Lecture 10 - Gauss Elimination

* AMA / Quiz / Prayer
* Midcourse evaluation, Exam 1 review.

0. Course Perspective

* Until now, we have been learning about numerical computing. We are now going to switch to solving math problems.
  - To those struggling: We won't have new computing ideas, but we are going to use the old ones!
  - To those bored: The next stuff is new! Pay attention

I. Systems of Linear Algebraic Equations

* A linear algebraic equation looks like:

$$\begin{aligned} 3x_0 - x_1 + x_2 &= -3 \\ -x_0 + 2x_1 - x_2 &= 8 \\ x_0 - x_1 - x_2 &= 1 \end{aligned}$$

* In linear algebra you will (or have already) talked more formally about these types of equations. Informally, if there are:
  - no powers (x_1^2)
  - no products ($x_1 x_2$)
  - no differentials ($\frac{dx_1}{dx_2}$)

The system is linear & algebraic.

* we can write a general linear system as:

$$a_{0,0}x_0 + a_{0,1}x_1 + \dots + a_{0,n-1}x_{n-1} = b_0$$

$$a_{1,0}x_0 + a_{1,1}x_1 + \dots + a_{1,n-1}x_{n-1} = b_1$$

$$\vdots$$

$$a_{n-1,0}x_0 + a_{n-1,1}x_1 + \dots + a_{n-1,n-1}x_{n-1} = b_{n-1}$$

* This can be written in vector-matrix notation as $\underline{\underline{A}}\,\underline{x} = \underline{b}$ where

$$\underline{\underline{A}} = a_{ij} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & & & \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix}$$

(size n matrix b/c counting starting at 0)

$$\underline{b} = b_i = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \qquad \underline{x} = x_i = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

* A couple of comments:

  - Our system is <u>square</u>. This is the case when the number of equations equals the number of unknowns. You'll talk <u>a lot</u> more about this in linear algebra.
  - Notice the index notation (e.g. a_{ij}). It is useful for writing vectors & matrices.

    e.g. a_{11} is the 1,1 element of the A matrix.

$\boxed{\text{Practice}}$ write our 3x3 system as a set of vectors and matrices.

$$\underline{\underline{A}} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

II. Solving linear systems

* How do you solve a linear system? You already know how to do this! (High School math). We are going to learn (review) a systematic way to do this called Gauss Elimination.

Example: $\underline{\underline{A}}\,\underline{x} = b$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Rules:

- can multiply an equation (a row) by a constant
- can add equations together (rows).

Objective:

- Make an upper triangular matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

- Then solve for bottom (x_3). Use x_3 to find x_2. Use x_3 & x_2 to find x_1.

col 1

make these 0 first

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ -1 & 2 & -1 & 8 \\ 1 & -1 & -1 & 1 \end{array}\right] \times(-3) \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 3 & -6 & 3 & -24 \\ 1 & -1 & -1 & 1 \end{array}\right] \quad R_2 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5 & -2 & 21 \\ 1 & -1 & -1 & 1 \end{array}\right] \times(3) \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5 & -2 & 21 \\ 3 & -3 & -3 & 3 \end{array}\right] \quad R_3 \leftarrow R_1 - R_3$$

only 1 to zero →
$$\begin{array}{c} \text{col 2} \\ \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5 & -2 & 21 \\ 0 & 2 & 4 & -6 \end{array}\right] \end{array} \times \left(\frac{5}{2}\right) \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5 & -2 & 21 \\ 0 & 5 & 10 & -15 \end{array}\right] \quad R_3 \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5 & -2 & 21 \\ 0 & 0 & -12 & 36 \end{array}\right] \quad \text{upper triangular!} \quad \begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \\ 36 \end{bmatrix}$$

* Now substitute and solve:

$$-12x_2 = 36 \rightarrow x_2 = \frac{36}{-12} = -3$$

$$5x_1 - 2x_2 = 21 \rightarrow x_1 = \frac{21 + 2x_2}{5} = \frac{21 + 2(-3)}{5} = \frac{21-6}{5}$$
$$= \frac{15}{5} = 3$$

$$3x_0 - x_1 + x_2 = -3 \rightarrow x_0 = \frac{-3 + x_1 - x_2}{3}$$
$$= \frac{-3 + 3 - -(3)}{3} = \frac{3}{3} = 1$$

$$\boxed{x_0 = 1, \; x_1 = 3, \; x_2 = -3}$$

* How are we going to code this? We need a systematic way of writing the process.

There were two steps.

(1) Forward Elimination

(2) Backward Substitution

(1) Forward Elimination

- start in 1st column; eliminate elements below a_{00}
- Find ratio with 1st element in 2nd row and a_{00}

- multiply all elements in 2^{nd} row by ratio (and b_1!)
- add all elements in 1^{st} row to 2^{nd} row (and b_1!)
- repeat for 3^{rd} row, 4^{th} row, ..., n^{th} row.
- repeat for 2^{nd} column (with a_{11}), 3^{rd} column (a_{22}), ... $(n-1)^{th}$ column ($a_{n-2,n-2}$).

... or in vector notation:

ratio

$$a_{ij} = a_{kj} - \left(\frac{a_{kk}}{a_{ik}}\right) a_{ij} \qquad b_i = b_k - \left(\frac{a_{kk}}{a_{ik}}\right) b_i$$

top row. current row

for $k = 0, 1, \ldots, n-2$ ← column we are eliminating

$i = k+1, k+2, \ldots, n-1$ ← row we are doing

$j = k, k+1, \ldots, n-1$ ← elements in the row.

(only need $k+1, k+2 \ldots, n$, but k gives the '0')

(This is a triple nested loop!)

(2) Backward Substitution

- start at bottom
- move any known x_i's and their coefficients to the right-hand side
- divide by coefficient to solve for x_i.
- repeat for each x_i

... or in vector notation

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n-1} a_{ij} x_j\right), \quad x_{n-1} = \frac{b_{n-1}}{a_{n-1,n-1}}$$

for $i = n-2, n-3, \ldots, 0$

Practice Solve this system by hand using Gauss Elimination

$$x_0 - x_1 + 2x_2 = -1$$

$$2x_0 + x_1 - x_2 = 5$$

$$0 + x_1 - x_2 = 1$$

(solution : $x_0 = 2$, $x_1 = -1$, $x_2 = -2$)