ChEn 374
Fluid Mechanics

Navier Stokes Equations

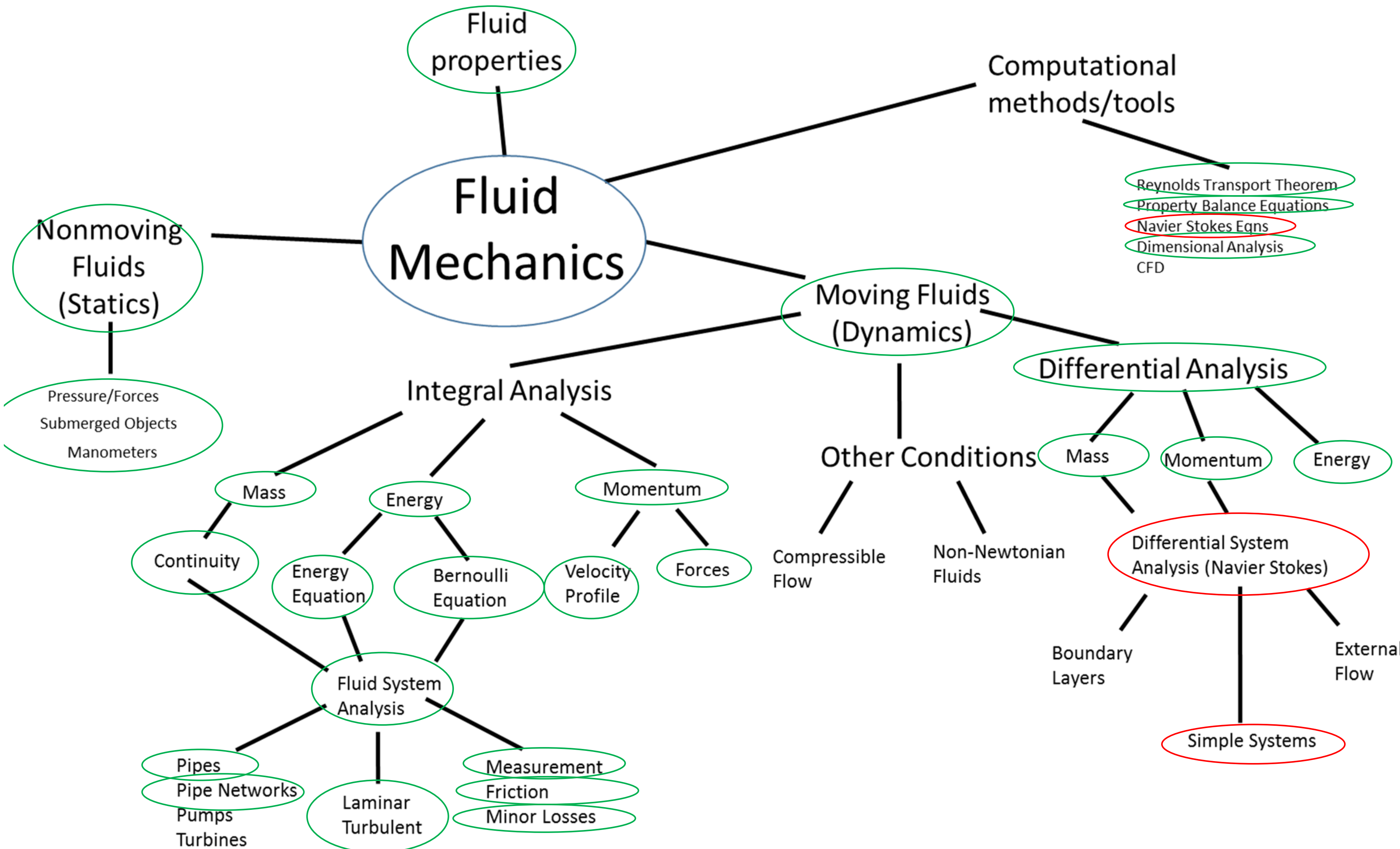
Spiritual Thought

I wish today to speak of forgiveness. I think it may be the greatest virtue on earth, and certainly the most needed. There is so much of meanness and abuse, of intolerance and hatred. There is so great a need for repentance and forgiveness. It is the great principle emphasized in all of scripture, both ancient and modern...

