Lecture 23 – Systems of ODEs

Outline

I. Systems of ODEs
II. Higher order ODEs
III. Examples

I. Systems of ODEs

* Last time: $\dfrac{dy}{dt} = f(y,t)$, $y(0) = y_0$

  · first-order · linear or non-linear

  · ODE (only t) · initial conditions $\rightarrow$ IVP

* Many cases where one cares about more than 1 dependent variable

$y(t)$ ← independent, ↑ dependent

dep. vars.: $y(t)$, $x(t)$, $z(t)$ ← ODEs

* Example: $A + B \rightarrow C$

(0) $\dfrac{dA}{dt} = -k\,A \cdot B$ $A(0) = 2\,\frac{mol}{L}$

(1) $\dfrac{dB}{dt} = -k\,A\,B$ $B(0) = 1\,{mol}/{L}$

(2) $\dfrac{dC}{dt} = k \cdot A\,B$ $C(0) = 0\,\frac{mol}{L}$ ← $t=0$

first order ✓
non-linear ✓
ODE (t) ✓
IVP ($t=0$) ✓
⇓
system!

$A \rightarrow y_0$ $B \rightarrow y_1$ $C \rightarrow y_2$

(0) $\dfrac{dy_0}{dt} = f_0(t, y_0, y_1, y_2) = -k\,y_0 y_1$, $y_0(0) = 2\,\frac{mol}{L}$

(1) $\dfrac{dy_1}{dt} = f_1(t, y_0, y_1, y_2) = -k\,y_0 y_1$, $y_1(0) = \frac{1\,mol}{L}$

(2) $\dfrac{dy_2}{dt} = f_2(t, y_0, y_1, y_2) = k\,y_0 y_1$, $y_2(0) = 0\,\frac{mol}{L}$

more compact: $\boxed{\dfrac{d\underline{y}}{dt} = \underline{f}(t, \underline{y}) , \quad \underline{y}(0) = \underline{y}_0}$

$\underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$, $\underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$

$\underline{y}_0 = \begin{bmatrix} y_0(0) \\ y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} {mol}/{L}$

* In general, the same equation applies for a system of n ODEs.

$$\underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}, \quad \underline{y_0} = \begin{bmatrix} y_0(0) \\ y_1(0) \\ \vdots \\ y_{n-1}(0) \end{bmatrix}$$

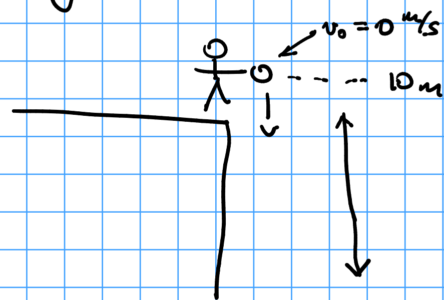
II. Higher order ODEs

* There are also a lot of examples where we want to solve higher-order ODEs.

* Example:

$g = -9.8\ m/s^2$

$$m\frac{d^2x}{dt^2} = -g$$



$x(0) = 10\ m \qquad x'(0) = 0\ m/s$

- 2^{nd} order, linear, ODE, IVP
  (non-system) .... yet.

$v \equiv \frac{dx}{dt}$ (define)

$$m\frac{d^2x}{dt^2} = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{dv}{dt}$$

initial cond.?

system of 1^{st} order, linear, ODE IVPs.

$$\begin{cases} m\dfrac{dv}{dt} = -g \\[2ex] \dfrac{dx}{dt} = v \end{cases}$$

$\frac{dx}{dt}(0) = 0\ m/s = v(0)$ ✓

$x(0) = 10\ m$ ✓

* In general, we can turn any higher-order ODE into a system of 1^{st} order ODEs.

  - many dependent vars
  - high order derivatives

  $\Longrightarrow$ system of first-order ODEs.

* What are the steps?

(1) Define an intermediate variable (v) for each order of derivative

(2) substitute our definition into orig. EQ & ICs.

Practice:

$$\frac{d^3y}{dt^3} = f(t,y) \qquad y''(0) = A, \quad y'(0) = B, \quad y(0) = C$$

(1) $\frac{d^2y}{dt^2}, \frac{dy}{dt} \rightarrow a = \frac{d^2y}{dt^2}, \quad v = \frac{dy}{dt}$ ✓

(2) $\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = f(t,y) \rightarrow$

$\frac{d}{dt}\left(\frac{dy}{dt}\right) = a \rightarrow$

$$\boxed{\begin{aligned} \frac{da}{dt} &= f(t,y), & a(0) &= A \\ \frac{dv}{dt} &= a, & v(0) &= B \\ \frac{dy}{dt} &= v, & y(0) &= C \end{aligned}}$$

2^{nd} order $\rightarrow$ 2 ODEs

3^{rd} order $\rightarrow$ 3 ODEs

$\vdots$

n^{th} order $\rightarrow$ n ODEs

Practice #2:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + ty = \cos(t)$$

$$y'(0) = -2, \quad y(0) = 2$$

$\frac{dy}{dt} \equiv v$

$$\frac{dv}{dt} = -3v - ty - \cos t, \quad v(0) = -2$$

$$\frac{dy}{dt} = v, \quad y(0) = 2$$

III. Python ODE solver (solve_ivp)

* learn to solve ODE IVPs of the form

$$\frac{d\underline{y}}{dt} = \underline{f}(t, \underline{y}) \qquad \underline{y}(0) = \underline{y}_0$$

$$\underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}$$