ChEn 374
Fluid Mechanics

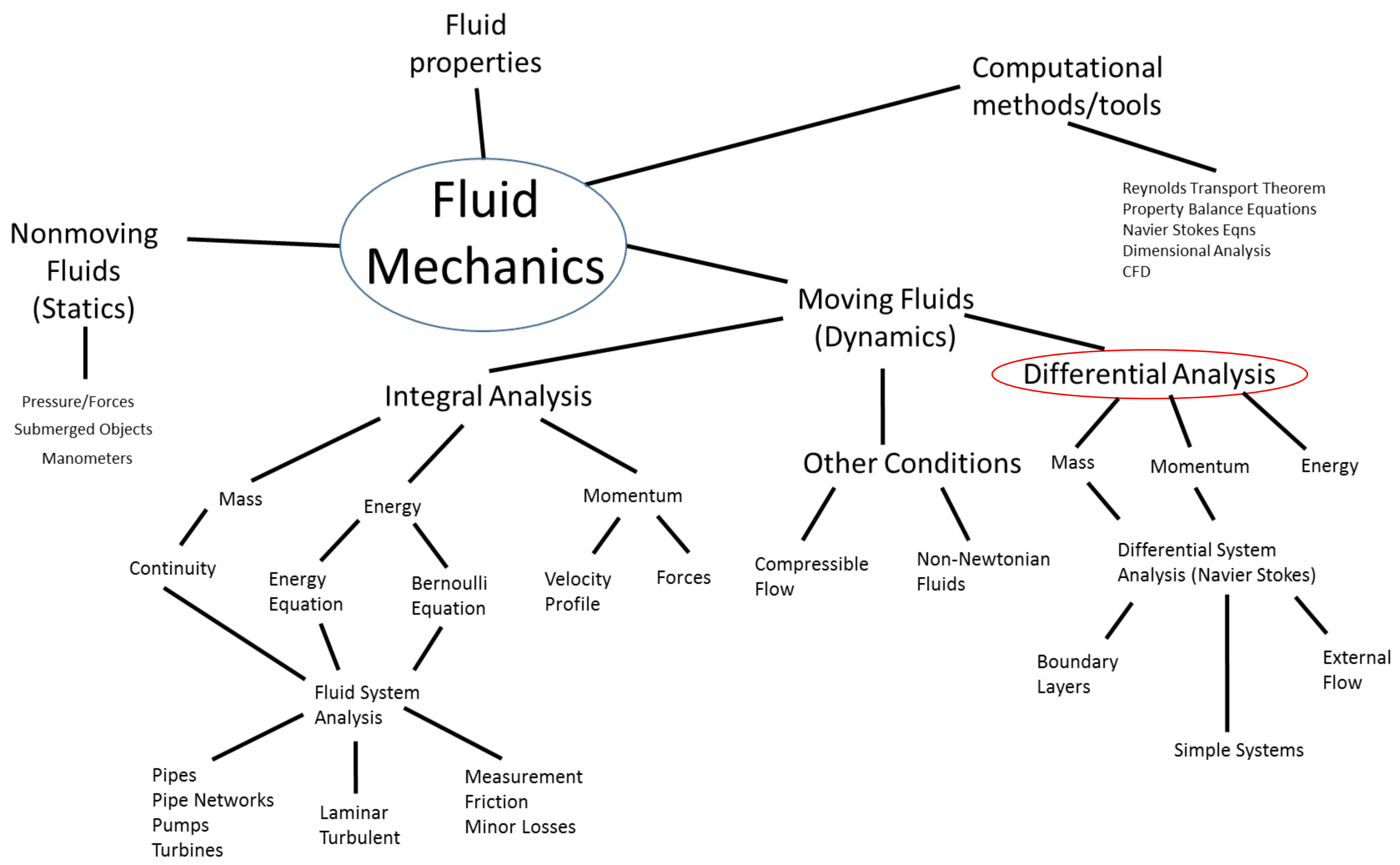
Differential Balances

Spiritual Thought

THE FAMILY is ordained of God. Marriage between man and woman is essential to His eternal plan. Children are entitled to birth within the bonds of matrimony, and to be reared by a father and a mother who honor marital vows with complete fidelity. Happiness in family life is most likely to be achieved when founded upon the teachings of the Lord Jesus Christ. Successful marriages and families are established and maintained on principles of faith, prayer, repentance, forgiveness, respect, love, compassion, work, and wholesome recreational activities. By divine design, fathers are to preside over their families in love and righteousness and are responsible to provide the necessities of life and protection for their families. Mothers are primarily responsible for the nurture of their children. In these sacred responsibilities, fathers and mothers are obligated to help one another as equal partners. Disability, death, or other circumstances may necessitate individual adaptation. Extended families should lend support when needed.

