Lecture 10 – Review of Matrix Algebra

Outline

0. Perspective
I. Linear Systems
II. Index Notation
III. Gauss Elimination
    A. Forward Elimination
    B. Back Substitution

0. Perspective

* turning point: Numerical computing ↓ solving Math problems

I. Systems of Linear Algebraic EQ's

* A linear system looks like:

$$\boxed{\begin{aligned} 3x_0 - x_1 + x_2 &= -3 \\ -x_0 + 2x_1 - x_2 &= 8 \\ x_0 - x_1 - x_2 &= 1 \end{aligned}}$$

short way
* no powers (x_1^2) (not rigorous/formal)
* no products ($x_1 x_2$)
* no differentials ($\frac{dx_1}{dx_2}$)

linear algebraic.

* General Case

EQs →
$$\begin{aligned} A_{0,0} x_0 + A_{0,1} x_1 + \dots + A_{0,n-1} x_{n-1} &= b_0 \\ A_{1,0} x_0 + A_{1,1} x_1 + \dots + A_{1,n-1} x_{n-1} &= b_1 \\ &\vdots \\ A_{n-1,0} x_0 + A_{n-1,1} x_1 + \dots + A_{n-1,n-1} x_{n-1} &= b_{n-1} \end{aligned}$$

coeffs, variables, RHS

* $n \times n$
  $0 \dots n-1$
* square: n EQ's, n unknowns → unique solution

* Matrix Notation

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

matrix, vectors

col, vars ↓

$$\underline{\underline{A}} = A_{ij} = \begin{bmatrix} A_{0,0} & A_{0,1} & \dots & A_{0,n-1} \\ A_{1,0} & A_{1,1} & \dots & A_{1,n-1} \\ & \vdots & & \\ A_{n-1,0} & A_{n-1,1} & \dots & A_{n-1,n-1} \end{bmatrix}$$ ← row = EQ

$$\underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = x_j \qquad \underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} = b_i$$

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b} \rightarrow \sum_j A_{ij} x_j = b_i$$

$\boxed{\text{Practice}}$ write the 3x3 matrix from above as $\underline{\underline{A}}, \underline{x}, \underline{b}$

$$\underline{\underline{A}} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

II. Index Notation

* By hand → vector notation: $\underline{\underline{A}} \cdot \underline{x} = \underline{b}$ $\qquad \underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$
* On a computer → index notation (loops!)

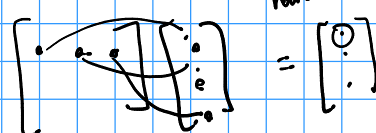
Examples:

vector norm: $|\underline{v}| = \sqrt{\sum_i v_i^2} = \sqrt{v_0^2 + v_1^2 + \dots v_{n-1}^2}$

dot product: $\underline{v} \cdot \underline{w} = \sum_i v_i w_i = v_0 w_0 + v_1 w_1 + \dots$

matrix-vector product: $\underline{\underline{A}} \cdot \underline{x} = \sum_j A_{ij} x_j = y_i = \underline{y}$

rows cols



matrix-matrix product: $\underline{\underline{A}} \cdot \underline{\underline{B}} = \sum_j A_{ij} B_{jk} = C_{ik}$

cols rows

Principles of Index Notation?

- 1 index = 1 dimension
- Dot product → sum over a repeated index
- Repeated index disappears
- If in doubt → write it out!
- Summation sign w/ index → loop.

$\boxed{\text{Example}}$

$$\underline{\underline{A}} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \qquad A_{ij}$$

$$\underline{\underline{A}} \cdot \underline{x} = \begin{bmatrix} A_{00} x_0 + A_{01} x_1 \\ A_{10} x_0 + A_{11} x_1 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{1} A_{0j} x_j \\ \sum_{j=0}^{1} A_{1j} x_j \end{bmatrix} = \sum_j A_{ij} x_j = y_i$$

$i = 0, 1$ ✓

In Python?

```
y = np.zeros(2)

for i in range(2):
    for j in range(2):
        y[i] += A[i,j] * x[j]
```

similar formulas

III. Solving Linear Systems.

* How do? Gauss Elimination ← systematic, same way
Efficient
Code-able.

