced echelon form of the matrix $A$.

If $m < n$, then the dimension of solution space is at least $n - m$.

**Theorem 16** The non-homogeneous system $Ax = b$ is consistent if and only if the RREF of the augmented matrix $A^\ast = [a_1 \ a_2 \ \cdots \ a_n \mid b]$ has no pivot in the last column (no row of the form $(0 \ 0 \ \cdots \ 0 \mid \square)$:

$$\begin{bmatrix} \square \ & 0 & 0 & \ast \\ 0 & \square & 0 & \ast \\ 0 & 0 & \square & \ast \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution set of the consistent linear system $Ax = b$ consists of

- a unique solution if there are no free variables in RREF;
- infinitely many solutions when there are free variables.

In this case the dimension of the solution space is equal to the dimension of the $\text{ker } A$ (number of free variables).
Theorem 17 (Solution of the non-homogeneous system $Ax = b$)

Let the vector $x_p \in \mathbb{R}^n$ be some particular solution of $Ax = b$. Then any solution of the non-homogeneous system $Ax = b$ is represented as

$$x = x_c + x_p$$

where $x_c = ker A$ is the solution of homogenous system $Ax = 0$ (complimentary solution).

Proof:  
1) Show that $x = x_c + x_p$ is a solution of $Ax = b$. Indeed

$$A(x_c + x_p) = Ax_c + Ax_p = 0 + b = b$$

2) Show that for any solution of $Ax = b$, the vector $x_c$ defined by the equation $x = x_c + x_p$ is a solution of the homogeneous system. Indeed

$$x = x_c + x_p$$
$$x_c = x_p - x$$

$$Ax_c = A(x_p - x) = Ax_p - Ax = b - b = 0$$

Row reduction of the augmented matrix $A^\oplus$ – procedure for solution of non-homogeneous system $Ax = b$:

The solution of non-homogeneous system $Ax = b$ consists of the following steps:

1) Write an augmented matrix $A^\oplus = (a_1 \ a_2 \ \ldots \ a_n \ | \ b)$

2) Use Gaussian elimination to obtain RREF – determine if linear system is consistent

3) Solve the system for basic variables

4) Rewrite solution in a vector form using free variables as the parameters
Example: Find the solution of the linear system
\[ \begin{align*}
12x + 2x_2 &= 3 \\
3x_1 + 6x_2 &= 9
\end{align*} \]

1) augmented matrix:
\[
A^* = \begin{bmatrix}
1 & 2 & 3 \\
3 & 6 & 9
\end{bmatrix}
\]

2) Gaussian elimination:
\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 6 & 9
\end{bmatrix} \to R_2 - 3R_1 \to R_2
\]
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

RREF
\[ x_1 \text{ basic variable} \]
\[ x_2 \text{ free parameter} \]

3) Equivalent system:
\[ \begin{align*}
x_1 + 2x_2 &= 3 \\
x_2 &= x_2
\end{align*} \]

Solve for basic variables:
\[ \begin{align*}
x_1 &= -2x_2 + 3 \\
x_2 &= x_2
\end{align*} \]

4) Rewrite in a vector form:
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
-2x_2 + 3 \\
x_2
\end{bmatrix}
\]

From which the general solution follows as:
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1
\end{bmatrix} + \begin{bmatrix}
3 \\
0
\end{bmatrix}, \quad x_2 \in \mathbb{R}
\]

The general solution of the non-homogeneous system can be written in a vector-parametric form:
\[ \mathbf{x} = t \mathbf{v} + \mathbf{x}_p \quad \text{where} \quad \mathbf{v} = \begin{bmatrix}
-2 \\
1
\end{bmatrix}, \quad \mathbf{x}_p = \begin{bmatrix}
3 \\
0
\end{bmatrix} \]

The solution has a geometrical visualization in the \( x_1,x_2 \) -plane:
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = t \begin{bmatrix}
-2 \\
1
\end{bmatrix} \quad \text{is a parametric equation of the line along the vector} \quad \mathbf{v} = \begin{bmatrix}
-2 \\
1
\end{bmatrix}, \quad \text{representing the solution set of the homogeneous system (kernel of the matrix} \ A \).

