Lecture 14 – Nomerical Integration

* prayer / spiritual thought
* announcements
* unit of the day summary

$1 \text{ m} = 3.28 \text{ ft}$

$1 \text{ kg} = 2.20 \text{ lbm}$

$1 \text{ lbf} = 32.2 \dfrac{\text{lbm ft}}{s^2}$

$K = {}^\circ C + 273.15$

${}^\circ R = {}^\circ F + 459.67$

$1 \text{ BTU} = 1055 \text{ J}$

$1 \text{ kcal} = 4184 \text{ J}$

$1 \text{ atm} = 101325 \text{ Pa}$

$1 \text{ atm} = 14.7 \text{ psi}$

$1 \text{ atm} = 760 \text{ torr}$

$1 \text{ hp} = 550 \dfrac{\text{ft lbf}}{s}$

$1 \text{ hp} = 745.7 \text{ W}$

I. The Trapezoidal Rule

* In calculus, you learned how to integrate a function

$$I = \int_a^b f(x)\, dx$$

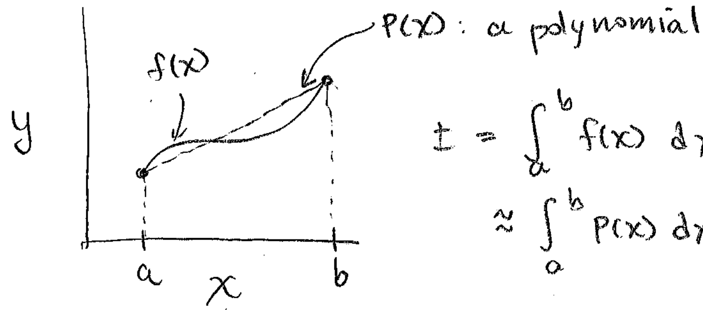
* Integrals can be difficult to evaluate analytically. Can we do it numerically? Yep.

* Numerical integration often goes by the alternate name "quadrature" or "numerical quadrature". Quadrature is a synonym with integration.

* How can we do the integral ?

$$I = \int_a^b f(x)\,dx$$

- One way to do it is based on the key idea that polynomials are easy to integrate.



$$I = \int_a^b f(x)\,dx$$
$$\approx \int_a^b P(x)\,dx$$

- In other words we interpolate between a & b with a polynomial and integrate our simple interpolation function.

Example : A Linear polynomial.

$$I = \int_a^b f(x)\,dx$$

- What line interpolates between $(a, f(a))$ & $(b, f(b))$?

$$\frac{P(x) - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad \text{(slope)}$$

$$\Rightarrow \quad P(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

- Approximate the integral using the interpolating polynomial. This gets more accurate when $|b-a|$ is small.

$$I \approx \int_a^b P(x)\,dx$$

$$I \approx \int_a^b \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right] dx$$

$$\approx \left[\frac{1}{2}\frac{f(b)-f(a)}{b-a}(x-a)^2 + f(a)\,x\right]_a^b$$

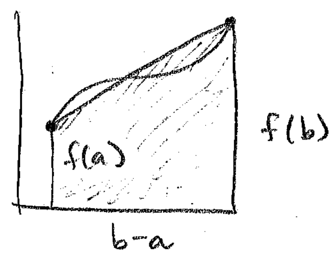
$$\approx \frac{\frac{1}{2}[f(b)-f(a)]\,(b-a)^2}{b-a} + b f(a) - a f(a)$$

$$\approx \frac{1}{2}[f(b)-f(a)](b-a) + b f(a) - a f(a)$$

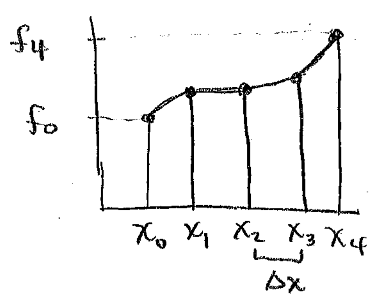
$$\approx \frac{1}{2}(b-a)[f(b)-f(a)+2f(a)]$$

$$\boxed{I \approx \frac{1}{2}(b-a)[f(b)+f(a)]}$$

* This is the area of a trapezoid, → so this method is called the trapezoid rule.



* Just like we saw with interpolation, it isn't very accurate to use one polynomial to fit the entire integrand. Instead we break the interval up into pieces.



This is called the composite trapezoid rule.

$$\boxed{I \approx \frac{1}{2}\Delta x\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right]}$$

* We can use higher order polynomials to get even more accurate methods.

  - quadratic polynomials → simpson's 1/3 rule
  - cubic polynomials → simpson's 3/8 rule

Practice: Use the composite trapezoid rule to evaluate the integral.

II. Numerical Quadrature on Python

* Scipy has an integration module:

  scipy.integrate

* There are two relevant functions for us.

  * scipy.integrate.quad
    - most accurate/fastest.
    - can only use if you know the function
    - uses a more advanced method called Gaussian quadrature

  * scipy.integrate.simps
    - use if you only have data points (no function)
    - uses composite simpson's rule
    - best if data is equally spaced.

See python scripts for examples & practice