The Family, A Proclamation to the World

Fluids Roadmap



Key Points

- Previous work – Integral From (RTT)
  - Bulk (average) properties
- Differential – give complete flow field
- Mass

$$-\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0$$

  - Impact of SS, const. ρ, uniform; coordinate systems

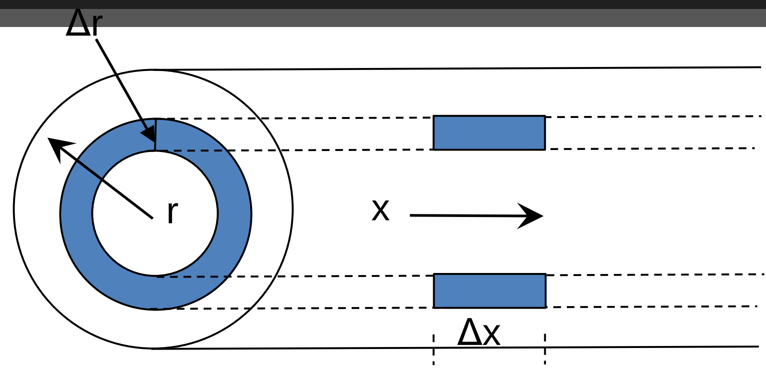
- Momentum

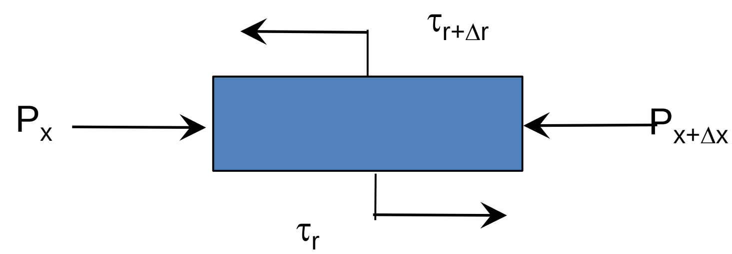
$$-\frac{\partial \rho\vec{v}}{\partial t} + \rho\vec{v} \cdot \nabla\vec{v} = -P + \rho\vec{g} - \nabla \cdot \bar{\bar{\vec{\tau}}}$$

Differential Form

- RTT for control volume (Eulerian View)
  - Net Mass
  - Net Force
  - Net Density
- Differential balance
  - Flow field
  - Start with integral and shrink CV to dV
    - Did this with laminar friction derivation…

Laminar Pipe Flow





Force Balance: Pressure, stress

$$\overbrace{(P_x - P_{x+\Delta x})(2\pi r \Delta r)} + \overbrace{(2\pi \Delta x r)}^{\text{Moving Fluid Shear Stress}}\tau_r - \underbrace{(2\pi \Delta x)(r+\Delta r)\tau_{r+\Delta r}}_{\text{Non-Moving Fluid (Resistive) Shear Stress}} = 0$$

Divide $2\pi\Delta r\Delta x$

$$r\frac{P_x - P_{x+\Delta x}}{\Delta x} + \frac{r\tau_r - (r+\Delta r)\tau_{r+\Delta r}}{\Delta r} = 0$$

Limit Δx, $\Delta r \rightarrow 0$

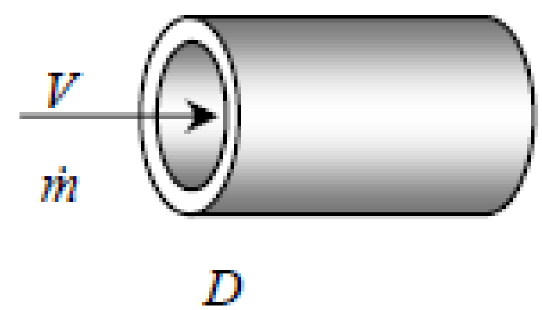
$$-\frac{dP}{dx} = \frac{1}{r}\frac{d(r\tau)}{dr} = C$$