Then by translation of this set by a constant vector \( \mathbf{x}_p = \begin{bmatrix}
3 \\
0
\end{bmatrix} \), the general solution of a non-homogeneous linear system is obtained as the parametric equation of the line:
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = t \begin{bmatrix}
-2 \\
1
\end{bmatrix} + \begin{bmatrix}
3 \\
0
\end{bmatrix}, \quad t \in \mathbb{R}
\]
LINEAR SYSTEMS WITH SQUARE MATRICES:

**Theorem 18**  Let \( A \in \mathbb{R}^{n \times n} \). The homogeneous linear system \( Ax = 0 \) has a non-trivial solution if and only if
\[
\det A = 0
\]

**Theorem 19**  If \( A \in \mathbb{R}^{n \times n} \) is invertible then for any \( b \in \mathbb{R}^{n} \) the linear system \( Ax = b \) has the unique solution
\[
x = A^{-1}b
\]

**Theorem 20**  (Cramer’s Rule)
If \( A \in \mathbb{R}^{n \times n} \) is invertible then for any \( b \in \mathbb{R}^{n} \) the unique solution of the linear system \( Ax = b \) is the vector
\[
x = \left[ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \in \mathbb{R}^{n} \]  the components of which are determined by the following formula:
\[
x_k = \frac{\det B_k}{\det A} \quad k = 1, 2, \ldots, n
\]
where matrices \( B_k \) are obtained from the matrix \( A \) by replacing the column-vector \( a_k \) by the vector of constants \( b \):
\[
B_k = \left[ a_1 \quad \cdots \quad a_{k-1} \quad b \quad a_{k+1} \quad \cdots \quad a_n \right] \quad k = 1, 2, \ldots, n
\]

**Example:**
Solve the linear system
\[
\begin{align*}
3x_1 + 2x_2 + 4x_3 &= 1 \\
2x_1 - x_2 + x_3 &= 0 \\
x_1 + 2x_2 + 3x_3 &= 1
\end{align*}
\]
Rewrite the system in the matrix form:
\[
Ax = b \quad \text{where} \quad A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]
Determinant of the matrix of coefficients \( \det A = -5 \), therefore:

1)  The solution of the linear system is given by
\[
x = A^{-1}b = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

2)  The same solution can be obtained with Cramer’s Rule:
\[
x_1 = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -\frac{1}{5}, \quad x_2 = \begin{vmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 0, \quad x_3 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \frac{2}{5}
\]
The following theorem combines some previous results and reveals the connection between different aspects of linear algebra:

**Theorem 21 (the inverse matrix theorem)**

Let $A \in \mathbb{R}^{n \times n}$ be a square $n \times n$ matrix, $A = (a_1, a_2, \ldots, a_n)$.

Then the following statements are equivalent:

1. $A$ is invertible
2. There exists $A^{-1} \in \mathbb{R}^{n \times n}$ such that $A^{-1} A = A A^{-1} = I$
3. $A x = 0$ has only the trivial solution $x = 0$
4. For any $b \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^n$ such that $A x = b$
5. The set of column-vectors $\{a_1, a_2, \ldots, a_n\}$ is linearly independent
6. $\text{span}\{a_1, a_2, \ldots, a_n\} = \mathbb{R}^n$
7. $\text{col } A = \mathbb{R}^n$
8. $\text{rank } A = n$
9. $A$ has $n$ pivots
10. $A$ is row equivalent to the identity matrix $I$
11. $\text{ker } A = \{0\}$
12. $\text{dim}(\text{ker } A) = 0$
13. $\det A \neq 0$
10. EIGENVALUE PROBLEM:

We have seen that linear transformation of a vector space can be defined with the help of matrices. Consider the linear transformation \( T(x) : \mathbb{R}^n \to \mathbb{R}^n \) defined by an \( n \times n \) matrix \( A \) as \( T(x) = Ax \). There is the following question concerning the output of the linear transformation: are there some vectors \( x \in \mathbb{R}^n \) which only are scaled under the given transformation but not rotated:

\[
Ax = \lambda x
\]

Such vectors if they exist are called the \textit{eigenvectors} and the corresponding scaling coefficients are called the \textit{eigenvalues} of \( A \). They can be found by solution of vector equation \( Ax = \lambda x \). It appears that these eigenvalues, in general, are the real or complex, \( \lambda \in \mathbb{R} \) or \( \mathbb{C} \), and the corresponding eigenvectors are the vectors over \( \mathbb{R} \) or \( \mathbb{C} \).

\textbf{Definition 1} \ (eigenvalues and eigenvectors)

Let \( A \in \mathbb{R}^{n \times n} \) be a \( n \times n \) matrix. The value \( \lambda \in \mathbb{R} \) or \( \mathbb{C} \) for which equation

\[
Ax = \lambda x
\]

has a non-trivial solution \( x \in \mathbb{R}^n, \mathbb{C}^n \), \( x \neq 0 \) is called an \textit{eigenvalue} of the matrix \( A \). Non-zero solution vector \( x \) is called an \textit{eigenvector} corresponding to eigenvalue \( \lambda \).

It is obvious that if \( x \) is an eigenvector corresponding to the eigenvalue \( \lambda \), then any scalar multiple \( cx \) also is an eigenvector corresponding to the eigenvalue \( \lambda \). Indeed:

\[
A(cx) = \lambda (cx) \Rightarrow cAx = c\lambda x \Rightarrow Ax = \lambda x \quad \text{provided } c \neq 0.
\]

And if \( x_1, ..., x_k \) are the eigenvectors corresponding to eigenvalue \( \lambda \), then any their linear combination \( c_1x_1 + ... + c_kx_k \) also is an eigenvector corresponding to eigenvalue \( \lambda \).