May God help us to be a little kinder, showing forth greater forbearance, to be more forgiving, more willing to walk the second mile, to reach down and lift up those who may have sinned but have brought forth the fruits of repentance, to lay aside old grudges and nurture them no more.

–President Gordon B. Hinkley

Fluids Roadmap



Key Points

- Potential Simplifications
  - SS, 1-D, constant properties, etc.
- Boundary Conditions
  - Known v
  - Symmetry (e.g. $\frac{\partial v}{\partial x} = 0$)
  - No slip (v = 0) at walls
- Examples (PRACTICE!!!)

Mass $$\nabla \cdot \vec{v} = 0$$

Momentum $$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g} - \nabla \cdot \tau$$

Mass $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-Mom $$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Y-Mom $$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Z-Mom $$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

How to deal with $\nabla \cdot \bar{\bar{\tau}}$?

- Inviscid Flow: $\nabla \cdot \bar{\bar{\tau}} = 0$ (Euler Equations)
  - Valid far from walls and obstructions
- Constant ρ: $\nabla \cdot \bar{\bar{\tau}} = -\mu \nabla^2 \vec{v}$
  - Reasonable for incompressible flow
    - Newtonian flow is implied… why?
- Reduce dimensions of the problem:
- Pipe Flow = u(y), v = w = 0

Problem Setup (SIMPLIFY!!!)

- Boundary Conditions:
  - Given velocity at inlet, outlet flow field
  - Symmetry points: pipe centerlines, etc. ($\frac{dv}{dx} = 0$)
  - v = 0 at walls (no slip condition)
- Initial Conditions (v field given at t = 0)

- Use these to transform N.S. into usable ODE

Big Picture Moment…

- Integral Balances
  - Energy, Mass & Momentum
- Differential Balances
  - Energy (Next Semester)
  - Mass & Momentum – Navier Stokes
    - Only 3 ways to solve Navier Stokes Eq.
      1. Simplify the problem until it's very basic (Today)
      2. Boundary Layer Approximation (Wednesday)
      3. CFD (Last week of Lectures)

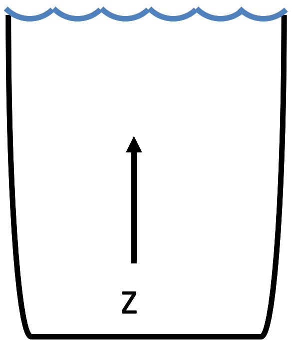
Navier-Stokes: Method 1

- New and Difficult… PRACTICE!!
- Don’t take shortcuts!!! Easy to get lost
- Methodology to maximize points (on tests)
  1. List and number all assumptions
  2. Write continuity equation
     - Reduce continuity equation using assumptions
     - Attach a number to the result
  3. Write Navier Stokes for non-main direction(s)
     - Eliminate terms using assumptions (number)
  4. Write Navier Stokes for main direction
     - Eliminate terms using assumptions (number)
  5. Solve Remaining Equation with BC’s

Example 1

- Barometric Equation (no velocity, just a fluid)

- $\rho \frac{\partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$

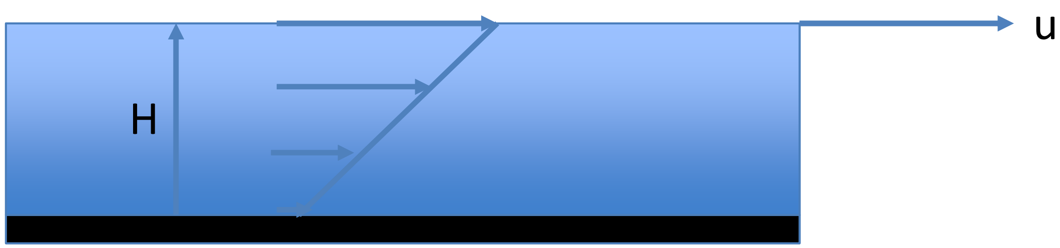


- $\nabla P = \rho g$
- $P_2 - P_1 = \rho g (z_2 - z_1)$

Couette Flow (example)