Separate variables and integrate with τ=0 at r=0

$$\tau = -\frac{r}{2}\frac{dP}{dx}$$

Laminar Friction Derivation

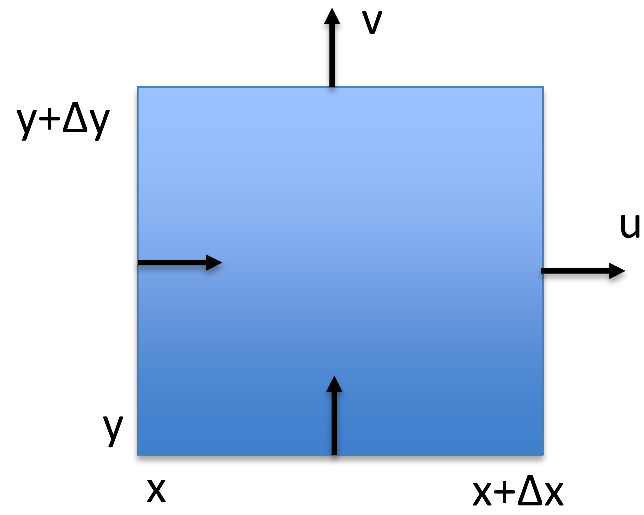
- $\frac{dP}{dx} = \frac{1}{r}\frac{d(\tau r)}{dr}$ == constant
- Solve for τ
- $\tau = -\frac{dP}{dx} \cdot \frac{r}{2}$, evaluate at wall:
- $\tau_w = -\frac{dP}{dx} \cdot \frac{R}{2} = -\frac{\Delta P}{L} \cdot \frac{D}{4}, \quad \therefore 4\tau_w = \frac{-\Delta P D}{L}$
- Insert sheer stress expression, $\tau = -\mu \frac{du}{dr}$:
- $-\mu \frac{du}{dr} = -\frac{dP}{dx} \cdot \frac{r}{2}$, solve for u, (bounds: r to R & 0 to u):
- $\boxed{\mathrm{u(r)} = \frac{-R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)}$, Laminar velocity profile,



Mass Balance (method 1)

- $0 = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \vec{v} \cdot \vec{n} dA$
  - Gauss Divergence Theorem:
    - $\int_A \vec{v} \cdot \vec{n} dA = \int_V \nabla \cdot \vec{v} dV$
    - $\frac{d}{dt}\int_{CV} \rho dV + \int_{CV} \nabla \cdot (\rho\vec{v}) dV$=0
  - Constant Control volume:
    - $\int_{CV} \frac{d\rho}{dt} dV + \int_{CV} \nabla \cdot (\rho\vec{v}) dV$=0
    - $\int_{CV} \left(\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{v})\right) dV$=0
  - $\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{v})$=0

Mass Balance (Method 2)



- Accum = in – out + gen
- $\frac{\partial(\rho \Delta x \Delta y)}{\partial t} = (\rho u \Delta y)_x$- $(\rho u \Delta y)_{x+\Delta x} + (\rho v \Delta x)_y$-$(\rho v \Delta x)_{y+\Delta y}$
- Divide by $\Delta x, \Delta y$
- $\frac{\partial \rho}{\partial t} = \frac{(\rho u)_x - (\rho u)_{x+\Delta x}}{\Delta x} + \frac{(\rho v)_y - (\rho v)_{y+\Delta y}}{\Delta y}$
- $Lim\ \Delta x, \Delta y$→0

Method 2 (cont.)

- $\frac{d\rho}{dt} = -\frac{d\rho u}{dx} - \frac{d\rho v}{dy} \rightarrow \frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} = 0$
- $\frac{\partial\rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0$

- Simplifications:
  - S.S. $\rightarrow$ $\nabla \cdot \rho\vec{v} = 0$
  - S.S. and uniform ρ $\rightarrow$ $\nabla \cdot \vec{v} = 0$
  - Const. ρ $\rightarrow$ $\nabla \cdot \vec{v} = 0$

Coordinate Systems

Cartesian

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Cylindrical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial r \rho u_r}{\partial r} + \frac{1}{r}\frac{\partial \rho u_\theta}{\partial \theta} + \frac{\partial \rho u_z}{\partial z} = 0$$

Spherical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u)}{\partial r} + \frac{1}{r sin\theta}\frac{\partial (\rho u_\theta sin\theta)}{\partial \theta} + \frac{1}{r sin\theta}\frac{\partial u_\emptyset}{\partial \emptyset} = 0$$