\textbf{Definition 2} \ (eigenspace)

The set of all eigenvectors corresponding to eigenvalue \( \lambda \) completed with the zero vector (which is the trivial solution of \( Ax = \lambda x \)) is called the \textit{eigenspace} of the matrix \( A \) corresponding to \( \lambda \):

\[
\text{Eig}_A(\lambda) = \{ x \neq 0 \mid Ax = \lambda x \} \cup \{ 0 \}
\]

Eigenvectors and eigenvalues of the matrix \( A \) can be found by solving the vector equation

\[
Ax = \lambda x
\]

\[
Ax - \lambda x = 0
\]

\[
Ax - \lambda Ix = 0
\]

This is the homogeneous linear system which has a non-trivial solution only if the matrix of the system has the zero determinant

\[
det(A - \lambda I) = 0
\]
Therefore, the eigenvalues of the matrix $A$ have to satisfy this equation, which after evaluation of the determinant becomes an algebraic equation in $\lambda$ of degree $n$:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \quad \text{with real coefficients } a_i \in \mathbb{R}$$

According to the **Fundamental Theorem of Algebra** there exist exactly $n$ roots of this equation which can be real distinct, repeated, or complex conjugate. This equation is called the **characteristic equation**:

$$\det (A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

**Theorem 22** The scalar $\lambda$ is an eigenvalue of the matrix $A \in \mathbb{R}^{n \times n}$ if and only if it satisfies the characteristic equation $\det (A - \lambda I) = 0$.

**Theorem 23** If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$, then the corresponding eigenvectors $x_1, x_2, \ldots, x_m$ are linearly independent.

**Theorem 24** If $\lambda$ is a root of multiplicity $k$ of the characteristic equation $\det (A - \lambda I) = 0$ and therefore it is an eigenvalue of the matrix $A \in \mathbb{R}^{n \times n}$, then the dimension of the corresponding eigenspace is $\dim \text{Eig}_A (\lambda) \leq k$.

So, if $\lambda$ is a root of multiplicity 1, then $\dim \text{Eig}_A (\lambda) = 1$. It means that only one linearly independent eigenvector corresponds to $\lambda$.

**Theorem 25** If matrix $A \in \mathbb{R}^{n \times n}$ is symmetric, $A^T = A$, then its eigenvalues are real.

**Theorem 26** If matrix $A \in \mathbb{R}^{n \times n}$ is invertible then 0 is not its eigenvalue.

If 0 is an eigenvalue of $A$ then $\det A = 0$.

**PROCEDURE FOR SOLVING AN EIGENVALUE PROBLEM:**

1. Construct the characteristic equation $\det (A - \lambda I) = 0$.
2. Find the roots of the characteristic equation $\lambda_k$ eigenvalues (can be distinct, repeated, complex conjugate).
3. For each eigenvalue $\lambda_k$ solve the linear system $(A - \lambda_k I)u = 0$ to find eigenvectors $u_k$ corresponding to $\lambda_k$. 
EXAMPLES:  

1. (distinct real eigenvalues)  

Solve the eigenvalue problem for \( A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \).

1) Characteristic equation:  

\[
\det \begin{pmatrix} 2 - \lambda & 0 \\ 1 & 3 - \lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6 = 0
\]

2) The roots are:  

\( \lambda_1 = 2 \)  
\( \lambda_2 = 3 \)

3) Vector equation for \( \lambda_1 = 2 \):  

\[
(A - \lambda_1 I) \mathbf{u}_1 = \mathbf{0}  
\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Row reduction of matrix of coefficients:  

\[
\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]

Corresponding equation includes one basic variable and one free variable:  

\( x_1 + x_2 = 0 \)

Solve for the basic variable  

\( x_1 = -x_2, \quad x_2 \in \mathbb{R} \)

Any value of the free parameter \( x_2 \) provides a solution, choose \( x_2 = -1 \),  
then \( x_1 = 1 \). Therefore, the solution vector (eigenvector) is  

\[
\mathbf{u}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

Vector equation for \( \lambda_2 = 3 \):  

\[
(A - \lambda_2 I) \mathbf{u}_2 = \mathbf{0}  
\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Row reduction of matrix of coefficients:  

\[
\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

The corresponding equation includes one basic variable and one free variable:  

\( x_1 + 0 \cdot x_2 = 0 \)

Solve for the basic variables  

\( x_1 = 0 \cdot x_2, \quad x_2 \in \mathbb{R} \)

Any value of the free parameter \( x_2 \) provides a solution, choose \( x_2 = 1 \),  
then \( x_1 = 0 \). Therefore, the solution vector (eigenvector) is  

\[
\mathbf{u}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
2. (real eigenvalues, repeated eigenvalue)

Solve the eigenvalue problem for \( \mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \).

