ChEn 374
Fluid Mechanics

Differential Balances

Spiritual Thought

D&C 89:18-20

18 And all saints who remember to keep and do these sayings, walking in obedience to the commandments, shall receive health in their navel and marrow to their bones;

19 And shall find wisdom and great treasures of knowledge, even hidden treasures;

20 And shall run and not be weary, and shall walk and not faint.

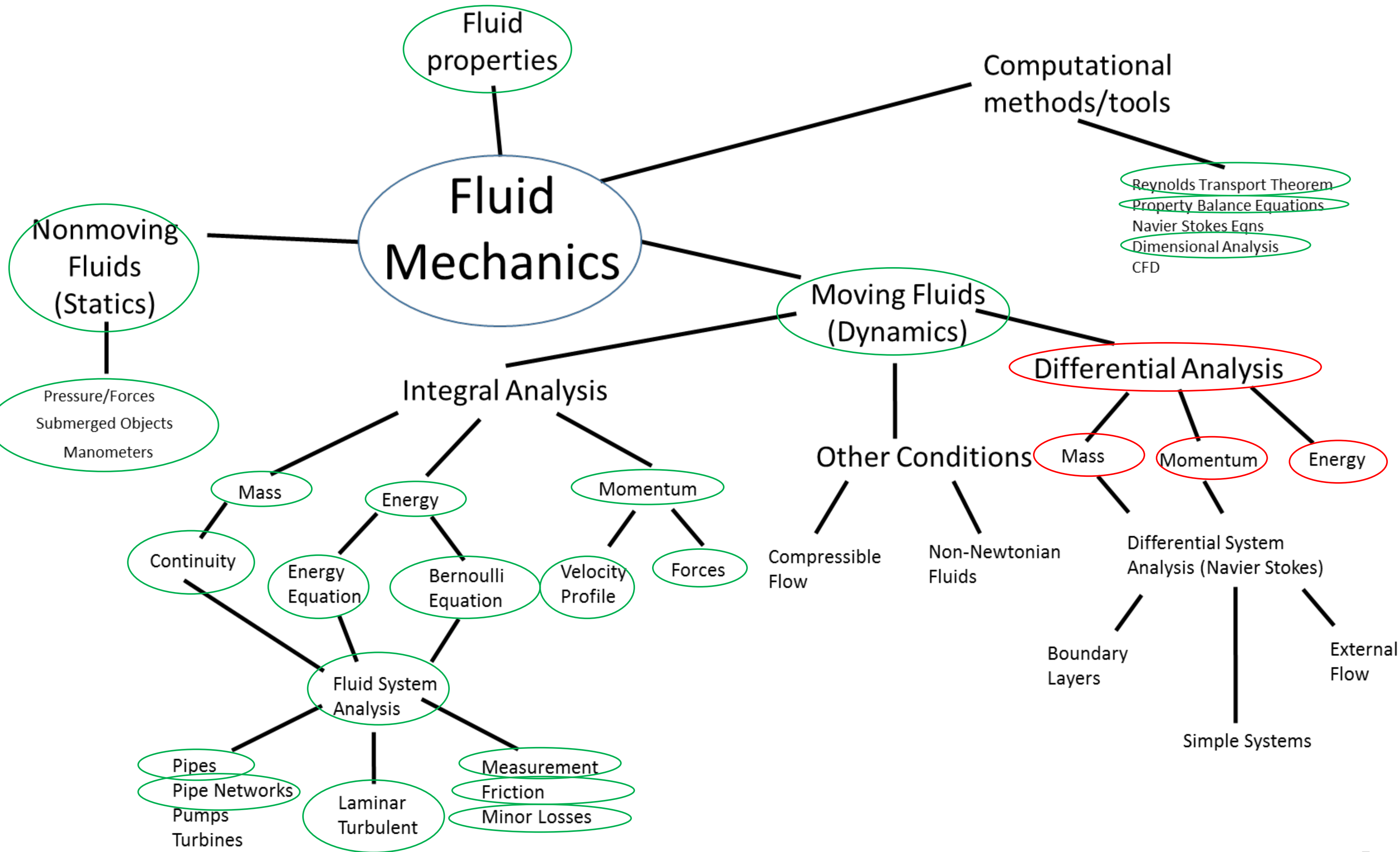
OEP # 7 Clip

OEP #7

Open Ended Problem #7
The REAL castaway
GROUP WORK OKAY, Due 11/6/24 at beginning of class

