Lecture 23 - Calculus and ODEs

- Prayer/Spiritual Thought
- Announcements

Outline

1. Symbolic Calculus
2. Numerical Integration
3. Differential Equations

1. Symbolic Calculus

A. Explanation

- In addition to the algebra we saw a few weeks ago, Mathcad's symbolic solver can also do some basic calculus.
- Derivatives:
    - two different operators -- derivative and prime
    - derivative: Ribbon -> Math -> Operators -> Derivative (d/dx). Shortcut: <ctrl><shift>D
    - prime: Ribbon -> Math -> Operators -> Prime (x'). Shortcut: <ctrl>'
- Integrals:
    - Definite and Indefinite Integrals
    - Ribbon -> Math -> Operators -> Integral
    - Shortcut: <ctrl><shift>I
    - For an indefinite integral, don't add limits of integration
- Remember to use the symbolic evaluation operator (the arrow).
    - Ribbon -> Math -> Symbolics -> Symbolic Evaluation (arrow)
    - Keyboard shortcut: <ctrl>.

B. Examples

(i) Derivatives

simple derivative

$$\frac{d}{dx}\sin(x) \rightarrow \cos(x)$$

partial derivatives

$$\frac{d}{dt}\sin(x \cdot t) \rightarrow x \cdot \cos(t \cdot x)$$

functions using symbolic derivatives

$$f(x) := \frac{d}{dx}\operatorname{atanh}\left(x^2\right) \rightarrow -\frac{2 \cdot x}{x^4 - 1} \qquad f(5) \rightarrow -\frac{5}{312} \qquad \frac{-2 \cdot 5}{5^4 - 1} \rightarrow -\frac{5}{312}$$

prime operator

$f(x) := x^3 + 4 \cdot x^2 + 4\ x - 1$ $\qquad f'(x) \rightarrow 3 \cdot x^2 + 8 \cdot x + 4$

$f''(x) \rightarrow 6 \cdot x + 8$ $\qquad f'''(x) \rightarrow 6$

(ii) Integrals

Definite Integrals

$$\int_1^5 3\ x^2 + 8\ x + 4\, \mathrm{d}x \rightarrow 236$$

$$\int_0^\infty \exp\left(-x^2\right) \mathrm{d}x \rightarrow \frac{\sqrt{\pi}}{2}$$

Infinity: Ribbon -> Math -> Symbols or <ctrl><shift>Z

Indefinite Integrals

$$\int 3\ x^2 + 8\ x + 4\, \mathrm{d}x \rightarrow x^3 + 4 \cdot x^2 + 4 \cdot x$$

Note: no constant!

$$\int f''(x)\, \mathrm{d}x \rightarrow 3 \cdot x^2 + 8 \cdot x$$

Using f''(x) from previous example

Defining functions using integrals

$g(x) := \int 6 \cdot x + 8\, \mathrm{d}x$ $\qquad g(4) \rightarrow 80$

2. Numerical Integration

A. Explanation

- Sometimes it is impractical to do a symbolic integral, and we want to do a numerical integration.
- Recall we have two cases: 1. You know the function) and 2. You have data points only.
- If you know the function (like quad in python):
    - Use a **numerical definite integral**. Use the integral operator (<ctrl><shift>I) but evaluate it using the "=" sign not the symbolic equals "->".
- If you don't know the function (like simps in python):
    - You must code the trapezoidal rule by hand. Mathcad has no built-in function for Newton-Coates quadratures (these types of integration methods) :(.

B. Examples

1. Numerical Definite Integrals

$$\int_0^1 \frac{2}{\sqrt{\pi}} \exp\left(-t^2\right) \mathrm{d}t = 0.8427$$

This is erf(1)

$$\mathrm{erf}(1) = 0.843$$

$$\frac{1}{\pi} \int_0^{\pi} \cos\left(2 \cdot \tau - 3 \cdot \sin(\tau)\right) \mathrm{d}\tau = 0.486$$

This is J_2(3), Bessel's function of the first kind

$$\mathrm{Jn}(2, 3) = 0.486$$

2. Trapezoidal rule for $\int_0^1 \frac{2}{\sqrt{\pi}} \, exp\left(-t^2\right) \mathrm{d}t$

$N_{pts} := 100$ $\quad t_{lo} := 0$ $\quad t_{hi} := 1$ $\quad$ <-- limits of integration