1) Characteristic equation:
\[
\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0
\]

2) The roots are:
\( \lambda_1 = 4 \\
\lambda_2 = \lambda_3 = 2 \quad \text{eigenvalue of multiplicity 2} \)

3) Vector equation for \( \lambda_1 = 4 \):
\[ (\mathbf{A} - \lambda_1 \mathbf{I}) \mathbf{u} = \mathbf{0} \]
\[
\begin{pmatrix} 2-4 & 1 & -1 \\ 0 & 3-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

Row reduction of matrix of coefficients:
\[
\begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow R_2 + R_3 \rightarrow R_2, \quad R_1 - R_2 \rightarrow R_1
\]
\[
\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{row reduced echelon form}
\]

Corresponding system of equation includes two basic variables \( x_1, x_2 \) and one free variable \( x_3 \):
\[
x_1 + x_2 = 0 \\
x_2 + x_3 = 0
\]
Solve for the basic variables
\[
x_1 = -x_3 \\
x_2 = -x_3 \quad x_3 \in \mathbb{R}
\]
Any value of the free parameter \( x_3 \) provides a solution, choose \( x_3 = -1 \), then \( x_1 = x_2 = 1 \). Therefore, the solution vector (eigenvector) is
\[
\mathbf{u}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]
Vector equation for $\lambda = 2$:

$$(A - \lambda I)u_2 = 0$$

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 & -1 \\ 0 & -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Row reduction of matrix of coefficients:

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\overset{R_2 - R_1 \rightarrow R_2}{\rightarrow}$$

$$\overset{R_1 + R_1 \rightarrow R_1}{\rightarrow}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ row reduced echelon form

The corresponding equation includes one basic variable $x_i$ and two free variables $x_2, x_3$ (the dimension of kernel is 2):

$$0 \cdot x_i + x_2 - x_3 = 0 \quad \Rightarrow \quad x_2 = x_3 \quad x_2, x_3 \in \mathbb{R}$$

It is possible to construct two independent vectors with arbitrary $x_i \in \mathbb{R}$ and $x_2 = x_3$. The simplest non-zero vectors satisfying these conditions are

$$u_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore, for the given matrix, there are two linearly independent vectors corresponding to eigenvalue of multiplicity two.

3. (complex eigenvalues)

If $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue of the matrix with real coefficients $A \in \mathbb{R}^{nxn}$ with the corresponding eigenvector $u \in \mathbb{R}^n$, then the complex conjugate of it $\bar{\lambda} = a - bi$ is also an eigenvalue of $A \in \mathbb{R}^{nxn}$ with the eigenvector $\bar{u}$. Indeed, let

$$Au = \lambda u$$

$$A\bar{u} = \bar{\lambda} \bar{u}$$

$$A\bar{u} = \bar{\lambda} \bar{u}$$

Example: Solve the eigenvalue problem for $A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$.

1) Characteristic equation:

$$\det \begin{pmatrix} 1 - \lambda & 5 \\ -2 & 3 - \lambda \end{pmatrix} = \lambda^2 - 4\lambda + 13 = 0$$
2) The roots are:
\[ \lambda_2 = 2 + 3i \]
\[ \lambda_2 = 2 - 3i \]

3) Vector equation for \( \lambda = 2 + 3i \):
\[
(A - \lambda I)u = 0
\]
\[
\begin{pmatrix}
1 - (2 + 3i) & 5 \\
-2 & 3 - (2 + 3i)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 0
\]
\[
\begin{pmatrix}
-1 - 3i & 5 \\
-2 & 1 - 3i
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 0
\]
Row reduction of matrix of coefficients:
\[
\begin{pmatrix}
-1 - 3i & 5 \\
-2 & 1 - 3i
\end{pmatrix}
\rightarrow (-1+3i) R_1
\]
\[
\begin{pmatrix}
10 & -5 + 15i \\
-2 & 1 - 3i
\end{pmatrix}
\rightarrow R_1 / 5
\]
\[
\begin{pmatrix}
2 & -1 + 3i \\
-2 & 1 - 3i
\end{pmatrix}
\rightarrow R_2 + R_1 \rightarrow R_2
\]
\[
\begin{pmatrix}
2 & -1 + 3i \\
0 & 0
\end{pmatrix}
\text{ row echelon form}
\]
The corresponding equation includes one basic variable and one free variable:
\[
2x_1 + (-1 + 3i)x_2 = 0
\]
Solve for the basic variable
\[
2x_1 = (1 - 3i)x_2 \quad x_2 \in \mathbb{R}
\]
Any value of free parameter \( x_2 \) provides a solution, choose \( x_2 = 2 \), then \( x_1 = 1 - 3i \). Therefore, the solution vector (eigenvector) is
\[
u_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix}i
\]
Then the eigenvector corresponding to the second eigenvalue \( \lambda_2 \) is
\[
u_2 = \overline{u_1} = \begin{pmatrix} \overline{1} \\ \overline{2} \end{pmatrix} - \begin{pmatrix} \overline{3} \\ \overline{0} \end{pmatrix}i = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}i
\]

