Lecture 15 – Numerical ODEs

* prayer / spiritual thought.
* announcements
* Unit of the day

$$°F = \frac{9}{5} °C + 32$$

I. Systems of Rate Equations

* Recall during our Excel unit, we learned about rate equations

$$\frac{dy}{dt} = f(y,t)$$

$$y(0) = y_0$$

- first-order differential equations → only 1st derivatives
- ordinary differential equation → derivative w.r.t. only 1 dependent variable (e.g. t)
- initial value problem (IVP) → know initial condition

* In your ODEs class you will learn how to solve analytically. Today we will remind you how to solve numerically.

* Aside: sometimes students get numerical integration & solving an ODE confused:

- An ODE is "harder" than an integral problem. In other words not all ODEs reduce to integrals, but some simple ones do. If you haven't taken an ODEs class, don't worry about this too much.

* finally, recall we derived the explicit Euler method for solving a rate equation:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \Delta t\, f(y_n, t_n) \\ y_0 &= y(0) \end{aligned}}$$

* Let's talk a little bit more about systems of rate equations

$$\frac{dy_1}{dt} = f_1(y_1, y_2, \ldots, y_m, t)$$

$$\frac{dy_2}{dt} = f_2(y_1, y_2, \ldots, y_m, t)$$

$$\vdots$$

$$\frac{dy_m}{dt} = f_m(y_1, y_2, \ldots, y_m, t)$$

$$y_1(0) = y_{1,0}$$

$$y_2(0) = y_{2,0}$$

$$\vdots$$

$$y_m(0) = y_{m,0}$$

- like algebraic equations, systems of ODEs can be linear or non-linear, coupled or uncoupled.
- An ODE is non-linear if there are non-linear terms of the independent variable (e.g. y - not t).

- the explicit Euler method can be directly applied to systems of ODEs

(in vector notation)

$$\underline{y}_{n+1} = \underline{y}_n + \Delta t \, \underline{f}(\underline{y}_n, t_n)$$

m = # of y's

n = # of time points

(written out)

$$y_{1,n+1} = y_{1,n} + \Delta t \, f_1(y_1, y_2, \cdots y_m, t)$$

$$y_{2,n+1} = y_{2,n} + \Delta t \, f_2(y_1, y_2, \cdots, y_m, t)$$

$$\vdots$$

$$y_{m,n+1} = y_{m,n} + \Delta t \, f_m(y_1, y_2, \cdots, y_m, t)$$

- $\boxed{\text{New idea}}$ We can turn higher order ODEs into systems of first order ODEs.

<u>example</u>:

$$\frac{d^2y}{dt^2} = -gt \qquad y(0) = y_0$$

$$\left.\frac{dy}{dt}\right|_{t=0} = v_0$$

higher order ODEs have more initial conditions - <u>one for each derivative</u>

* we can add a new variable, v.

$$v \equiv \frac{dy}{dt} \quad \text{(we define, velocity)}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dv}{dt}$$

$$\frac{dv}{dt} = -gt$$ (*) Rate Equation

$$\frac{dy}{dt} = v$$ (*) Also a rate equation!

Initial conditions?

$$\frac{dy}{dt} = v \qquad y(0) = y_0 \qquad ✓$$

$$\frac{dv}{dt} = -gt \qquad \left.\frac{dy}{dt}\right|_{t=0} = v_0 \qquad v(0) = v_0 \quad ✓$$

So the steps are

(1) define a new variable for each derivative until only have 1st order derivatives

e.g. 3rd order

$$\frac{d^3y}{dt^3}: \quad v = \frac{dy}{dt} \quad a = \frac{dv}{dt}$$

(2) substitute definitions onto original equation

$$a = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = \frac{da}{dt}$$

(3) Find initial conditions

(4) Solve system of rate equations

Practice 1: classify & re-arrange sytems of ODEs

II. ODE solvers in Python

* The ODE solver in python is in the scipy.integrate module (same as simps, quad)

* The solver is called "odeint"
  - much more accurate than Explicit Euler
  - default method is "Runge-Kutta 45"

* See Example 2 file for syntax & usage

* See Practice 2 for a practice problem.