Name:

ECEn 370

Quiz 12 Solutions

Friday, April 2, 2010.

Binomial: $p_S(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $\quad k = 0, 1, \ldots, n$; $\mathrm{E}[S] = np$; $\mathrm{var}(S) = np(1-p)$
Geometric: $p_T(t) = (1-p)^{t-1}p$, $\quad t = 1, 2, \ldots$; $\mathrm{E}[T] = \frac{1}{p}$; $\mathrm{var}(T) = \frac{1-p}{p^2}$
Pascal: $p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$, $\quad t = k, k+1, \ldots,$
Poisson with λ: $p_Z(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $\quad k = 0, 1, 2, \ldots$; $\mathrm{E}[Z] = \lambda$; $\mathrm{var}(Z) = \lambda$
Poisson with $\lambda\tau$: $p_{N_\tau}(k) = P(k, \tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$, $\quad k = 0, 1, \ldots,$; $\mathrm{E}[N_\tau] = \lambda\tau$; $\mathrm{var}(N_\tau) = \lambda\tau$
Exponential: $f_T(t) = \lambda e^{-\lambda t}$, $\ t \geq 0$; $\mathrm{E}[T] = \frac{1}{\lambda}$; $\mathrm{var}(T) = \frac{1}{\lambda^2}$
Erlang: $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$, $y \geq 0$

1. Imagine that your missionary friend receives mail every day with a probability of 1/5 according to a Bernoulli process.
a) How many days can he expect to receive mail in any 30-day month?
$\mathrm{E}[S] = np = 30 \cdot \frac{1}{5} = 6$

b) How many days is it expected to take until he receives his third mail day?
$\mathrm{E}[Y_3] = \mathrm{E}[T_1 + T_2 + T_3] = \mathrm{E}[T_1] + \mathrm{E}[T_2] + \mathrm{E}[T_3] = 3 \cdot 5 = 15$

c) Given that the missionary has not received mail for two weeks, how many days is it expected until he receives a mail day?
$\mathrm{E}[T] = \frac{1}{p} = 5$

d) What is the probability of receiving the third mail day on day 5?
$p_{Y_3}(5) = \binom{5-1}{3-1} p^3 (1-p)^{5-3} = \binom{4}{2} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2 = 6 \cdot \frac{16}{5^5} \approx 0.0307$

e) Given that the missionary "knows" that he will receive at least one mail day in the next two days, what is the probability that he will receive mail on the second day?
Let B be the event that the missionary receives at least one mail day in the next three days.
$P(B) = 1 - P\,(\text{no mail days}) = 1 - \binom{2}{0}\left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$
$P\left(X_2|B\right) = \frac{P(X_2 \cap B)}{P(B)} = \frac{\frac{1}{5}}{\frac{9}{25}} = \frac{5}{9}$

2. Another polling problem - or pole-cutting. Imagine that you have a process where you have a feed of wood into your cutting machine at a constant rate of 10 meters per minute. Now you cut the wood according to a Poisson process with a rate of 2 cuts/minute.
a) What is the average length of wood that you cut?
The arrival times are exponential with parameter λ. $\mathrm{E}[T] = \frac{1}{\lambda} = \frac{1}{2}$ minutes. This translates into 5 meter poles.

b) What is the probability of making 3 cuts in one minute?
$P(3, 1) = e^{-2(1)}\frac{(2 \cdot 1)^3}{3!} = e^{-2}\frac{8}{6} = \frac{4}{3}e^{-2}$

c) What is the probability of randomly selecting three poles and the length totaling less than 16 meters? (You can leave it in integral form)
This corresponds to $Y_3 = T_1 + T_2 + T_3$ and an Erlang random variable where $P(Y_3 \leq 1.6)$.
$P(Y_3 \leq 1.6) = \int_0^{1.6} \frac{2^3 y^2 e^{-2y}}{(3-1)!} = \int_0^{1.6} \frac{2^3 y^2 e^{-2y}}{(3-1)!} = \int_0^{1.6} 4y^2 e^{-2y} \approx 0.6201$