4. (system of ODEs) For application of the eigenvalue problem for solution of the System of Linear Ordinary Differential equations see Chapter V (Section 5.5).
11. LINEAR ALGEBRA WITH MAPLE:

Many linear algebra operations can be performed only using `linalg` package which is downloaded by the command:

```maple
> with(linalg):
```

**Entering vectors and matrices:**

by array:

```maple
> u:=array([3,2,-3]);
```

```maple
u := [3, 2, -3]
```

```maple
> A:=array([[-1,2,1],[3,2,1]]);
```

```maple
A :=
\begin{bmatrix}
-1 & 2 & 1 \\
3 & 2 & 1
\end{bmatrix}
```

by command:

```maple
> v:=vector([1,2,-1]);
```

```maple
v := [1, 2, -1]
```

```maple
> B:=matrix(3,2,[1,2,3,4,5,6]);
```

```maple
B :=
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
```

**Operations with vectors:**

```maple
> w:=vector([1,0,1]);
```

```maple
w := [1, 0, 1]
```

dot product:

```maple
> dotprod(u,v);
```

```maple
10
```

cross product:

```maple
> crossprod(u,v);
```

```maple
[4, 0, 4]
```

angle between two vectors:

```maple
> beta:=evalf(angle(u,v));beta:=evalf(convert(beta,degrees));
```

```maple
β := 0.5148059555
```

```maple
β := 29.49620851 degrees
```

**Operations with matrices:**

Arithmetic operations are performed by applying command `evalm`

addition:

```maple
> c:=evalm(u+v);
```

```maple
c := [4, 4, -4]
```

scalar multiplication:

```maple
> C:=evalm(2*B);
```

```maple
C :=
\begin{bmatrix}
2 & 4 \\
6 & 8 \\
10 & 12
\end{bmatrix}
```

matrix multiplication:

```maple
> F:=evalm(A*B);
```

```maple
F :=
\begin{bmatrix}
10 & 12 \\
14 & 20
\end{bmatrix}
```

transpose:

```maple
> transpose(C);
```

```maple
\begin{bmatrix}
2 & 6 & 10 \\
4 & 8 & 12
\end{bmatrix}
```
determinant:
  \( \text{det}(F) \);

inverse:
  \( \text{inverse}(F) \);
  \[
  \begin{bmatrix}
  \frac{5}{8} & -\frac{3}{8} \\
  -\frac{7}{16} & \frac{5}{16}
  \end{bmatrix}
  \]

adjoint:
  \( \text{adjoint}(F) \);
  \[
  \begin{bmatrix}
  20 & -12 \\
  -14 & 10
  \end{bmatrix}
  \]

row reduced echelon form (RREF):
  \( \text{rref}(A) \);
  \[
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & \frac{1}{2}
  \end{bmatrix}
  \]

rank of matrix
  \( \text{rank}(A) \);
  2

kernel of matrix:
  \( \text{kernel}(A) \);
  \{[0, 1, -2]\}

column space of a matrix:
  \( \text{colspan}(A) \);
  \{[-1, 3], [0, -8]\}

Solution of linear system of algebraic equations
  \( A := \text{matrix}(2, 2, [2, 4, -1, 3]); \)
  \[
  A := \begin{bmatrix}
  2 & 4 \\
  -1 & 3
  \end{bmatrix}
  \]
  \( b := \text{array}([[1], [-2]]); \)
  \[
  b := \begin{bmatrix}
  1 \\
  -2
  \end{bmatrix}
  \]
solution with \text{linsolve}:
  \( x := \text{linsolve}(A, b); \)
  \[
  x := \begin{bmatrix}
  11 \\
  10 \\
  -3 \\
  10
  \end{bmatrix}
  \]
solution with inverse matrix:
  \( x := \text{evalm}(\text{inverse}(A) \&* b); \)
  \[
  x := \begin{bmatrix}
  11 \\
  10 \\
  -3 \\
  10
  \end{bmatrix}
  \]

The Eigenvalue Problem:

eigenvalues:
  \( G := \text{matrix}(3, 3, [1, 2, 1, 6, -1, 0, -1, -2, -1]); \)
  \[
  G := \begin{bmatrix}
  1 & 2 & 1 \\
  6 & -1 & 0 \\
  -1 & -2 & -1
  \end{bmatrix}
  \]
  \( \text{eigenvalues}(G); \)
  0, 3, -4
with notations: [eigenvalue, multiplicity of eigenvalue, [corresponding eigenvector]]

