Name:

ECEn 370

Quiz 10 Solutions

Friday, March 19, 2010.

FYI: For iid X_i r.v.s with mean $\mathbf{E}[X]$ and $\text{var}(X)$, where $Y = X_1 + \cdots + X_N$,
$\mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X]$
$\text{var}(Y) = \mathbf{E}[N]\text{var}(X) + (\mathbf{E}[X])^2 \text{ var }(N)$
Markov Inequality: For non-negative R.V. X, $P(X \geq a) \leq \frac{\mathbf{E}[X]}{a}$, for all $a > 0$.
Chebyshev Inequality: $\text{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$

1. You have lightbulbs with a lifetime described by an exponential random variable with parameter λ. You decide to buy either 8 or 9, or 10 of the bulbs with equal probability. You then take your bulbs home and use them sequentially until they all burn out.

a) What is the mean of the total lifetime of the bulbs?

$\mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X] = \frac{1}{\lambda} \cdot 9 = \frac{9}{\lambda}$

b) What is the variance of the total lifetime of the bulbs?

$\text{var}(Y) = \mathbf{E}[N]\text{var}(X) + (\mathbf{E}[X])^2 \text{ var }(N) = 9 \cdot \frac{1}{\lambda^2} + \frac{1}{\lambda^2}(\frac{2}{3}) = \frac{27+2}{3\lambda^2} = \frac{29}{3\lambda^2}$

2. Suppose you arrive at the bus stop and you know that the average waiting time is posted as 5 minutes until the next bus arrives.
a) What is an upper bound on the probability that you wait more than 10 minutes?

$P(X \geq 10) \leq \frac{\mathbf{E}[X]}{10}$ so the upper bound is 1/2.

b) What is a lower bound on the probability that you wait less than 15 minutes?

$P(X \leq 15) = 1 - P(X \geq 15) = 1 - \leq \frac{5}{15} = \geq \frac{2}{3}$

c) Suppose you now know that the standard deviation of the arrival time is 1 minute. What is the lower bound on the probability of the bus arriving between 3 and 7 minutes?

$P(|X - 5| \leq 2) = 1 - P(|X - 5| \geq 2) = 1 - \leq \frac{1}{4} = \geq \frac{3}{4}$

3. Suppose you have a whole bunch of wood poles. Suppose you want to make an estimate of their height by computing the sample mean. You know the standard deviation of the height is 0.01 m. How many pole measurements do you need to take to be 95% confident that your measurement is within 0.01 m of the true mean?

$P(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$ which can be derived from the Chebyshev Inequality above.
$P(|M_n - \mu| \leq \epsilon) = 1 - \left(\leq \frac{\sigma^2}{n\epsilon^2}\right)$
For $\epsilon = 0.01$ m, we need to make sure that $1 - \left(\leq \frac{\sigma^2}{n\epsilon^2}\right) \geq 0.95$.
This occurs when $0.05 = \frac{(0.01)^2}{n(0.01)^2}$, so we nee to take at least $n = 20$ measurements.