Non-Newtonian Fluid Math

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This document summarizes equations for computing pressure drop of a power-law fluid through a pipe. This is important to determine required pumping power for many fluids of practical importance, such as crude oil, melted plastics, and particle/liquid slurries. To simplify the analysis, I will assume the use of smooth and relatively long pipes (i.e. neglect pipe roughness and entrance effects). I further assume the reader has had exposure to principles taught in a college-level fluid mechanics course.

Types of Fluids

To begin, one must understand the difference between Newtonian and non-Newtonian fluids. There are many types of non-Newtonian fluids and this is a vast topic, which is studied under a branch of physics known as rheology. Rheology concerns itself with how materials (principally liquids, but also soft solids) flow under applied forces.

Some non-Newtonian fluids exhibit time-dependent or viscoelastic behavior. This means their flow behavior depends on the history of what forces were applied to the fluid. Such fluids include thixotropic (shear-thinning over time) and rheopectic (shear-thickening over time) types. Viscoelastic fluids exhibit elastic or solid-like behavior when forces are first applied, and then transition to viscous flow under continuing force. These include egg whites, mucous, shampoo, and silly putty. We will not be analyzing any of these time-dependent fluids in this discussion.

Time-independent or inelastic fluids flow with a constant rate when constant forces are applied to them. They are the focus of this discussion. To understand time-independent non-Newtonian fluids we must first understand Newtonian fluids. A Newtonian fluid exhibits a simple linear relationship between shear stress τ and strain rate $\dot{\gamma}$. The ratio between these two quantities is the viscosity μ, a quantity that depends on temperature but is otherwise a constant:

$$\tau = \mu\,\dot{\gamma} \tag{1}$$

In contrast, for a non-Newtonian fluid, the viscosity is a function of $\dot{\gamma}$. A common way to describe this is with an apparent viscosity μ_{ap} that is measured *at a particular shear rate*. For a so-called power-law fluid, we use the relationship

$$\mu_{\text{ap}}(\dot{\gamma}) = K\,\dot{\gamma}^{n-1} \tag{2}$$

where K and n are constants that nevertheless depend on the fluid and the temperature. K has been called the consistency index and n the flow behavior index. Because μ_{ap} has units of pressure $\times$ time and $\dot{\gamma}$ has units of inverse time, the units on K depend on the value of n, such as $\mathrm{Pa} \cdot \mathrm{s}^n$.

By substituting Eq. 2 into Eq. 1, one can see that for a power-law fluid

$$\tau = K\,\dot{\gamma}^n \tag{3}$$

One can further see that if $n = 1$ then Eq. 3 is equivalent to Eq. 1 and Newtonian behavior is recovered. If $n > 1$ the fluid is known as dilatant or shear-thickening. If $n < 1$ the fluid is known as pseudoplastic or shear-thinning, which is the more common case for non-Newtonian fluids.

The mechanical behaviors of Newtonian, pseudoplastic, and dilatant fluids are illustrated in Fig. 1. Note that these classifications are based only on the value of n; the consistency index K could be large or small for a particular fluid regardless of its classification, and this would change its relative value of apparent viscosity.

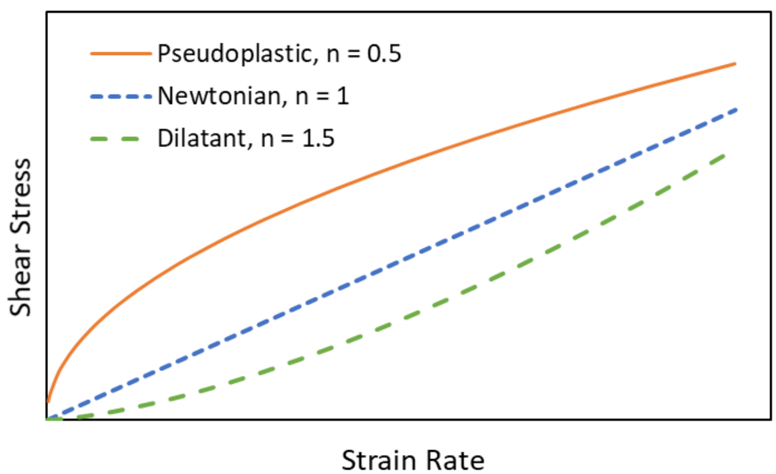

Fig. 1. Illustration of shear stress vs. strain rate for three types of fluids.