Differentiation of matrices (each entry is differentiated):

$$> A:=\text{matrix}(2,2,[3*t,\sin(3*t),\exp(2*t),2*t*\exp(4*t)]);$$

$$A := \begin{bmatrix} 3t & \sin(3t) \\ e^{(2t)} & 2t e^{(4t)} \end{bmatrix}$$

$$> Ap:=\text{map}(\text{diff},A,t);$$

$$Ap := \begin{bmatrix} 3 & 3 \cos(3t) \\ 2 e^{(2t)} & 2 e^{(4t)} + 8 t e^{(4t)} \end{bmatrix}$$

Integration of matrices (each entry is integrated)

$$> Ai:=\text{map}(\text{int},A,t);$$

$$Ai := \begin{bmatrix} \frac{3t^2}{2} & \frac{1}{3} \cos(3t) \\ \frac{1}{2} e^{(2t)} & \frac{1}{2} t e^{(4t)} - \frac{1}{8} e^{(4t)} \end{bmatrix}$$

Evaluation of values of the functions inside of the matrix A:

$$> A:=\text{matrix}([[\sin(x),\cos(x)],[\exp(x),2*x]]);$$

$$A := \begin{bmatrix} \sin(x) & \cos(x) \\ e^x & 2 x \end{bmatrix}$$

$$> \text{for n from 1 to 2 do } A[n,1]:=\text{subs}(x=\pi/2,A[n,1]) \text{ od;}$$

$$A_{1,1} := \sin\left(\frac{\pi}{2}\right)$$

$$A_{2,1} := e^{\pi/2}$$

$$> \text{for n from 1 to 2 do } A[n,2]:=\text{subs}(x=\pi/2,A[n,2]) \text{ od;}$$

$$A_{1,2} := \cos\left(\frac{\pi}{2}\right)$$

$$A_{2,2} := \pi$$

$$> \text{evalm}(A);$$

$$\begin{bmatrix} \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \\ e^{\pi/2} & \pi \end{bmatrix}$$

$$> \text{evalf}(\text{evalm}(A));$$

$$\begin{bmatrix} 1.0 & -0.2051033808 \times 10^{-9} \\ 4.810477382 & 3.141592654 \end{bmatrix}$$

$$> \text{map}(\text{evalf},A);$$

$$\begin{bmatrix} 1.0 & -0.2051033808 \times 10^{-9} \\ 4.810477382 & 3.141592654 \end{bmatrix}$$
12. REVIEW QUESTIONS:

1) What is a vector space?

2) What operations with vectors are defined in vector spaces?

3) What is the meaning of closure axioms for vector spaces i) and ii)?

4) What is a vector subspace?

5) What conditions should be satisfied by the subset of a vector space to be a subspace?

6) Recall examples of vector spaces.

7) Is the union of vector subspaces of the vector space $V$ a vector space?

8) Is the intersection of vector subspaces of the vector space $V$ a vector space?

9) What is a linear combination of vectors?

10) What conditions should be satisfied by the subset of a vector space to be a subspace?

11) What is a span of a set of vectors?

12) What vectors are called linearly independent?

13) Is the set of vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ linearly independent?

14) What is the basis of a vector space?

15) What is the dimension of a vector space?

16) What is a linear map between vector spaces?

17) What is the image of a linear map?

18) What is the kernel of a linear map?

19) What is the rank of a linear map?

20) What is the relationship between the rank and the dimension of the kernel of a linear map?

21) When are two matrices equal?

22) How is the sum of two matrices defined?

23) What is a symmetric matrix?

24) For what matrices is the product of matrices defined?

25) Give an example of two non-zero $2 \times 2$ matrices such that $AB = 0$?

26) Show that if $AB = 0$, for $A, B \in \mathbb{R}^{nn}$, then $(BA)^2 = 0$.

27) How is the transformation of vector space defined with the help of matrices?

28) What is the determinant?

29) What is the matrix inverse?

30) What are the elementary row operations?

31) What is the Row Reduced Echelon Form (RREF)?

32) What is a consistent linear system?

33) When does a homogeneous system $Ax = 0$ have a non-trivial solution?

34) What is the method of solution of linear systems of equations?

35) What are eigenvalues and eigenvectors?
EXERCISES:

1) Prove the following statements for the vector space $V$:
   a) The negative element $-u \in V$ is unique.
   b) Zero vector $0 \in V$ is unique.
   c) $0u = 0$ for any $u \in V$.

2) Let $U_i \subset V$ be the subspaces of the vector space $V$. Show that the intersection of subspaces
   \[ \bigcap_{i=1}^{n} U_i = \{ u \in V \mid u \in U_i , i=1,...,n \} \]
   is a subspace of $V$.

3) Let $U$ be a vector space, and $V$ and $W$ be subspaces of $U$.
   Prove or disapprove the following statements:
   a. Intersection $V \cap W$ is a subspace of $U$.
   b. Union $V \cup W$ is a subspace of $U$.

4) Let $V$ be a vector space and let $u_1, u_2, ..., u_n \in V$.
   Show that $span\{u_1, u_2, ..., u_n\}$ is a subspace of $V$.

