Second Order Systems

Second Order Equations

Standard Form

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

K = Gain
τ = Natural Period of Oscillation
ζ = Damping Factor (zeta)

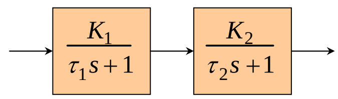
Note: this has to be 1.0!!!

Corresponding Differential Equation

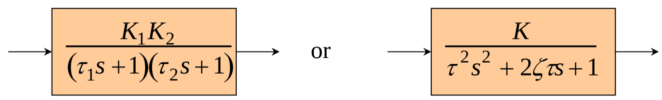
$$\tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau\frac{dy}{dt} + y = Kf(t)$$

Origins of Second Order Equations

1. Multiple Capacity Systems in Series



become



2. Controlled Systems (to be discussed later)
3. Inherently Second Order Systems
   - Mechanical systems and some sensors
   - Not that common in chemical process control

Examination of the Characteristic Equation

$$\tau^2 s^2 + 2\zeta\tau s + 1 = 0$$

$\zeta > 1$	Overdamped	Two distinct real roots
$\zeta = 1$	Critically Damped	Two equal real roots
$0 < \zeta < 1$	Underdamped	Two complex conjugate roots

Response of 2nd Order System to Step Inputs

Overdamped Eq. 5-48 or 5-49	Sluggish, no oscillations
Critically damped Eq. 5-50	Faster than overdamped, no oscillation
Underdamped Eq. 5-51	Fast, oscillations occur

Ways to describe underdamped responses:
- Rise time
- Time to first peak
- Settling time
- Overshoot
- Decay ratio
- Period of oscillation

Response of 2nd Order Systems to Step Input ($0 < \zeta < 1$)

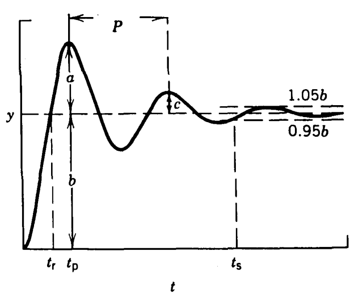


Figure 5.10. Performance characteristics for the step response of an underdamped process.

1. **Rise Time:** t_r is the time the process output takes to first reach the new steady-state value.
2. **Time to First Peak:** t_p is the time required for the output to reach its first maximum value.
3. **Settling Time**: t_s is defined as the time required for the process output to reach and remain inside a band whose width is equal to ±5% of the total change in *y*. The term 95% response time sometimes is used to refer to this case. Also, values of ±1% sometimes are used.
4. **Overshoot:** ***OS*** = *a*/*b* (% overshoot is 100*a*/*b*).
5. **Decay Ratio**: ***DR*** = *c*/*a* (where *c* is the height of the second peak).
6. **Period of Oscillation**: ***P*** is the time between two successive peaks or two successive valleys of the response.

Eq. 5-51

$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\}$$

Response of 2^{nd} Order Systems to Step Input

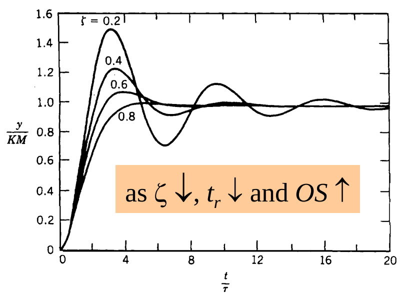


Figure 5.8. Step response of underdamped second-order processes.

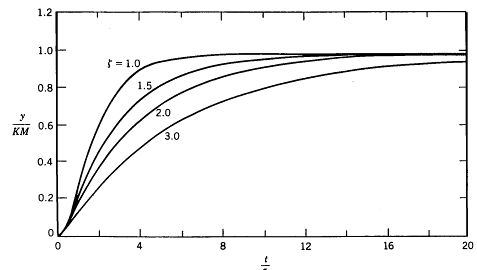


Figure 5.9. Step response of critically-damped and overdamped second-order processes.