Determining Power-Law Constants under Laminar Flow

To fully understand a power-law fluid one must determine the unknown constants K and n in Eq. 2 or 3. One way to do this is to produce a flow with a known value of strain rate $\dot{\gamma}$ and to then measure shear stress τ. One can then repeat this for different values of $\dot{\gamma}$. For instance, in Couette flow a moving surface is placed parallel to a stationary surface, with a fixed gap of distance H, as shown in Fig. 2 (left). The fluid in the gap moves in a laminar fashion as shown by the velocity lines. In Couette flow there is a single strain rate determined by $\dot{\gamma} = \Delta v / H$, where Δv is the velocity of the upper surface relative to the lower surface. τ can be determined by measuring the force required on the moving surface to maintain steady motion.

Fig. 2. Couette flow (left) generates a single strain rate determined by the motion of the upper plate. In contrast, pressure-driven pipe flow (right) generates multiple strain rates relating to local velocities.

Pressure-driven pipe flow is how we generally move fluids in the real world, but it is more difficult to analyze than Couette flow, especially when turbulence is present. Fig. 2 (right) illustrates average local velocities in a pipe. A range of strain rates is exhibited, as given by the magnitude of the slope of v_z (local axial flow velocity) vs. r (radial position). Strain rate is zero in the center of the pipe and at a maximum value next to the wall of the pipe. Because there is a range of $\dot{\gamma}$ values in pipe flow, there is a range of τ values as well, in accord with Eq. 3.

In pipe flow the things one generally knows (i.e. that can be measured) are the pressure drop ΔP for a section of pipe with length L and diameter D, and the total volumetric flow rate $\dot{V}$. (In this document we express ΔP as a positive quantity, meaning upstream pressure minus downstream pressure.) Our goal is to determine enough about the non-Newtonian fluid that we could predict ΔP for any values of L, D, and $\dot{V}$, so that we can estimate required pumping power.

For pipe flow the most important shear stress to know is the value at the wall (τ_w) because this determines the pressure drop down the pipe, ΔP. A force balance on the fluid gives

$$\left(\frac{\pi}{4}D^2\right)\ \Delta P = (\pi D L)\ \tau_w \tag{4}$$

where the first set of parentheses gives cross-sectional area (the area that pressure acts on) and the second set of parentheses gives circumferential area (the area that wall shear stress acts on) for a section of pipe. Rearranging Eq. 4 gives

$$\Delta P = \left(\frac{4L}{D}\right)\ \tau_w \tag{5}$$

As shown by Metzner and Reed (1955) for laminar flow one can solve for the wall shear stress for power-law fluids in terms of either pipe-average velocity v or flow rate $\dot{V}$. Applying Eq. 3 at the wall gives

$$\tau_w = K\ \dot{\gamma}_w^n \tag{6}$$

where

$$\dot{\gamma}_w = \left(\frac{3n+1}{4n}\right)\frac{8v}{D} = \left(\frac{3n+1}{4n}\right)\frac{32\dot{V}}{\pi D^3} \tag{7}$$

Eqs. 5-7 can be combined to get

$$\Delta P = K\left(\frac{4L}{D}\right)\left(\frac{3n+1}{4n}\right)^n\left(\frac{32\dot{V}}{\pi D^3}\right)^n \tag{8}$$

To experimentally use the results of Metzner and Reed, one could run a series of laminar pipe flow experiments (on the same pipe) and measure ΔP as a function of $\dot{V}$. By taking the logarithm of both sides of Eq. 8, one can show that

$$n = \frac{d\log(\Delta P)}{d\log(\dot{V})} \tag{9}$$

In other words, one could plot $\log(\dot{V})$ vs. $\log(\Delta P)$, and n would be the slope of a straight-line fit to the data. Or, equivalently, one could perform a power-law fit to $\dot{V}$ vs. ΔP, in which the fitted exponent is n. So it is possible to determine n, and once that is known, to determine K from Eq. 8 and the same set of laminar pipe flow experiments.

Once n and K are known, Eq. 8 could be used again to predict pressure drop for new flow rates, pipe diameters, and lengths.

Power-Law Fluid Behavior under Turbulent Flow

Turbulent flow is more complicated than laminar flow, but similar principles can be used to analyze pressure drop or to determine constants K and n, though with perhaps less certainty. It is helpful to define pressure drop for a section of pipe in terms of a *friction factor*, and in turn how friction factor depends on *Reynolds number*.