$i := 0 \,..\, N_{pts}$ $\quad t_i := t_{lo} + \left(t_{hi} - t_{lo}\right) \cdot \frac{i}{N_{pts}}$ $\quad \Delta t := t_1 - t_0$

$f(t) := \frac{2}{\sqrt{\pi}} \exp\left(-t^2\right)$ $\quad$ <-- integrand

$$I := \frac{\Delta t}{2} \cdot \left(f\left(t_0\right) + 2 \sum_{i=1}^{N_{pts}-1} f\left(t_i\right) + f\left(t_{N_{pts}}\right) \right)$$

<-- composite trapezoidal rule

$$I = 0.8427$$

3. Differential Equations

A. Explanation

- ODEs are solved in Mathcad using a **solve block**
- Three parts of a solve block:

    1. Guesses. No guesses needed
    2. Constraints. Add the ODE(s) and initial condition(s) here.
    3. Solver: y := odesolve(y(t), <end time>)

- Range of solution is from the initial condition to the end time given in odesolve.
- Odesolve returns a function. Can evaluate at any time.
- Can also add nonlinear equations (with guesses) as appropriate
- Multiple ODEs require vector notation in odesolve (see example).

B. Examples

1. Solve the following second order ODE from t = 0 to t = 20:

Guess Values

Constraints

$$3 \cdot x''(t) + x'(t) + 5 \cdot x(t) = sin(\pi \cdot t)$$

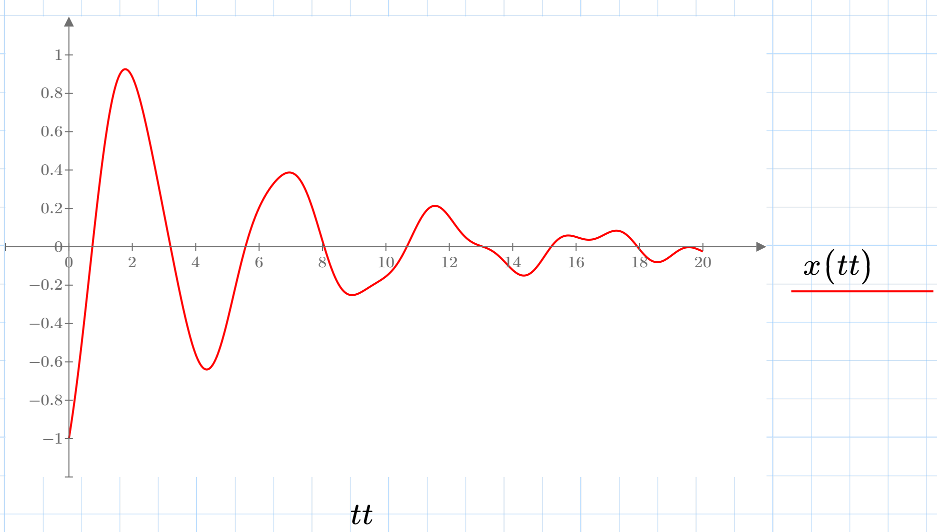
$$x(0) = -1$$

$$x'(0) = 1$$

Solver

$$x := \mathbf{odesolve}(x(t), 20)$$

Tip: Can use prime or derivative notation. If you use prime notation, make sure to use <ctrl>' not a simple apostrophe.



2. Solve the following set of rate equations from t = 0 to t = 12:

$\varepsilon := 2.5 \qquad t_{end} := 12$

Guess Values

Constraints

$$\frac{d}{dt}x(t) = \varepsilon \cdot \left(1 - y(t)^2\right) \cdot x(t) - y(t)$$

$$\frac{d}{dt}y(t) = x(t)$$

$$x(0) = 0$$

$$y(0) = 2$$

Solver

$$\begin{bmatrix} X \\ Y \end{bmatrix} := \mathbf{odesolve}\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t_{end}\right)$$

$t := 0, 0.01 .. t_{end}$ <-- This gives discrete t points to plot for Mathcad, which can get confused by the implicit plot.