5) Show that if $0 \in \{u_1, u_2, ..., u_n\}$ then the set $\{u_1, u_2, ..., u_n\}$ is linearly dependent.

6) Determine if the following sets are linearly independent or dependent:
   a) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
   b) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

7) What is the standard basis for the vector space
   \[ \mathbb{R}_2 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a,b,c,d \in \mathbb{R} \right\} \]

8) Let $V$ be an $n$ dimensional vector space then (fill in the blank with $>,=,<$):
   a) if $\{u_1, ..., u_p\}$ spans $V$, then $p \underline{\underline{=}} n$
   b) if $\{u_1, ..., u_p\}$ is a basis for $V$, then $p \underline{\underline{=}} n$
   c) if $\{u_1, ..., u_p\}$ is a linearly independent set in $V$, then $p \underline{\underline{\leq}} n$

9) Show that if $\{u, v\}$ is a linearly independent subset of $\mathbb{R}^n$, then $\{u + v, u - v\}$ is also linearly independent.

10) Suppose that $v_1, v_2, ..., v_m$ are linearly independent vectors in $\mathbb{R}^n$ and let $u$ be any vector that is not in $span\{v_1, v_2, ..., v_m\}$. Show that $v_1, v_2, ..., v_m, u$ are linearly independent.
11) Let \( A \) be a square matrix, \( A \in \mathbb{R}_{nn} \). Show that

\[
A_s = \frac{1}{2}(A + A^T)
\]

is a symmetric matrix and

\[
A_a = \frac{1}{2}(A - A^T)
\]

is a skew-symmetric matrix (antisymmetric).

12) Fill in the blank with the best symbol or number:

a) If \( A \in \mathbb{R}_{nn} \) is an invertible \( n \times n \) matrix, then \( \text{rank } A = \) 

b) If \( A \in \mathbb{R}_{nn} \) is an invertible \( n \times n \) matrix and \( \det A = d \), then \( \det A^{-1} = \) 

13) True – False questions:

a) If \( A, B \in \mathbb{R}_{nn} \) are invertible \( n \times n \) matrices, then \( (AB)^T = A^TB^T \) __

b) For any \( m \times n \) matrix \( A \in \mathbb{R}_{mn} \), the matrix \( AA^T \) is symmetric ___

c) For any \( m \times n \) matrix \( A \in \mathbb{R}_{mn} \), the matrix \( A^TA \) is symmetric ___

d) If \( A \in \mathbb{R}_{nn} \) is invertible, the columns of \( A \) are linearly independent ___

e) All eigenvalues of a symmetric matrix are real ___

f) The set of all invertible \( n \times n \) matrices is a subspace of the vector space \( \mathbb{R}_{nn} \) of all \( n \times n \) matrices ___

14) Prove the following statement by mathematical induction:

Let \( A_1, A_2, \ldots, A_m \in \mathbb{R}_{nn} \) be invertible \( n \times n \) matrices. Then \( A_1A_2\cdots A_m \) is invertible and \( (A_1A_2\cdots A_m)^{-1} = A_m^{-1}A_{m-1}^{-1}\cdots A_1^{-1} \). 

15) Let \( A = \begin{pmatrix}
3 & 1 & 4 \\
2 & -1 & 0 \\
2 & 1 & 1 \\
\end{pmatrix} \) and \( B = \begin{pmatrix}
1 & 3 & 0 \\
2 & -1 & 3 \\
0 & -1 & 2 \\
\end{pmatrix} \), then compute

a) \( 3A \)  
b) \( A + B \)  
c) \( 2A - B \)  
d) \( AB \)  
e) \( A^TB \)  
f) \( (BA)^T \)  

16) Let \( A = \begin{pmatrix}
2 & 1 & 0 \\
-1 & 0 & 3 \\
\end{pmatrix} \) and \( B = \begin{pmatrix}
1 & 0 \\
1 & 2 \\
0 & 1 \\
\end{pmatrix} \). Determine if it is possible to perform the following operations, and if yes, find the result:

a) \( AB \)  
b) \( BA \)

c) \( BA^T \)  
d) \( A^TB^T \)

e) \( (A + B^T)B \)  
f) \( AA^T \)
17) Let \( \mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \). Compute \( \mathbf{A}^2 = \mathbf{A} \mathbf{A} \), \( \mathbf{A}^4 = \mathbf{A}^2 \mathbf{A} \). What will be \( \mathbf{A}^n \)?