Continuity

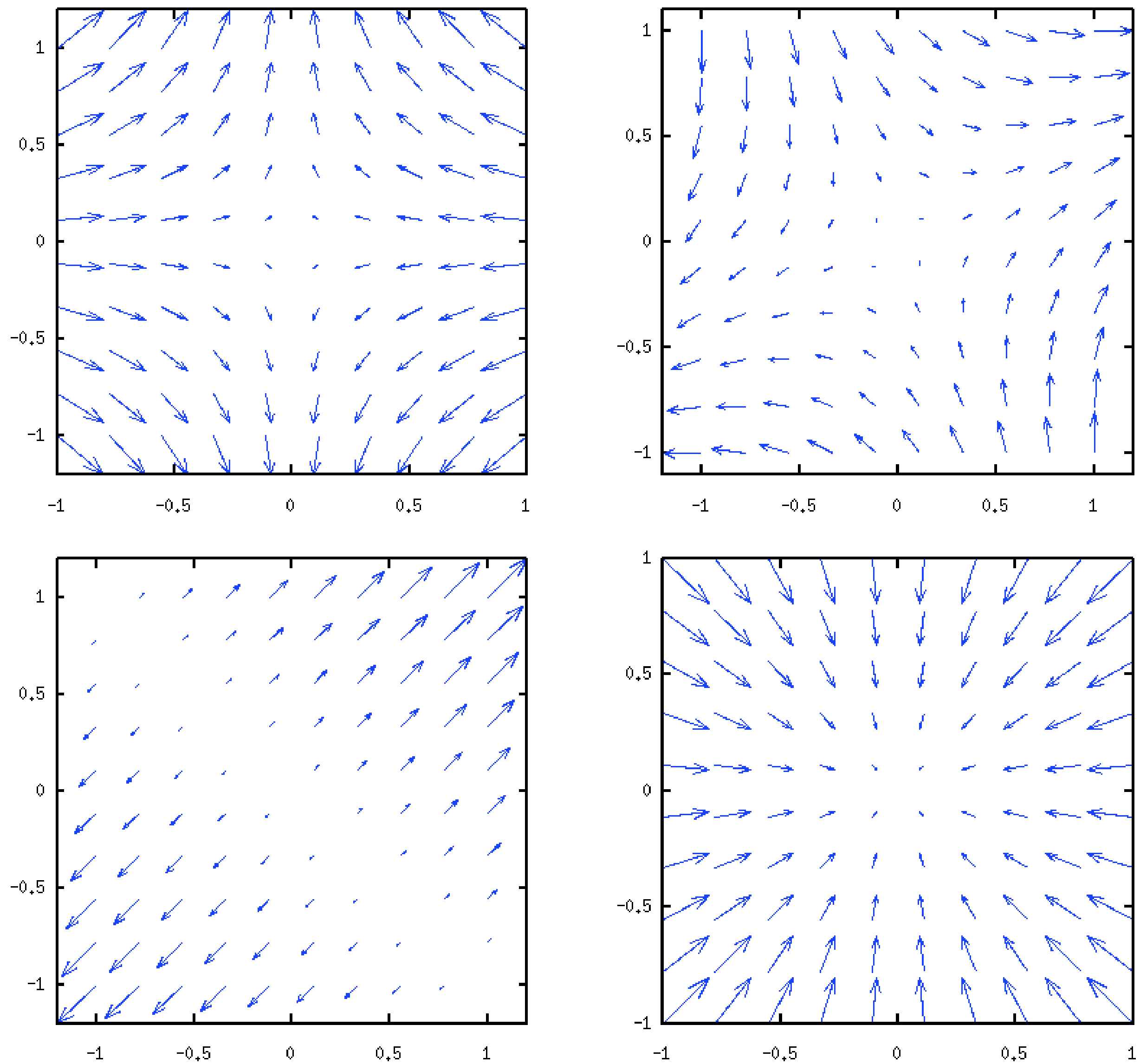
- Does not SOLVE for velocity!
  - Just "constrains" it….
    - Tells us what conditions must occur for it to work
- In order to find velocity, we need to do a differential momentum balance!
  - Momentum balance along isn't enough
  - Too many unknowns
  - Momentum/mass balances done together
  - Energy gives Temperature field – Next semester

Example (mass balance)

- 2-D flow, S. S. , constant ρ; u=u(x,y), v=v(x,y)
- u = ax+by, v = cx+dy,

What are the constraints on a, b, c, and d?