After 18 days on Mars (well, actually 18 sol, but it's close enough), a massive storm forces the abortion of the first manned mission to mars. Astronaut Mark Watney is unfortunately standing in the wrong place at the wrong time, and he is smashed out of sight by a communications array that was knocked off its structure by the wind. Apparently the storm wind force of 8600 N was enough to knock this array loose. What is the wind speed required (on the Martian planet) in order to create 8.6 kN of force on the communications array dish? (hint – we're talking about a force caused by fluids here…)

Fluids Roadmap



Key Points

- Previous work – Integral From (RTT)
  - Bulk (average) properties
- Differential – give complete flow field
- Mass

$$-\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0$$

  - Impact of SS, const. ρ, uniform; coordinate systems

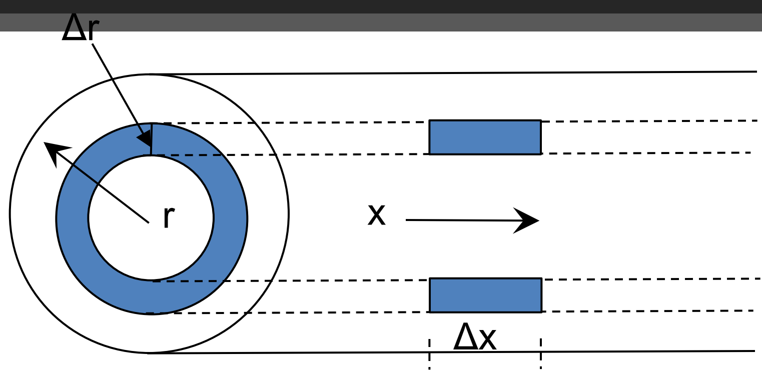
- Momentum

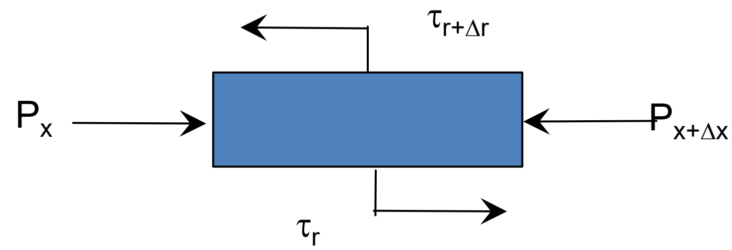
$$-\frac{\partial \rho\vec{v}}{\partial t} + \rho\vec{v} \cdot \nabla\vec{v} = -P + \rho\vec{g} - \nabla \cdot \bar{\bar{\tau}}$$

Differential Form

- RTT for control volume (Eulerian View)
  - Net Mass
  - Net Force
  - Net Density
- Differential balance
  - Flow field
  - Start with integral and shrink CV to dV
    - Did this with laminar friction derivation…

Laminar Pipe Flow





Force Balance: Pressure, stress

Moving Fluid Shear Stress

$$(P_x - P_{x+\Delta x})(2\pi r \Delta r) + (2\pi \Delta x r)\tau_r - (2\pi \Delta x)(r + \Delta r)\tau_{r+\Delta r} = 0$$

Non-Moving Fluid (Resistive) Shear Stress

Divide $2\pi\Delta r\Delta x$

$$r\frac{P_x - P_{x+\Delta x}}{\Delta x} + \frac{r\tau_r - (r+\Delta r)\tau_{r+\Delta r}}{\Delta r} = 0$$

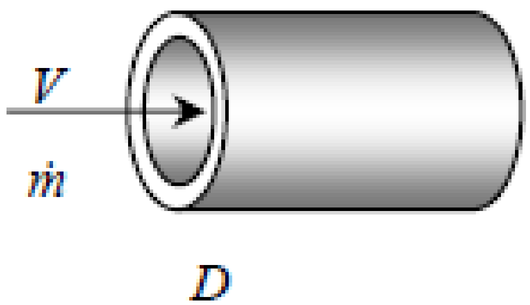
Limit Δx, $\Delta r \rightarrow 0$

$$-\frac{dP}{dx} = \frac{1}{r}\frac{d(r\tau)}{dr} = C$$

Separate variables and integrate with τ=0 at r=0 $\quad \tau = -\frac{r}{2}\frac{dP}{dx}$

