Lecture 10 - Review of Matrix Algebra

* AMA / Quiz / Prayer
* Exam 1 Review

0. Course Perspective

* Until now, we have been learning about numerical computing. We are now going to switch to solving math problems.
    - To those struggling: we won't have new computing ideas, but we are going to use the old ones!
    - To those bored: The next stuff is new! Pay attention.

I. Systems of Linear Algebraic Equations

* A linear algebraic equation looks like:

$$
\begin{aligned}
3x_0 - x_1 + x_2 &= -3 \\
-x_0 + 2x_1 - x_2 &= 8 \\
x_0 - x_1 - x_2 &= 1
\end{aligned}
$$

* In linear algebra you will (or already have) talked more formally about these types of equations.

  Informally, if there are:

    - no powers (x_1^2)
    - no products $(x_0 x_1)$
    - no differentials $\left(\frac{dx_1}{dx_2}\right)$

  the system is linear and algebraic.

* we can write a general linear system as

$$A_{0,0}\,x_0 + A_{0,1}\,x_1 + \dots + A_{0,n-1}\,x_{n-1} = b_0$$

$$A_{1,0}\,x_0 + A_{1,1}\,x_1 + \dots + A_{1,n-1}\,x_{n-1} = b_1$$

$$\vdots$$

$$A_{n-1,0}\,x_0 + A_{n-1,1}\,x_1 + \dots + A_{n-1,n-1}\,x_{n-1} = b_{n-1}$$

* This can be re-written in vector-matrix notation as

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b} \qquad \text{or} \qquad \sum_j A_{ij}\,x_j = b_i$$

$$\underline{\underline{A}} = A_{ij} = \begin{bmatrix} A_{0,0} & A_{0,1} & \dots & A_{0,n-1} \\ A_{1,0} & A_{1,1} & \dots & A_{1,n-1} \\ & \vdots & & \\ A_{n-1,0} & A_{n-1,1} & \dots & A_{n-1,A_{n-1}} \end{bmatrix}$$

$$\underline{b} = b_i = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \qquad \underline{x} = x_j = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

* This system is square. This is the case when the number of equations (# of rows) equals the number of unknowns (# of cols). This is needed for solvability. You will talk a lot more about this in a linear algebra class.

Practice: write the 3x3 matrix from the beginning of the notes as a set of vectors & matrices.

$$\underline{\underline{A}} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

II. Index notation

* when using arrays and matrices by hand it is useful to use vector notation as a shorthand.

$$\underline{\underline{A}} \cdot \underline{x} = b \quad \text{or} \quad \underline{\underline{A}} \cdot \underline{\underline{B}}$$

* when programming, it is more useful to use index notation:

vector norm: $|\underline{v}| = \sqrt{\sum_i v_i^2}$ (2-norm)

dot product: $\underline{v} \cdot \underline{w} = \sum_i v_i w_i$

matrix-vector product: $\underline{\underline{A}} \cdot \underline{x} = \sum_j A_{ij} x_j = y_i$

matrix-matrix product: $\underline{\underline{A}} \cdot \underline{\underline{B}} = \sum_j A_{ij} B_{jk} = C_{ik}$

* Some tips:
  - The sum is always over the repeated index
  - The repeated index disappears
  - If in doubt, write it out!
  - Each unique index is a loop.

Example:

$$\underline{\underline{A}} = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

$$\underline{\underline{A}} \cdot \underline{x} = \begin{bmatrix} A_{00} x_0 + A_{01} x_1 \\ A_{10} x_0 + A_{11} x_1 \end{bmatrix} = \begin{bmatrix} \sum_j A_{0j} x_j \\ \sum_j A_{1j} x_j \end{bmatrix} = \sum_j A_{ij} x_j$$

what is repeated?

i is not repeated

In Python:

```
y = np.zeros(2)
for i in range(2):
    for j in range(2):
        y[i] += A[i,j] * x[j]
```

II. Solving Linear Systems

* How do you solve a linear system? You already know how to do this (hopefully from high school)! We are going to review a systematic way of doing this called Gauss Elimination. Next time we will learn to code it.

Example: $\underline{\underline{A}} \cdot \underline{x} = \underline{b}$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Rules:

- can multiply any row (an equation) by a constant
- can add rows (equations) together

Objective:

- step 1: make an upper triangular matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

- step 2: solve for x_2, then use x_2 to find x_1, then use x_2 & x_1 to find x_0

Col 0: get zeros below diagonal

Row 1 first →
$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ -1 & 2 & -1 & 8 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

• take ratio w/ diagonal : $\frac{-1}{3}$ ← current row / ← diagonal

$$\text{Row 1} - \underbrace{\left(\tfrac{-1}{3}\right)}_{\text{ratio}} \text{Row 0} \rightarrow \text{Row 1}$$

(replace)

need minus sign

↳ should immediately cancel element we want to make 0.

Row 2 →
$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

$-1 - (-1/3)(3) = 0$

$2 - (-1/3)(-1) = 5/3$

$-1 - (-1/3)(1) = -2/3$

$8 - (-1/3)(-3) = 7$

don't forget 'b'

ratio: $(1/3)$

Row 2 – $(1/3)$ Row 0 → Row 2

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ 0 & -2/3 & -4/3 & 2 \end{array}\right]$$

col 1

• now move to column 1.

• start below diagonal

• Row 2 – Ratio · $\underbrace{\text{Row 1}}_{\text{diagonal}}$ → Row 2

$\frac{\text{current row}}{\text{diagonal}}$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -3 \\ 0 & 5/3 & -2/3 & 7 \\ 0 & 0 & \frac{-24}{15} & \frac{24}{5} \end{array}\right]$$

$-2/3 - \left(\frac{-2/3}{5/3}\right) 5/3 \rightarrow 0$ ✓

$-4/3 - \left(\frac{-2}{5}\right) \cdot \left(\frac{-2}{3}\right) \rightarrow -24/15$

$2 - \left(\frac{-2}{5}\right)(7) \rightarrow \frac{24}{5}$

✱ Upper triangular! (end step 1)

Review: (of step 1)

- Start in col 0. Loop over rows below diagonal
- Find ratio : current row / diagonal
- Multiply row w/ diagonal & add to current row → replace row.
- Loop over columns.

<u>step 2</u>: solve for x's, start w/ $x_2 \to x_1 \to x_0$

$$-\frac{24}{15}x_2 = \frac{24}{5} \quad \to \quad x_2 = \frac{24}{5}\left(\frac{-15}{24}\right) = -3, \quad \boxed{x_2 = -3}$$

$$\frac{5}{3}x_1 - \frac{2}{3}x_2 = 7 \quad \to \quad x_1 = \left(\frac{3}{5}\right)\left(7 + \frac{2}{3}x_2\right)$$

$$= \frac{3}{5}\left(7 + \frac{2}{3}\cdot -3\right) = \frac{3}{5}(5)$$

$$\boxed{x_1 = 3}$$

$$3x_0 - x_1 + x_2 = -3$$

$$x_0 = \frac{1}{3}(-3 + x_1 - x_2)$$

$$= \frac{1}{3}(\cancel{-3} + \cancel{3} + (+3)) = 1$$

$$x_0 = 1$$

$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

<u>Review</u>: (of step 2)
- start at bottom and solved for x_2
- worked backwards up the rows, substituting each one we learned.

<u>Comments</u>:
- There are easier ways, but they are not <u>systematic</u> enough to code.
- Next time we will code this solver!
- Can always check your answer!

$$\sum_j A_{ij}\, x_j \overset{?}{=} b_i$$