$0 < \zeta < 1$

$\zeta \geq 1$

Note that $\zeta < 0$ gives an unstable solution

Relationships between OS, DR, P and τ, ζ

for step input to 2^{nd} order system, underdamped solution $Y(s) = \dfrac{KM}{s\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)}$, $\zeta < 1$

(5-52)	$t_p = \dfrac{\pi\tau}{\sqrt{1-\zeta^2}}$		
(5-53	$OS = \exp\left(-\dfrac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$	$\zeta = \sqrt{\dfrac{\left[\ln(OS)^2\right]}{\pi^2 + \left[\ln(OS)^2\right]}}$	Above (5-56)
(5-54)	$DR = (OS)^2 = \exp\left(-\dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$		
(5-55)	$P = \dfrac{2\pi\tau}{\sqrt{1-\zeta^2}}$	$\tau = \dfrac{\sqrt{1-\zeta^2}}{2\pi} P$	Above (5-57)
(5-60)	$t_r = \dfrac{\tau}{\sqrt{1-\zeta^2}}\left(1-\cos^{-1}\zeta\right)$		

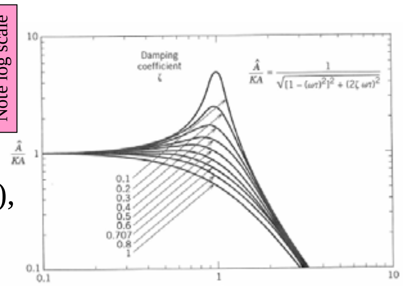
Response of 2nd Order System to Sinusoidal Input

- Output is also oscillatory
- Output has a different amplitude than the input
- Amplitude ratio is a function of ζ, τ
  (see Eq. 5-63)
- Output is phase shifted from the input
- Frequency ω must be in radians/time!!!
  (2π radians = 1 cycle)
- P = time/cycle = $1/(\nu)$, $2\pi\nu = \omega$, so $P = 2\pi/\omega$
  (where ν = frequency in cycles/time)

Sinusoidal Input, 2nd Order System (Section 5.4.2)

Note log scale

- Input = $A \sin \omega t$, so
  - A is the amplitude of the input function
  - ω is the frequency in radians/time
- At long times (so exponential dies out),



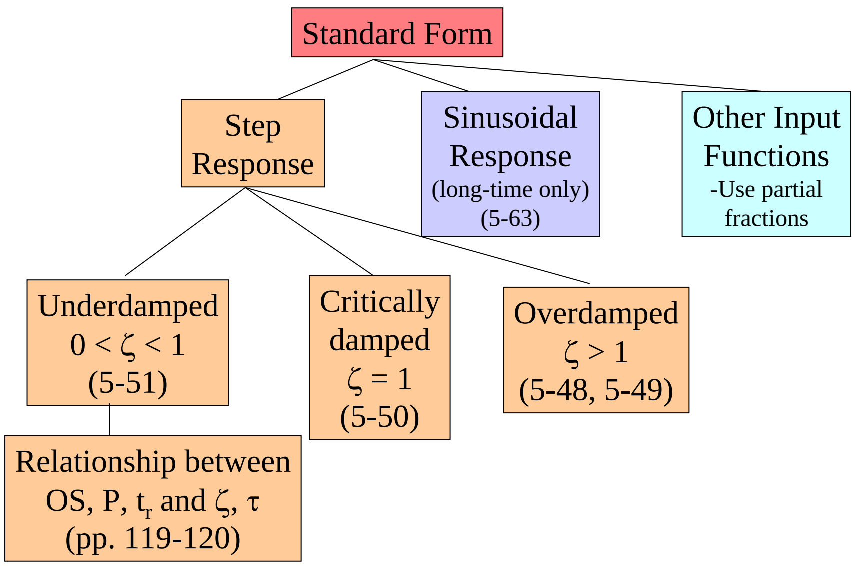
$\hat{A}$ is the output amplitude

$$\hat{A} = \frac{KA}{\sqrt{\left[1-(\omega\tau)^2\right]^2 + (2\zeta\omega\tau)^2}} \qquad (5\text{-}63)$$

Note: There is also an equation for the maximum amplitude ratio (5-66)

Bottom line: We can calculate how the output amplitude changes due to a sinusoidal input