Laminar Friction Derivation

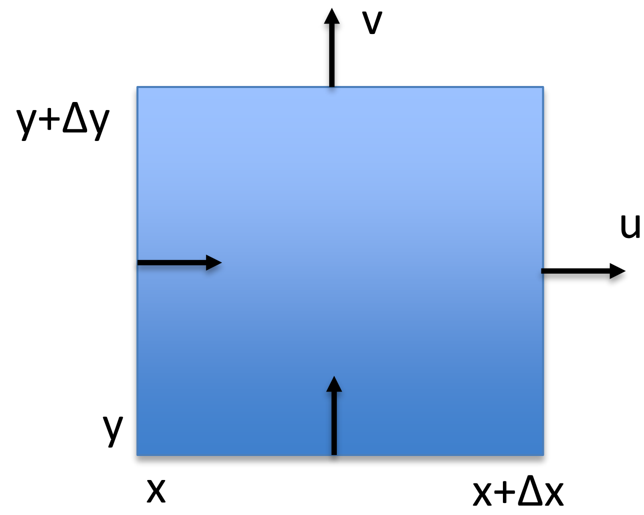
- $\frac{dP}{dx} = \frac{1}{r}\frac{d(\tau r)}{dr}$ == constant
- Solve for τ
- $\tau = -\frac{dP}{dx} \cdot \frac{r}{2}$, evaluate at wall:
- $\tau_w = -\frac{dP}{dx} \cdot \frac{R}{2} = -\frac{\Delta P}{L} \cdot \frac{D}{4}, \quad \therefore 4\tau_w = \frac{-\Delta P D}{L}$
- Insert sheer stress expression, $\tau = -\mu \frac{du}{dr}$:
- $-\mu \frac{du}{dr} = -\frac{dP}{dx} \cdot \frac{r}{2}$, solve for u, (bounds: r to R & 0 to u):
- $u(r) = \frac{-R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)$, Laminar velocity profile,



Mass Balance (method 1)

- $0 = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho\vec{v} \cdot \vec{n} dA$
  - Gauss Divergence Theorem:
    - $\int_A \vec{v} \cdot \vec{n} dA = \int_V \nabla \cdot \vec{v} dV$
    - $\frac{d}{dt}\int_{CV} \rho dV + \int_{CV} \nabla \cdot (\rho\vec{v}) dV = 0$
  - Constant Control volume:
    - $\int_{CV} \frac{d\rho}{dt} dV + \int_{CV} \nabla \cdot (\rho\vec{v}) dV = 0$
    - $\int_{CV} \left(\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{v})\right) dV = 0$
  - $\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{v}) = 0$

Mass Balance (Method 2)



- Accum = in – out + gen
- $\frac{\partial(\rho\Delta x\Delta y)}{\partial t} = (\rho u\Delta y)_x - (\rho u\Delta y)_{x+\Delta x} + (\rho v\Delta x)_y - (\rho v\Delta x)_{y+\Delta y}$
- Divide by $\Delta x, \Delta y$
- $\frac{\partial \rho}{\partial t} = \frac{(\rho u)_x - (\rho u)_{x+\Delta x}}{\Delta x} + \frac{(\rho v)_y - (\rho v)_{y+\Delta y}}{\Delta y}$
- $Lim\ \Delta x, \Delta y \rightarrow 0$

Method 2 (cont.)

- $\frac{d\rho}{dt} = -\frac{d\rho u}{dx} - \frac{d\rho v}{dy} \rightarrow \frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} = 0$
- $\frac{\partial\rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0$

- Simplifications:
  - S.S. → $\nabla \cdot \rho\vec{v} = 0$
  - S.S. and uniform ρ → $\nabla \cdot \vec{v} = 0$
  - Const. ρ → $\nabla \cdot \vec{v} = 0$

Coordinate Systems

Cartesian

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Cylindrical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho u_r}{\partial r} + \frac{1}{r}\frac{\partial \rho r u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Spherical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u)}{\partial r} + \frac{1}{r sin\theta}\frac{\partial (\rho u_\theta sin\theta)}{\partial \theta} + \frac{1}{r sin\theta}\frac{\partial u_\emptyset}{\partial \emptyset} = 0$$