Example: $\underline{\underline{A}} \cdot \underline{x} = \underline{b}$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Rules
- can multiply any (RHS!) row by a const
- add rows together.

(rows = Eqs)

GE → (1) Make an upper triangular matrix

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

Forward Elimination

(2) Solve for x_2 (bottom) $\quad x_2 = \frac{\bullet}{\bullet}$

↳ Solve for x_1 (plug in x_2) $\quad x_1 = \frac{\bullet - \bullet x_2}{\bullet}$

⋮

Solve for x_0 (plug in x_2, x_1) $\quad x_0 = \frac{\bullet - \bullet x_2 - \bullet x_1}{\bullet}$

Back Substitution

$$\begin{array}{cc} & \text{col } 0 \\ \text{Row1} & \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ \boxed{-1} & 2 & -1 & 8 \\ 1 & -1 & -1 & 1 \end{array}\right] \end{array}$$

- take ratio w/ diagonal : $-1/3$ → Ro

$$\text{Row 1} - \left(-\tfrac{1}{3}\right)\text{Row 0} \rightarrow \text{Row 1}$$

$$-1 - \left(-\tfrac{1}{3}\right)3 = 0$$

$$2 - (-1/3)\,{-1} = \tfrac{6}{3} - \tfrac{1}{3} = \tfrac{5}{3}$$

$$-1 - (-1/3)\,1 = -\tfrac{3}{3} + \tfrac{1}{3} = -2/3$$

$$8 - (-1/3)\,{-3} = 8-1 = 7$$

$$\begin{array}{cc} & 0 \\ \text{Row 2} & \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ \boxed{1} & -1 & -1 & 1 \end{array}\right] \end{array}$$

- take ratio w/ diagonal : $1/3$

$$\text{Row 2} - (1/3)\,\text{Row 0} \rightarrow \text{Row 2}$$

$$\begin{array}{cc} & \quad\text{col } 1 \\ \text{Row 2} & \left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ 0 & \boxed{-2/3} & -4/3 & 2 \end{array}\right] \end{array}$$

- take ratio w/ <u>Row 1</u> : $\dfrac{-2/3}{5/3} = -2/5$

$$\text{Row 2} - (-2/5)\cdot\text{Row 1} \rightarrow \text{Row 2}$$

$$-2/3 - (-2/5)\cdot 5/3 = 0 \quad ✓$$

$$-\tfrac{4}{3} - \left(\tfrac{-2}{5}\right)\cdot\tfrac{-2}{3} = \tfrac{-24}{15}$$

$$2 - \left(\tfrac{-2}{5}\right)\cdot 7 = \tfrac{24}{5}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ 0 & 0 & \tfrac{-24}{15} & \tfrac{24}{5} \end{array}\right]$$

<u>End of Step 1</u> (upper triangular matrix)

- start in col 0: loop over rows below diagonal
- ratio: current row / diagonal
- mult. row w/ diagonal by ratio & add to current row → replace row.
- loop over cols.

<u>Step 2:</u> (back sub)

$$-\tfrac{24}{15}\,x_2 = \tfrac{24}{5} \rightarrow x_2 = \frac{24/5}{-24/15} = -3 \quad ✓ \qquad \boxed{x_2 = -3}$$

$$5/3\,x_1 - 2/3\,x_2 = 7$$

$$x_1 = \frac{7 + \frac{2}{3}x_2}{5/3} = \frac{7 + \frac{2}{3}(-3)}{5/3} = \frac{5 \cdot 3}{5} = 3$$

$$\boxed{x_1 = 3}$$

$$3x_0 - x_1 + x_2 = -3$$

$$x_0 = \frac{-3 + x_1 - x_2}{3} = \frac{-\cancel{3} + \cancel{3} - -3}{3} = 1 \quad \boxed{x_0 = 1}$$

$$\boxed{\underline{x} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}}$$

Review:
- start @ bottom, solve for x_2
- work back to x_0, subbing in for what we know.

Comments:

- Easier if you have a special case. But this way is systematic.
- Next time we'll code this!
- Can always check answer!

$$\sum_j A_{ij} x_j \stackrel{?}{=} b_i$$