Lecture 14 – Systems of NLEs

* Prayer / Quiz

I. Methods for systems of NLEs

   A. Picard

   B. Newton

II. Example : 2comp. system

I. Methods for systems of NLEs

* The problem

$$f_0(x_0, x_1, \ldots, x_{n-1}) = 0$$
$$f_1(x_0, x_1, \ldots, x_{n-1}) = 0$$
$$\vdots$$
$$f_{n-1}(x_0, x_1, \ldots, x_{n-1}) = 0$$

} standard / residual form

$$\Downarrow$$

$$\underline{f}(\underline{x}) = \underline{0} \qquad \underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \qquad \underline{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

* Fixed-Point Methods

$$\underline{x}^{(k+1)} = \underline{g}(\underline{x}^{(k)}) \quad \leftarrow \text{ diff methods, diff. } \underline{g}$$

A. Picard's method

$$\underline{g} = \underline{x} + \underline{f}(\underline{x})$$

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{f}(\underline{x}^{k})$$

B. Newton's method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

analogously:

$$\underline{g}(\underline{x}) = \underline{x} - \underline{\underline{J}}^{-1} \cdot \underline{f}(\underline{x})$$

inverse of matrix of all the partial derivatives of $\underline{f}$. $\Rightarrow$ Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_0}{\partial x_{n-1}} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n-1}} \\ \vdots & & & \\ \frac{\partial f_{n-1}}{\partial x_0} & \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{bmatrix} = \frac{\partial f_i}{\partial x_j}$$

$i \leftarrow$ rows, $j \leftarrow$ cols.

Partial derivative: $\frac{\partial}{\partial x}(x^2+y^2) = 2x$

$$\frac{\partial}{\partial y}(\underbrace{x^2}_{\text{const}}+y^2) = 2y$$

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} - \underline{\underline{J}}^{-1}(\underline{x}^{(k)}) \cdot \underline{f}(\underline{x}^{(k)})$$

$$\underbrace{\underline{x}^{(k+1)} - \underline{x}^{(k)}}_{\underline{\underline{J}} \cdot \underline{\delta}} = \underline{\underline{J}}^{-1}(\underline{x}^{(k)}) \cdot (-\underline{f}(\underline{x}^{(k)}))$$

$\underline{\underline{J}} \cdot$

$$\boxed{\underline{\underline{J}} \cdot \underline{\delta} = -\underline{f}(\underline{x}^{(k)})}$$

$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$:)

$$\boxed{x^{(k+1)} = x^{(k)} + \underset{\sim}{\delta}}$$

- eval f $\leftarrow x^{(k)}$
- eval J
- solve lin. System $\Rightarrow \delta$
- update $x \to x^{(k+1)}$

II. Example: 2 EQ's, 2 Unknowns

* Picard's method & Newton's method for:

$$z = e^{-y}$$
$$y = 3z - z^2$$

(1) convert to std form

$$\to e^{-y} - z = 0 \qquad x_0 = y$$
$$\to y - 3z + z^2 = 0 \qquad x_1 = z$$

$$f_0 \to e^{-x_0} - x_1 = 0$$
$$f_1 \to x_0 - 3x_1 + x_1^2 = 0$$

$$\left.\begin{matrix} f_0\ (x_0, x_1) = 0 \\ f_1\ (x_0, x_1) = 0 \end{matrix}\right\} \ \underline{f}\ (\underline{x}) = \underline{0}$$

(2) write in the form of: $\underline{x} = g(\underline{x})$

(Picard's method)

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{f}(\underline{x}^{(k)})$$

$$\begin{bmatrix} x_0^{k+1} \\ x_1^{k+1} \end{bmatrix} = \begin{bmatrix} x_0^{k} \\ x_1^{k} \end{bmatrix} + \begin{bmatrix} e^{-x_0^{(k)}} - x_1^{(k)} \\ x_0^{(k)} - 3x_1^{(k)} + x_1^{(k)^2} \end{bmatrix}$$

$$\boxed{\begin{aligned} x_0^{(k+1)} &= x_0^{(k)} + e^{-x_0^{(k)}} - x_1^{(k)} \\ x_1^{(k+1)} &= x_1^{(k)} + x_0^{(k)} - 3x_1^{(k)} + \left(x_1^{(k)}\right)^2 \end{aligned}}$$

(Newton's method)

$$\underline{\underline{J}} \cdot \underline{\delta} = -\underline{f}$$

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{\delta}$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} \end{bmatrix}$$

$$\frac{\partial f_0}{\partial x_0} = \frac{\partial}{\partial x_0}\left(e^{-x_0} - x_1\right) = -e^{-x_0}$$

$$\frac{\partial f_0}{\partial x_1} = \frac{\partial}{\partial x_1}\left(e^{-x_0} - x_1\right) = -1$$

$$\frac{\partial f_1}{\partial x_0} = \frac{\partial}{\partial x_0}\left(x_0 - 3x_1 + x_1^2\right) = 1$$

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_0 - 3x_1 + x_1^2\right) = -3 + 2x_1$$

$$\begin{bmatrix} e^{-x_0^{(k)}} & -1 \\ 1 & 2x_1^{(k)} - 3 \end{bmatrix} \cdot \begin{bmatrix} \delta_0 \\ \delta_1 \end{bmatrix} = -\begin{bmatrix} -x_1^{(k)} + e^{-x_0^{(k)}} \\ x_0^{(k)} - 3x_1^{(k)} + \left(x_1^{(k)}\right)^2 \end{bmatrix}$$

$$x_0^{k+1} = \delta_0 + x_0^{(k)}$$

$$x_1^{(k+1)} = \delta_1 + x_1^{(k)}$$