ChEn 374
Fluid Mechanics

Navier Stokes Equations

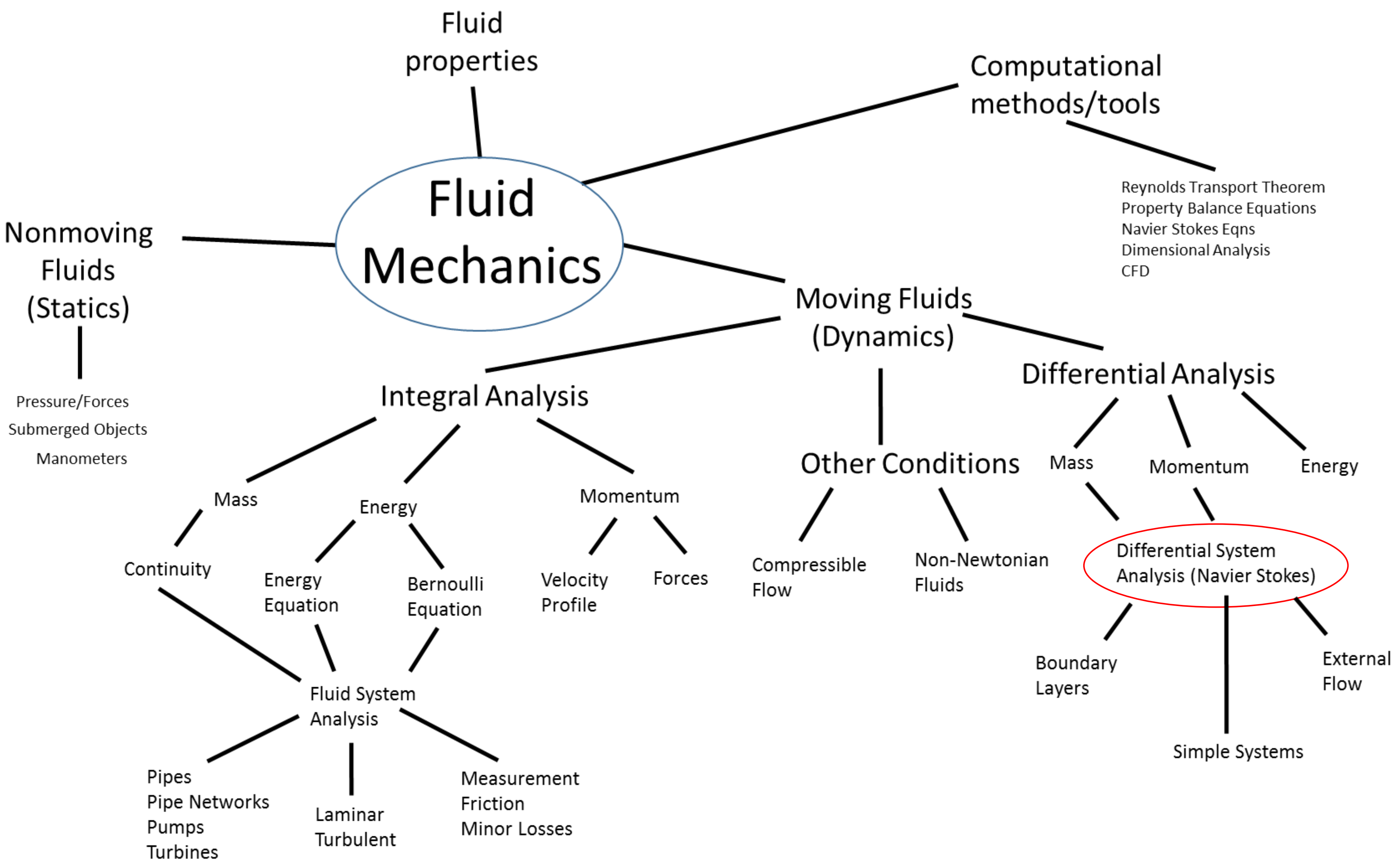
Spiritual Thought

1 Nephi 1:4-5

4 For it came to pass in the commencement of the first year of
the reign of Zedekiah, king of Judah, (my father, Lehi, having
dwelt at Jerusalem in all his days); and in that same year there
came many prophets, prophesying unto the people that they
must repent, or the great city Jerusalem must be destroyed.

5 Wherefore it came to pass that my father, Lehi, as he went
forth prayed unto the Lord, yea, even with all his heart, in behalf
of his people.