Continuity

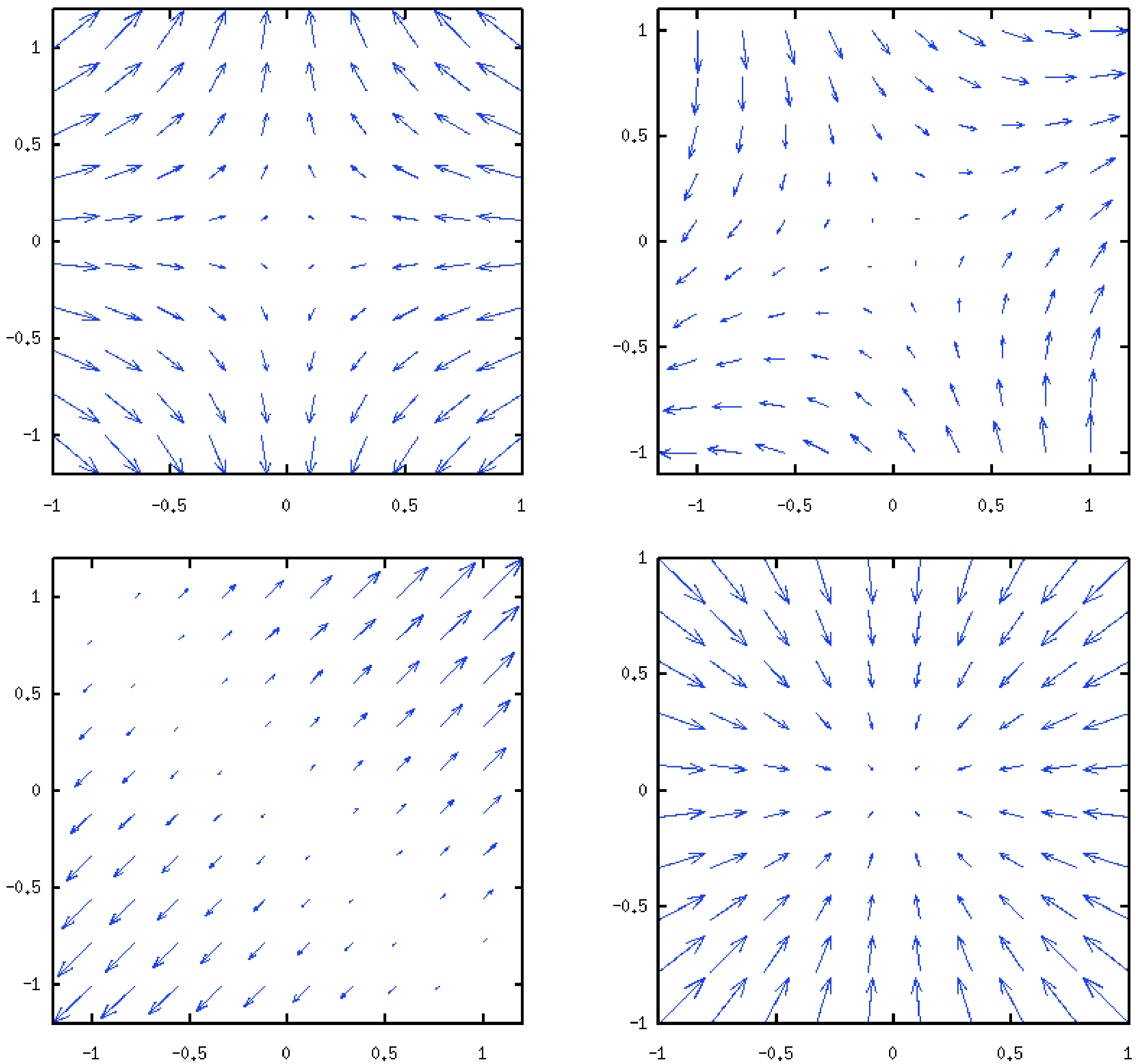
- Does not SOLVE for velocity!
  - Just “constrains” it….
    - Tells us what conditions must occur for it to work
- In order to find velocity, we need to do a differential momentum balance!
  - Momentum balance alone isn’t enough
  - Too many unknowns
  - Momentum/mass balances done together
  - Energy gives Temperature field – Next semester