18) Find the Row Reduced Echelon Form of the following matrices:

\[
\begin{pmatrix}
2 & 0 & 3 \\
0 & 1 & 2 \\
4 & 2 & 0
\end{pmatrix}
\]

a) 
\[
\begin{pmatrix}
5 & 3 & 1 \\
5 & 1 & 0
\end{pmatrix}
\]

b) 
\[
\begin{pmatrix}
2 & 1 & 2 & 0 & 2 \\
-4 & 2 & 0 & 1 & 0 \\
-4 & 2 & 4 & 1 & 0
\end{pmatrix}
\]

c) 
\[
\begin{pmatrix}
-1 & 2 & -1 & 2 \\
-2 & 2 & 1 & 4 \\
3 & 2 & 2 & 5 \\
-3 & 8 & 5 & 17
\end{pmatrix}
\]

d) 
\[
\begin{pmatrix}
2 & 1 & 3 & -1 & 2 \\
1 & 2 & 0 & 3 & 1 \\
-4 & -2 & -6 & 2 & -4 \\
3 & 1 & 0 & -1 & 0
\end{pmatrix}
\]

19) Solve the following linear system of equations:

a) 
\[
\begin{align*}
2x_1 - 4x_2 + 3x_3 &= 0 \\
x_1 + x_2 - 2x_3 &= 0
\end{align*}
\]

b) 
\[
\begin{align*}
x_1 + 2x_2 - 3x_3 &= 0 \\
x_1 + x_2 - 2x_3 &= 0
\end{align*}
\]

c) 
\[
\begin{align*}
x_1 - x_2 + 2x_3 + x_4 &= 0 \\
x_1 + 3x_2 - x_3 - x_5 &= 0 \\
2x_1 + x_2 - 3x_3 - x_5 &= 0
\end{align*}
\]

d) 
\[
\begin{align*}
x_1 - 3x_2 + 3x_4 &= 0 \\
2x_1 + x_2 + x_5 &= 0 \\
3x_1 + x_3 &= 0
\end{align*}
\]

e) 
\[
\begin{align*}
7x_1 - 4x_2 + x_3 + 3x_4 &= 5 \\
5x_1 + 7x_2 - 4x_3 - 6x_4 &= 3
\end{align*}
\]

f) 
\[
\begin{align*}
6x_1 - 4x_2 + 4x_3 + 3x_4 &= 3 \\
9x_1 - 6x_2 + 3x_3 + 2x_4 &= 4
\end{align*}
\]

g) 
\[
\begin{align*}
x_1 + 6x_2 - x_3 + x_4 + 2x_5 &= 0 \\
x_1 + 16x_2 - 6x_3 + 6x_4 + 7x_5 &= 0
\end{align*}
\]

h) 
\[
\begin{align*}
3x_1 - 8x_2 + x_3 + 2x_4 &= 5 \\
2x_1 - 5x_2 - 3x_3 + 2x_4 - 3x_5 &= 3
\end{align*}
\]

20) Find the determinant of the following matrices:

a) 
\[
\mathbf{A} = \begin{pmatrix} -2 & -6 \\ 2 & 5 \end{pmatrix}
\]

b) 
\[
\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 9 \end{pmatrix}
\]

c) 
\[
\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ -4 & 2 & -5 \\ 2 & 3 & 1 \end{pmatrix}
\]

d) 
\[
\mathbf{A} = \begin{pmatrix} 8 & 5 & -1 \\ 4 & 1 & -2 \\ -1 & 2 & 3 \end{pmatrix}
\]

e) 
\[
\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 & 3 \\ 4 & 0 & 0 & 0 \\ 0 & -1 & 4 & 1 \\ -2 & 0 & 2 & 3 \end{pmatrix}
\]

f) 
\[
\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 5 & 1 & 2 & -1 \\ 0 & 1 & 0 & 5 \\ -3 & 1 & 0 & 0 \end{pmatrix}
\]
21) Find the inverse of the following matrices:

a) \[ A = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} \]

b) \[ A = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} \]

c) \[ A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix} \]

d) \[ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \]

22) Solve the following linear systems of equations using matrix inverse and Cramer’s Rule:

a) \[ \begin{align*}
2x_1 - x_2 &= 1 \\
x_1 + x_2 &= 2
\end{align*} \]

b) \[ \begin{align*}
4x_1 + 5x_2 &= -3 \\
-5x_1 + x_2 &= 2
\end{align*} \]

c) \[ \begin{align*}
2x_1 + 4x_2 &= 1 \\
x_1 + 4x_2 &= 2
\end{align*} \]

d) \[ \begin{align*}
2x_1 + 5x_2 &= -2 \\
3x_1 + x_2 + 5x_3 &= 1 \\
x_1 + 2x_2 &= -1
\end{align*} \]

23) Solve the eigenvalue problem for the following matrices:

a) \[ A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \]

b) \[ A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \]

c) \[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

d) \[ A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \]

e) \[ A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \]

f) \[ A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \]

g) \[ A = \begin{pmatrix} 3 & 1 & 1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{pmatrix} \]

h) \[ A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \]

i) \[ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \]

k) \[ A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \]

24) Show that the matrix (the rotation matrix)

\[ A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

has complex eigenvalues when \( \theta \) is not a multiple of \( \pi \)
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