Fluids Roadmap



Key Points

- Potential Simplifications
  - SS, 1-D, constant properties, etc.
- Boundary Conditions
  - Known v
  - Symmetry (e.g. $\frac{\partial v}{\partial x} = 0$)
  - No slip (v = 0) at walls
- Examples (PRACTICE!!!)

Mass $$\nabla \cdot \vec{v} = 0$$

Momentum $$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g} - \nabla \cdot \tau$$

Mass $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-Mom $$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Y-Mom $$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Z-Mom $$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

How to deal with $\nabla \cdot \bar{\bar{\tau}}$?

- Inviscid Flow: $\nabla \cdot \bar{\bar{\tau}} = 0$ (Euler Equations)
  - Valid far from walls and obstructions
- Constant ρ: $\nabla \cdot \bar{\bar{\tau}}$ = -μ $\nabla^2 \vec{v}$
  - Reasonable for incompressible flow
    - Newtonian flow is implied… why?
- Reduce dimensions of the problem:
- Pipe Flow = u(y), v = w = 0

Problem Setup (SIMPLIFY!!!)

- Boundary Conditions:
  - Given velocity at inlet, outlet flow field
  - Symmetry points: pipe centerlines, etc. ($\frac{dv}{dx} = 0$)
  - v = 0 at walls (no slip condition)
- Initial Conditions (v field given at t = 0)

- Use these to transform N.S. into usable ODE

Big Picture Moment…

- Integral Balances
  - Energy, Mass & Momentum
- Differential Balances
  - Energy (Next Semester)
  - Mass & Momentum – Navier Stokes
    - Only 3 ways to solve Navier Stokes Eq.
      1. Simplify the problem until it’s very basic (Today)
      2. Boundary Layer Approximation (Wednesday)
      3. CFD (Last week of Lectures)

Navier-Stokes: Method 1

- New and Difficult… PRACTICE!!
- Don't take shortcuts!!! Easy to get lost
- Methodology to maximize points (on tests)

1) SS
2) Constant ρ
3) C.E. result

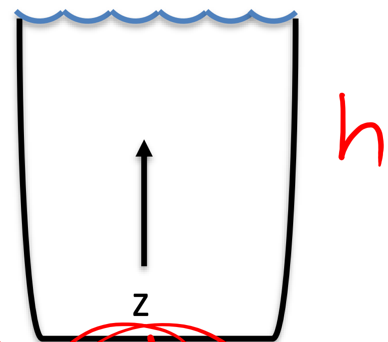
1. List and number all assumptions
2. Write continuity equation
   - Reduce continuity equation using assumptions
   - Attach a number to the result
3. Write Navier Stokes for non-main direction(s)
   - Eliminate terms using assumptions (number)
4. * Write Navier Stokes for main direction P
   - Eliminate terms using assumptions (number)
5. Solve Remaining Equation with BC's

Example 1

- Barometric Equation (no velocity, just a fluid)

- $\cancel{\rho \frac{\partial \vec{v}}{\partial t}}^{\,0} + \cancel{\rho\vec{v}\cdot\nabla\vec{v}}^{\,0} = -\nabla P - \nabla\bar{\bar{\tau}} + \rho\vec{g}$

$-\mu\cancel{\nabla^2\vec{v}}^{\,0}$



$-\nabla P = -\rho g$

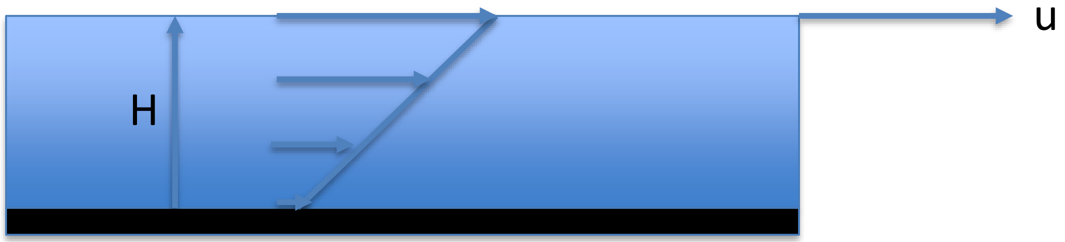
$\cancel{\frac{dP}{dx}} + \cancel{\frac{dP}{dy}} + \frac{dP}{dz}$

- $\nabla P = \rho g$ $\quad \underline{\frac{dP}{dz} = -\rho g}$
- $\underline{P_2 - P_1} = \underline{\rho(z_2 - z_1)g}$

Couette Flow (example)