$$\nabla \cdot \vec{v} = 0$$


1
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-0.5
-1
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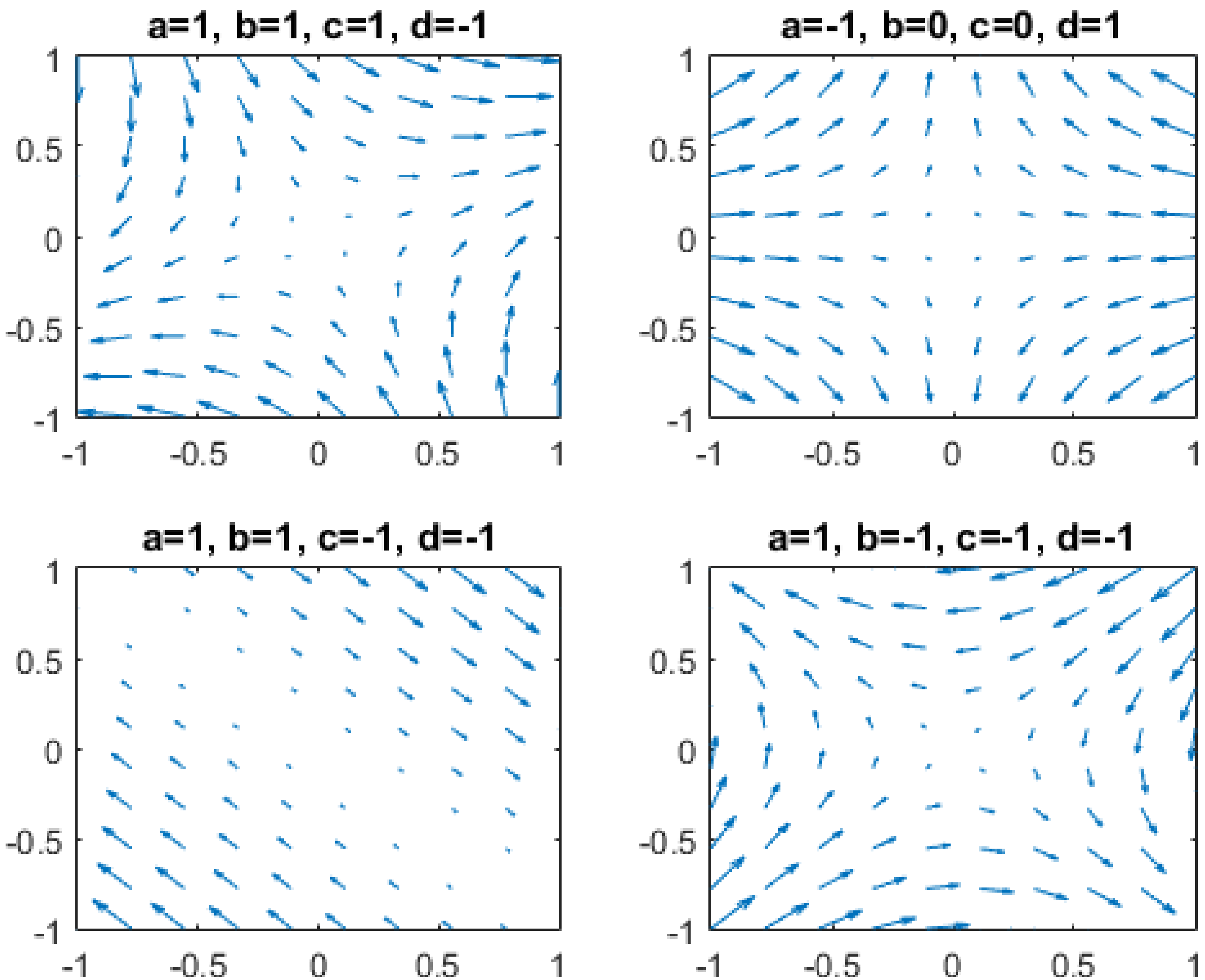
1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a=-1, b=0, c=0, d=1
a=1, b=1, c=-1, d=-1
a=1, b=-1, c=-1, d=-1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Momentum Balance

- Integral:
  - $\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{v}dV + \int_{CS} \rho\vec{v}\,\vec{v}\cdot\vec{v}dA$
    - Gauss divergence Theorem: $\int_{CV} \nabla(\rho\vec{v}v)\,dV$
  - Now find expression for forces:
    - $Gravity = \int_{CV} \rho\vec{g}dV$
    - Pressure =$-\int_{CV} \nabla P\,dV$ (surface F, used GDT)
    - Viscous = $-\int_{CV} \nabla\,\bar{\bar{\tau}}dV$ (surface f, used GDT)
  - $\frac{\partial\rho v}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P - \nabla\bar{\bar{\tau}} + \rho\vec{g}$
  - $\rho\frac{\partial v}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P - \nabla\bar{\bar{\tau}} + \rho\vec{g}$

Simplify

- Inviscid: $\nabla\bar{\bar{\tau}}=0$
- Const ρ: $\nabla\bar{\bar{\tau}} = -\mu\nabla^2\vec{v}$
- Reduce Dimensions: u(y), v = w = 0

Mass

$$\nabla \cdot \vec{v} = 0$$

Momentum

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g} - \nabla \cdot \tau$$

Mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-Mom

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Y-Mom

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Z-Mom

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$