Road Map for 2nd Order Equations

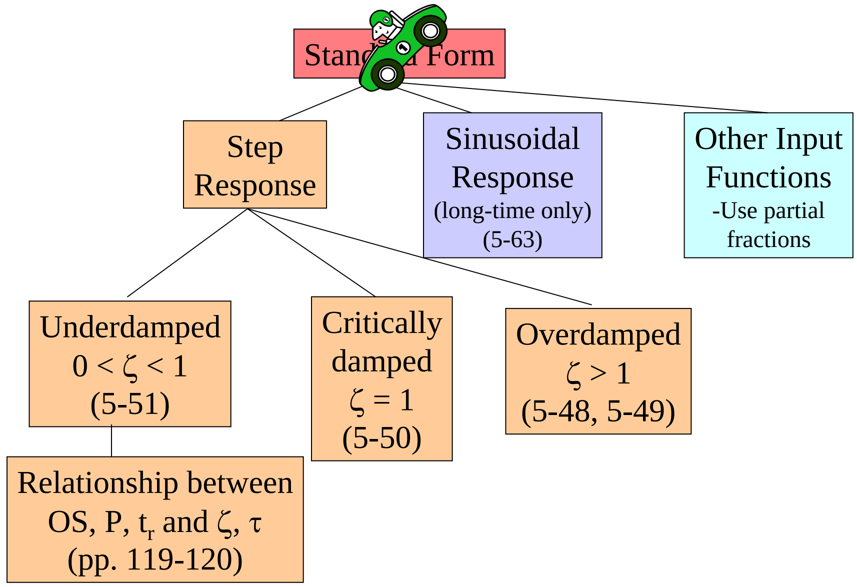


Example 5.5

- Heated tank + controller = 2nd order system

(a) When feed rate changes from 0.4 to 0.5 kg/s (step function), T_{tank} changes from 100 to 102°C.
Find gain (K) of transfer function:

Road Map for 2nd Order Equations



Example 5.5

- Heated tank + controller = 2nd order system

(a) When feed rate changes from 0.4 to 0.5 kg/s (step function), T_{tank} changes from 100 to 102°C.
Find gain (K) of transfer function:

Example 5.5

- Heated tank + controller = 2nd order system

(b) Response is slightly oscillatory, with first two maxima of 102.5 and 102.0°C at 1000 and 3600 S.

What is the complete process transfer function?

Example 5.5

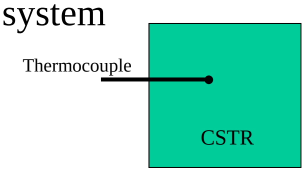
- Heated tank + controller = 2nd order system

(c) Predict t_r:

Example 5.6

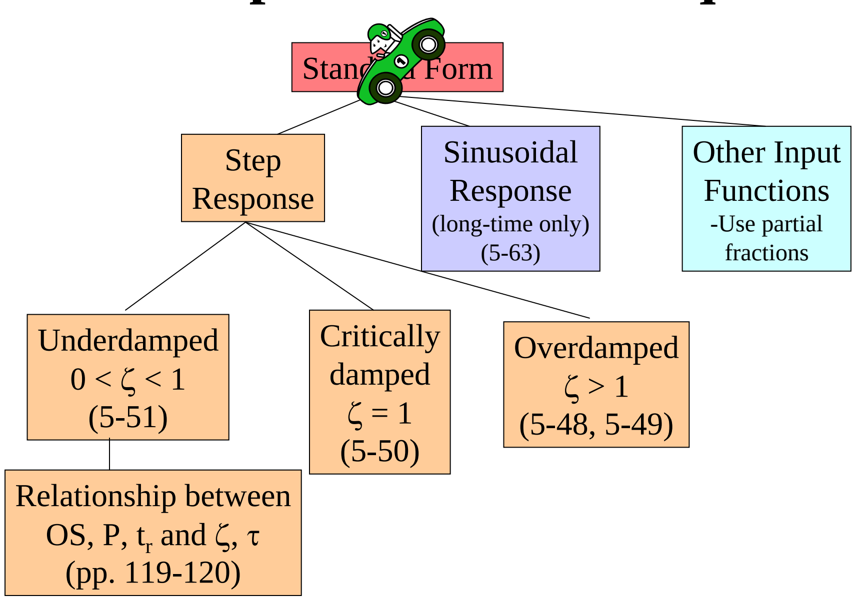
- Thermowell + CSTR = 2nd order system

(a) $\dfrac{T'_{meas}(s)}{T'_{reactor}(s)} = \dfrac{1}{(3s+1)(10s+1)}$



Find τ, ζ:

Road Map for 2nd Order Equations



Example 5.6

- Thermowell + CSTR = 2nd order system

(a) $\frac{T'_{meas}(s)}{T'_{reactor}(s)} = \frac{1}{(3s+1)(10s+1)}$

Find τ, ζ:

Example 5.6

- Thermowell + CSTR = 2nd order system

(b) Temperature cycles between 180 and 183°C, with period of 30 s.
Find ω, $\hat{A}$:

Example 5.6

- Thermowell + CSTR = 2nd order system

(c) Find A (actual amplitude of reactor sine wave):