Example 2

- Couette Flow



- $\rho \frac{\partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$
- Boundary Conditions:
  - u = 0 at y = 0
  - U = U at y = H
- Constant ρ, μ
- 1-D Flow (only in x-direction)

Example 2 (cont)

- $\rho\frac{\partial v}{\partial t}+\rho\vec{v}\cdot\nabla\vec{v}=-\nabla P-\nabla\bar{\bar{\tau}}+\rho\vec{g}$

Material Derivative

$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}$

$\nabla\bar{\bar{\tau}}$

- Only x-direction:
- $\frac{Du}{Dt}=-\frac{1}{\rho}\nabla P+\frac{\mu}{\rho}\nabla^2\vec{v}+\vec{g}$
- $\cancel{\frac{Du}{Dt}}=-\cancel{\frac{1}{\rho}\nabla P}+\nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)+\cancel{g_x}$
- $\frac{\partial^2 u}{\partial y^2}=0$ ⇒ linear

$u=ay+b$

$y=0 \quad u=0$

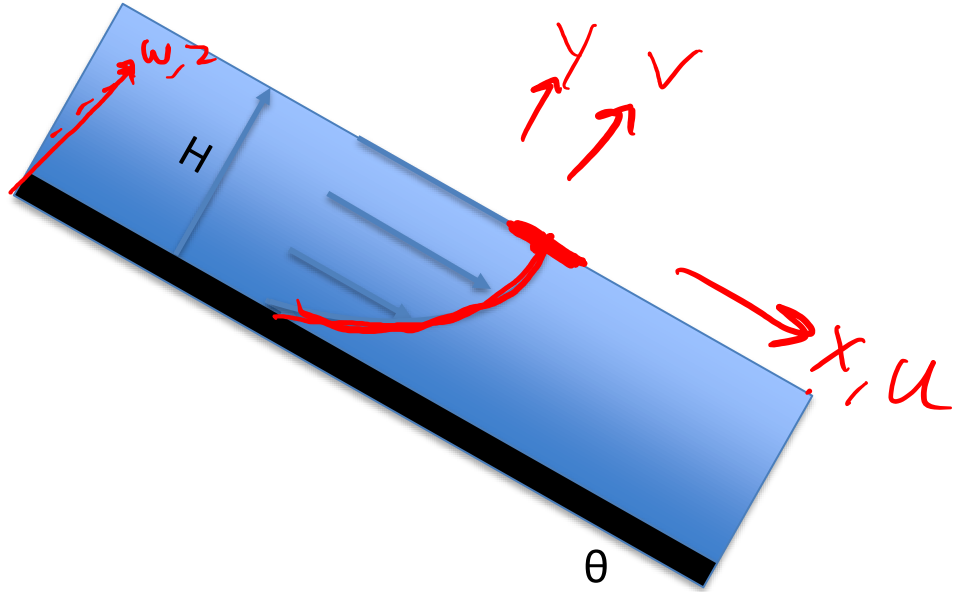
$y=H \quad u=U$

$b=0 \quad a=\frac{U}{H}$

$u=\frac{U}{H}y$

Example 3

- Flow down incline
- Steady
- Laminar
- 1-D (y-dir)
- v = w= 0
- u(0) = 0
- $\left.\frac{du}{dy}\right]_{y=H} = 0$
- $\frac{Du}{Dt} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\vec{v} + \vec{g}$



Example 3 (cont.)

- $\cancel{\frac{Du}{Dt}}^{0} = -\cancel{\frac{1}{\rho}\nabla P}^{0} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) + g_x$

$g_x = g\cos\theta$

- $\frac{\partial^2 u}{\partial y^2} = -\frac{g_x}{\nu} = -\frac{g}{\nu}\cos\theta$

$\rightarrow \int d^2u = \int \frac{-g}{\nu}\cos\theta \, dy\, dy$

$du = \left(-\frac{g}{\nu}\cos\theta(y + C_1)\right)dy$

$\frac{du}{dy} = -\frac{g}{\nu}\cos\theta(y + C_1)$

$@\ y = H,\ \frac{du}{dy} = 0$

$\therefore C_1 = -H$

$\int du = \int \frac{-g}{\nu}\cos\theta\,(y - H)\,dy$

$u = \frac{-g}{\nu}\cos\theta\left(\frac{y^2}{2} - yH + C_2\right)$

$@\ y = 0,\ u = 0 \rightarrow C_2 = 0$

$\boxed{u = \frac{-g\cos\theta}{\nu}\left(\frac{y^2}{2} - yH\right)}$