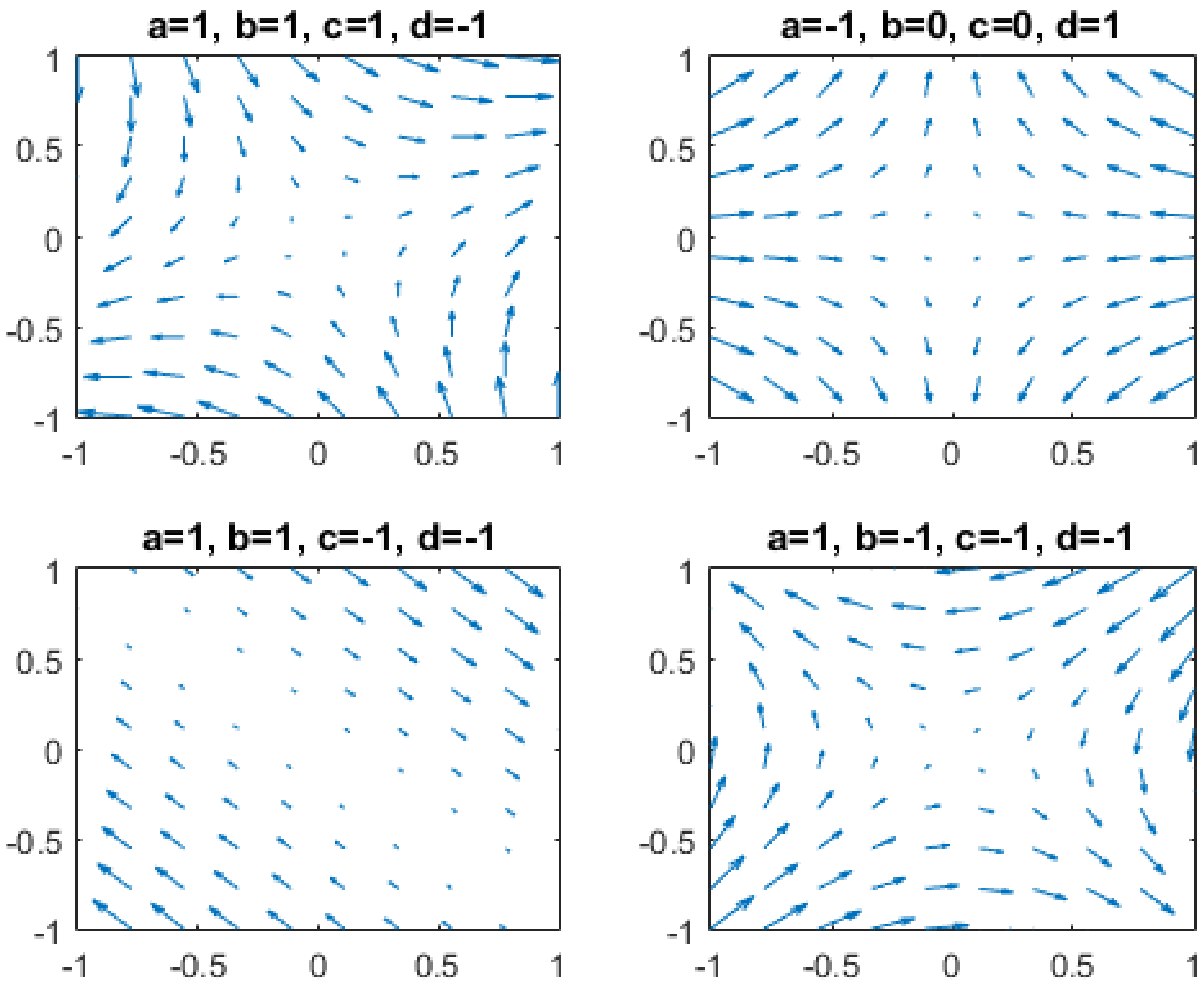
Example (mass balance)

- 2-D flow, S. S. , constant ρ; u=u(x,y), v=v(x,y)
- u = ax+by, v = cx+dy,

What are the constraints on a, b, c, and d?

$$\nabla \cdot \vec{v} = 0$$


1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1


a=1, b=1, c=1, d=-1
a=-1, b=0, c=0, d=1
a=1, b=1, c=-1, d=-1
a=1, b=-1, c=-1, d=-1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Momentum Balance

- Integral:
  - $\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}\, \vec{v} \cdot \vec{v} dA$
    - Gauss divergence Theorem: $\int_{CV} \nabla(\rho \vec{v} v)\, dV$
  - Now find expression for forces:
    - $Gravity = \int_{CV} \rho \vec{g} dV$
    - Pressure $= -\int_{CV} \nabla P\, dV$ (surface F, used GDT)
    - Viscous = $-\int_{CV} \nabla\, \bar{\bar{\tau}} dV$ (surface f, used GDT)
  - $\frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$
  - $\rho \frac{\partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$

Simplify

- Inviscid: $\nabla \bar{\bar{\tau}}=0$
- Const ρ: $\nabla \bar{\bar{\tau}} = -\mu \nabla^2 \vec{v}$
- Reduce Dimensions: u(y), v = w = 0

Mass $$\nabla \cdot \vec{v} = 0$$

Momentum $$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g} - \nabla \cdot \tau$$

Mass $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-Mom $$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Y-Mom $$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Z-Mom $$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$