Lecture 13 - Fixed Point Methods

* AMA / Quiz / Prayer

I. Non-linear Equations

* Today we move on from linear systems to non-linear systems. What is a non-linear equation? Any algebraic equation that isn't linear. (e.g. non-mormon is any one who isn't mormon - there are many kinds).
* There is a formal definition of a linear system I will leave for your math classes, but typically a non-linear equation has either polynomial terms (e.g. x^2, x^3) and/or transcendental terms (e.g. $e^x, \sin x$).
* Non-linear equations are harder to solve than linear equations, but they are very common in engineering.
* If the number of independent equations equals the number of unknowns in a linear system, we are guaranteed a single, unique solution. We don't have any such guarantee in a non-linear equation.
* Consider two examples:
  - an implicit equation: $\cos(2\pi x) - \frac{x}{3} = 0$

    (see python plot)

- No matter how we re-arrange, we can't isolate x.
- There are many solutions (I count 13).

- A polynomial : $x^2 - p = 0$
  ↖ a number

(see python plot)

- we can solve by hand using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

  in this case : $x = \pm\sqrt{p}$

- we can have 0, 1, or 2 solutions.

* Our general problem looks like this:

$$f(x) = 0$$

1 equation ↗ ↖ 1 unknown

- This is called the standard form or the residual form.
- The solution or solutions, x^*, are called the roots of the function.

$\boxed{\text{Activity}}$ write the equation in standard form:

$$\log_{10} P = A - \frac{B}{T+C}$$

where P is the dependent variable, T is the independent variable and A, B, C are parameters.

II. Fixed Point methods

* Because there are no general ways to solve a nonlinear equation, we don't have any direct methods(*). Instead we only have iterative methods.

(*) we can of course code the quadratic formula or other known solved problems.

* The most common class of methods are called fixed-point methods.

* we re-write $f(x) = 0$ as $x = g(x)$

then we guess: $x^{(0)}$ and use it to find $x^{(1)}$

$$x^{(1)} = g(x^{(0)})$$
$$x^{(2)} = g(x^{(1)})$$
$$\vdots$$
$$x^{(k+1)} = g(x^{(k)})$$

when $x^{(k+1)} \to x^{(k)}$ then we have:

$$x^* = g(x^*) \Rightarrow f(x^*) = 0$$

x^* is a "fixed point" because you get back x^*.

x^* is a root.

* How do we get $g(x)$? We'll see in a minute.

Example Pseudo-code for a fixed-point method.

```
x = x_guess
tol = 10^-8
k = 0
res = f(x)
while (res > tol):
    x = g(x)
    k = k+1
    res = f(x)
```

* looks the same as the iterative method in the linear case!

* Again, we can:

(1) converge

(2) "stall out" (same as converge in this case!)

(3) Diverge

* Methods for getting $g(x)$:

A. Picard's method.

* In Picard's method we add an x to both sides of the standard form:

$$f(x) = 0$$

$$\underbrace{x + f(x)}_{g(x)} = x$$

$$\boxed{x^{(k+1)} = x^{(k)} + f(x^{(k)})}$$

* Picard's method is easy, but it doesn't always converge quickly (or sometimes at all!)

B. Newton's method

* In Newton's method, we have

$$g(x) = x - \frac{f(x)}{f'(x)} \leftarrow \frac{df}{dx}$$

(*) See supplemental notes for derivation of Newton's method.

so,

$$x^{(k+1)} = g(x^{(k)})$$

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}}$$

* Newton's method is much better than Picard's. We will talk more about this later.

Example Solve $x^2 = 4$ using Newton's method.

$f(x) = x^2 - 4$

$f'(x) = 2x$

$$x^{(k+1)} = x^{(k)} - \frac{[x^{(k)}]^2 - 4}{2x^{(k)}}$$

guess!

k	$x^{(k)}$	$g(x^k)$
0	1	$1 - \frac{1^2 - 4}{2(1)} = 1 - \frac{-3}{2} = \frac{5}{2}$
1	$5/2$	$5/2 - \frac{(5/2)^2 - 4}{2(5/2)} = 2.05$
2	2.05	$g(2.05) = 2.0006097561$
3	2.0006	$g(2.0006) = 2.0000$ (to 7 digits)

Activity Solve $e^x = 2$ using Newton's method.
Start w/ a guess of $x^{(0)} = 1$

Lecture 13 - Supplemental Notes

Derivation of Newton's method:

$f(x) = 0$ is our equation in standard form.

First, expand $f(x)$ in a Taylor Series about point $x^{(k)}$

$$f(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2} f''(x^{(k)})(x - x^{(k)})^2 + \dots$$

if x is the root and $x^{(k)}$ is close to the root, then:

$f(x) = 0$

$x - x^{(k)}$ is small $\Rightarrow$ $(x - x^{(k)}) >> (x - x^{(k)})^2$

$\Rightarrow$ we can neglect all of the higher-order terms.

$$0 \approx f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)})$$

solve for x

$$f'(x^{(k)})(x - x^{(k)}) = -f(x^{(k)})$$

$$x = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

make this our next guess

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}}$$