The friction factor f is a dimensionless quantity to express the pressure drop across a section of pipe:

$$\Delta P = \frac{L}{D}\, f\, P_{\text{dyn}} \tag{10}$$

where dynamic pressure is

$$P_{\text{dyn}} = \frac{1}{2}\rho v^2 \tag{11}$$

The f used in Eq. 10 is the Darcy/Moody/Blasius friction factor and not the Fanning friction factor (which is 4 times smaller). Eqs. 10 and 11 apply to any flow situation (e.g. laminar or turbulent, Newtonian or power-law) because they essentially just define f.

In laminar flow of a Newtonian fluid, it can be shown that

$$f = \frac{64}{\text{Re}} \tag{12}$$

where the dimensionless Reynolds number is

$$\mathrm{Re} = \frac{\rho v D}{\mu} \tag{13}$$

Metzner and Reed defined a Reynolds number for power-law fluids, Re_{MR}. It is defined so that Eq. 12 continues to be true for laminar flow of a power-law fluid. By comparing Eqs. 10-12 with Eq. 8 it can be shown that

$$\mathrm{Re}_{MR} = \frac{\rho v^{2-n} D^n}{K \left(\frac{3n+1}{4n}\right)^n 8^{n-1}} \tag{14}$$

In a subsequent paper, Dodge and Metzner (1959) provided experimental data for turbulent flow of multiple power-law fluids. They presented a Moody-type chart for relating f to Re_{MR}. Based on my interpretation of their results, the critical pipe Reynolds number at which flow transitions from laminar to turbulent is around

$$\mathrm{Re}_{MR,\mathrm{crit}} = 2200\, n^{-0.15} + 100\, n^{-2} \tag{15}$$

An explicit formula for turbulent friction factor I likewise developed from their data is

$$f = (a\, \mathrm{Re}_{MR})^{-b} \tag{16}$$

where

$$a = \frac{42 n^{1.4}(n^{1.4} + 2) + 0.033}{n^{1.4} + 0.211} \tag{17}$$

$$b = (1 + 413.6\, n)^{-0.23} \tag{18}$$

These expressions become equivalent, in the case of $n = 1$, to the friction factor formula of Blasius (1913) for turbulent flow of a Newtonian fluid in a smooth pipe. Eqs. 16-18 can be considered physically reasonable for $0 < n < 2$ and $\mathrm{Re}_{MR,\mathrm{crit}} < \mathrm{Re}_{MR} < 10^5$, which encompass most situations of industrial interest.

Eqs. 16-18 can be combined with Eqs. 10, 11, and 14, to produce a master equation that relates ΔP to v, D, n, and K. By this means, one could perform a fit of experimental data to determine n and K as was done in the laminar flow case. One could also predict ΔP for new turbulent flow situations once n and K are known.

Fig. 3 illustrates predicted ΔP behavior for a power-law fluid in a particular section of pipe. Both the laminar and turbulent regimes are shown, where Eqs. 12 and 16 were respectively used. The transition region was determined from Eq. 15. As expected, pressure drop always increases with flow rate regardless of the type of fluid. The values of n and K and the flow regime (laminar vs. turbulent) determine the degree to which this happens.

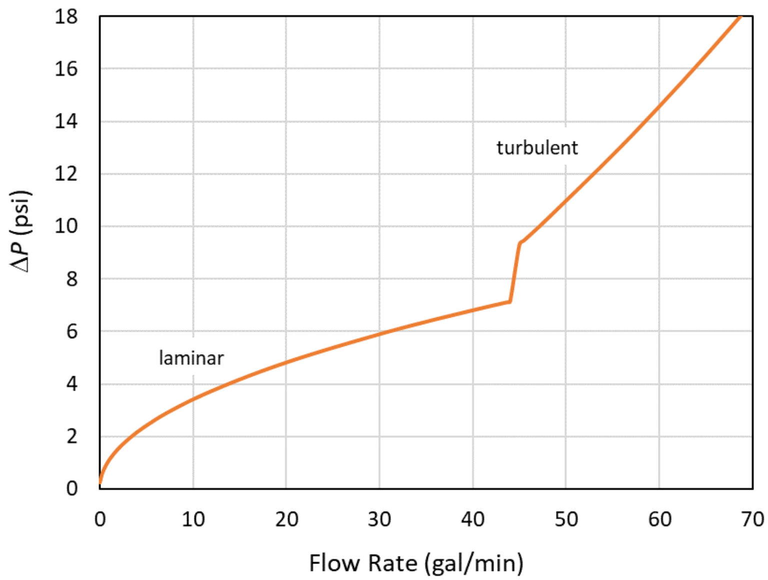

Fig. 3. Predicted behavior of a power-law fluid ($n = 0.5$) in a section of pipe.

References

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