Example 2

- Couette Flow



- $\rho \frac{\partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$
- Boundary Conditions:
  - u = 0 at y = 0
  - U = U at y = H
- Constant ρ, μ
- 1-D Flow (only in x-direction)

Example 2 (cont)

- $\rho \frac{\partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \bar{\bar{\tau}} + \rho \vec{g}$

Material Derivative

$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial z}$

$\nabla \bar{\bar{\tau}}$

- Only x-direction:
- $\frac{Du}{Dt} = -\frac{1}{\rho} \nabla P + \frac{\mu}{\rho} \nabla^2 \vec{v} + \vec{g}$
- $\frac{Du}{Dt} = -\frac{1}{\rho} \nabla P + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g_x$
- $\frac{\partial^2 u}{\partial y^2} = 0$ $\Rightarrow$ linear

$u = ay + b$

$y = 0 \quad u = 0$

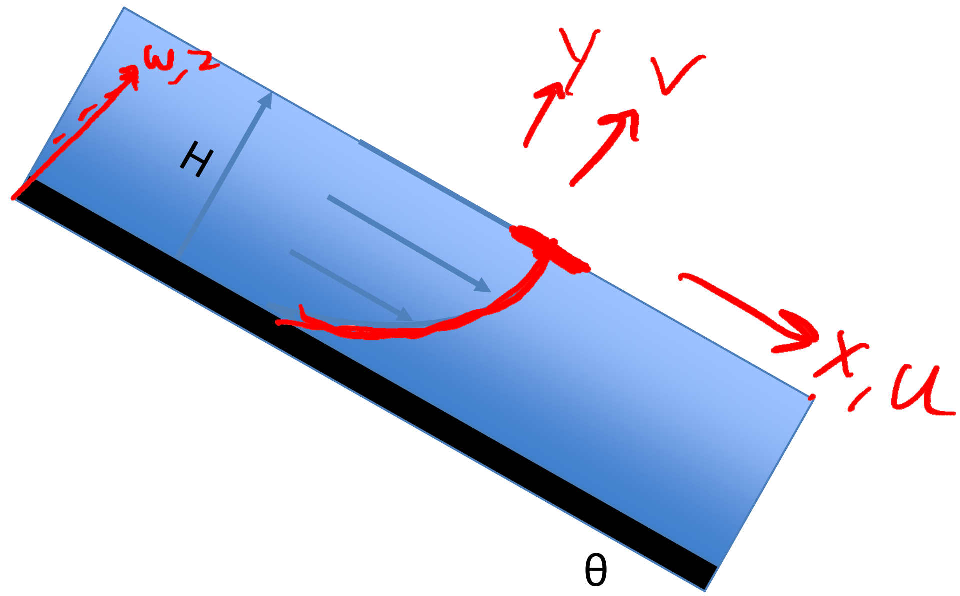
$y = H \quad u = U$

$b = 0 \quad a = \frac{U}{H}$

$u = \frac{U}{H} y$

Example 3

- Flow down incline
- Steady
- Laminar
- 1-D (y-dir)
- v = w= 0
- u(0) = 0
- $\left.\frac{du}{dy}\right]_{y=H} = 0$
- $\frac{Du}{Dt} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\vec{v} + \vec{g}$



Example 3 (cont.)

- $\frac{Du}{Dt} = -\frac{1}{\rho}\nabla P + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x$
- $\frac{\partial^2 u}{\partial y^2} = -\frac{g_x}{\nu} = -\frac{g}{\nu}\cos\theta$

$g_x = g\cos\theta$

$\rightarrow \int d^2u = \int \frac{-g}{\nu}\cos\theta \, dy\, dy$

$du = \left(\frac{-g}{\nu}\cos\theta(y + C_1)\right)dy$

$\frac{du}{dy} = -\frac{g}{\nu}\cos\theta(y + C_1)$

@ $y = H$, $\frac{du}{dy} = 0$

$\therefore C_1 = -H$

$\int du = \int \frac{-g}{\nu}\cos\theta(y - H)\,dy$

$u = \frac{-g}{\nu}\cos\theta\left(\frac{y^2}{2} - yH + C_2\right)$

@ $y = 0$, $u = 0 \rightarrow C_2 = 0$

$$u = \frac{-g\cos\theta}{\nu}\left(\frac{y^2}{2} - yH\right)$$