Name:

ECEn 370

Quiz 11 Solutions

Friday, March 26, 2010.

FYI: For a uniform random variable, the variance is $\frac{(b-a)^2}{12}$.
For an exponential random variable, the mean is $\frac{1}{\lambda}$ and the variance is $\frac{1}{\lambda^2}$.
Convergence in probability: $\lim_{n\to\infty} \mathrm{P}\left(|Y_n - a| \geq \epsilon\right) = 0$
Convergence in mean square: $\lim_{n\to\infty} \mathrm{E}\left[(X_n - c)^2\right] = 0$

1. You have random variables, $X_1, X_2, \ldots$ which are uniformly distributed over $[0, 1]$. Now suppose you have $Y_1, Y_2, \ldots$ which are given by the formula:

$$Y_n = \frac{X_n}{n} + 1$$

a) Show that Y_n converges to 1 in probability.

First we will find $\mathrm{P}\left(|Y_n - a| \geq \epsilon\right) = \mathrm{P}\left(\left|\frac{X_n}{n} + 1 - 1\right| \geq \epsilon\right) = \mathrm{P}\left(\frac{X_n}{n} \geq \epsilon\right)$

$$\mathrm{P}\left(\frac{X_n}{n} \geq \epsilon\right) = \begin{cases} \int_\epsilon^{1/n} n\,dx, & \epsilon \leq \frac{1}{n} \\ 0, & \epsilon > \frac{1}{n} \end{cases} = \begin{cases} 1 - n\epsilon, & \epsilon \leq \frac{1}{n} \\ 0, & \epsilon > \frac{1}{n} \end{cases}$$

For some $\epsilon > 0$, $\lim_{n\to\infty} \mathrm{P}\left(\frac{X_n}{n} \geq \epsilon\right) = 0$.

b) Show that Y_n converges to 1 in mean square.

$$\mathrm{E}\left[(X_n - c)^2\right] = \mathrm{E}\left[\left(\frac{X_n}{n} + 1 - 1\right)^2\right] = \mathrm{E}\left[\left(\frac{X_n}{n}\right)^2\right] = \frac{1}{n^2}\mathrm{E}\left[X_n^2\right] = \frac{1}{n^2}\left(\mathrm{var}(X_n) + (\mathrm{E}[X])^2\right) = \frac{1}{n^2}\left(\frac{1}{12} + \frac{1}{4}\right) = \frac{1}{3n^2}$$

$$\lim_{n\to\infty} \mathrm{E}\left[(X_n - 1)^2\right] = \lim_{n\to\infty} \frac{1}{3n^2} = 0$$

2. Another pole-ing problem. Suppose that you have a very strange saw mill that cuts the length of poles according to an exponential distribution producing an average length of 10 inches (essentially someone must be just putting the logs through the mill and someone else pulls a lever to cut it at a predefined rate without observing the log and with no other constraints on the process). Use the Central Limit Theorem. You can leave answers in terms of Φ, the CDF of the standard normal random variable.

a) You want to make a tower of sixteen of these poles by stacking them end-to-end. What is the probability that your tower is taller than 164 inches?

$S_n = X_1 + \cdots + X_{16}$ where X_i is an exponential random variable with mean $10 = \frac{1}{\lambda}$. $\lambda = \frac{1}{10}$. $\sigma^2 = 100$.
$z = \frac{c - n\mu}{\sigma\sqrt{n}} = \frac{164 - (16)(10)}{10\sqrt{16}} = \frac{4}{40} = \frac{1}{10}$.
$\mathrm{P}\left(S_n \geq 164\right) = 1 - \mathrm{P}\left(S_n \leq 164\right) \approx 1 - \Phi(\frac{1}{10})$

b) How many of these poles will you need to have a 50% probability that you can reach at least 200 inches?

This is the case when $\Phi(z) = 0.5$ which happens when $z = 0$.
This is when $c = n\mu$, which corresponds to $200 = n \cdot 10$.
This corresponds to $n = 20$ poles.