Lecture 21 – ODEs & Explicit Euler

* Prayer / Quiz / AMA

I. Introduction to Differential Equations

* This is our last type of problem we are going to solve in the class!

* Don't worry if you haven't had differential equations yet, I am going to cover what you need to know for this class. All you would need is the first day or two of Math 303 anyway :)

  Math 303 / 334 – Analytical methods for ODEs

  This class – Numerical methods for ODEs

* Differential Equations are equations which contain ... derivatives in them! They are very common in physics / chemistry.

  e.g. $$m\frac{d^2x}{dt^2} = F$$ (Newton's 2nd law)

  $$\frac{dc}{dt} = -kc^2$$ (2nd order chemical reaction rate)

* It is important to be able to classify differential equations, so you can identify how to solve them.

- <u>order</u> : The order of a differential equation is the order of the highest derivative.

  e.g. $m \frac{d^2x}{dt^2} = F$ — 2^{nd} order

  $\frac{dc}{dt} = -kc^2$ — 1^{st} order

  what is : $\frac{d^2x}{dt} + \frac{dx}{dt} - x^2 = 5$ (2^{nd} order)

- <u>linearity</u> : Just like algebraic equations, DEs can be linear or non-linear. In your DEs class you will only solve <u>linear</u> equations.

  $m \frac{d^2x}{dt^2} = F$ linear

  $\frac{dc}{dt} = -kc^2$ non-linear

  $y \frac{dy}{dt} + 5 = 0$ non-linear

  $\frac{dy}{dt} = -t^2$ <u>linear</u>! (in y)

- <u>Ordinary vs. Partial</u> : An ordinary differential equation (ODE) has derivatives w.r.t only one variable (ordinary derivatives)
  A Partial differential equation has derivatives w.r.t. 2 or more variables.

  ⊛ we will only look at ODEs in this class (& in 303/334).

  $\frac{dc}{dt} = -kc^2$ ODE

  $\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0$ PDE

- Initial / boundary conditions : All ODEs need auxiliary conditions to find a unique solution.

e.g.

$$\frac{dx}{dt} = v$$

speed ↗ ↖ constant

← special case, can solve by separating & integrating (not possible for all ODEs)

$$\int dx = \int v \, dt$$

indefinite integral

$$x = vt + C$$

only know $x(t)$ to within a constant.

- if I know $x(0) = x_0$, then

$$x(0) = v(0) + C = x_0 \Rightarrow C = x_0$$

$$x(t) = vt + x_0$$

- The auxiliary condition $x(0) = x_0$ is needed to uniquely specify $x(t)$.

- You need one auxiliary condition for each order of derivative (e.g. 1 for 1^{st} order, 2 for 2^{nd})

- If auxiliary conditions are specified at one point (e.g. $x(0)=0$, $\left.\frac{dx}{dt}\right|_{x=0}$) then the problem is an initial value problem (IVP) and the conditions are initial conditions.

- If not & the conditions are specified at different points, then the problem is a boundary value problem (BVP) & the conditions are boundary conditions (e.g. $x(0)=0$, $x(L)=0$).

II. The Explicit Euler method

* We will focus today only on a single, first order ODE IVP.

  Next time we will do higher order IVPs & systems of IVPs.

  (No PDEs, no BVPs, yes nonlinear!)

* A single, first order IVP has the form:

$$\left.\begin{aligned} \frac{dy}{dt} &= f(y,t) \\ y(0) &= y_0 \end{aligned}\right\}$$

You can think of this as a "rate equation"

LHS: rate of change of y

RHS: some function

IC: initial "position"

* How solve?

  (i) make a discrete grid for time

  (ii) approximate the derivative

  (iii) Use grid & approximate derivative to rewrite ODE for generic time point.

  } Explicit Euler method (formula below)

(i) discrete grid for time

n	0	1	2	3	4	5	6	7	8
t	0	0.5	1.0	1.5	2.0				
y	$y(0)$	$y(0.5)$	...						

(ii) approximate the derivative

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{y_{n+1} - y_n}{t_{n+1} - t_n} \qquad \text{e.g. } \frac{y_1 - y_0}{t_1 - t_0}$$

(iii) Putting (i) & (ii) together gives

$$\frac{dy}{dt} = f(y,t)$$

$$\downarrow$$

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} = f(y_n, t_n)$$

use the y & t we know on the grid.

Δt is normally a constant on our grid.

Solving for y_{n+1}

$$\boxed{y_{n+1} = y_n + \Delta t \, f(y_n, t_n)}$$

Explicit Euler method

III. Excel & Python

Activity

* Demonstration of coding the Explicit Euler method in Excel & Python

* Show the effect of step size on the accuracy & stability of the solution.

better for small Δt

"blows up" at large Δt

* Solve: $\frac{dc}{dt} = -\frac{c}{\tau}$ $\tau = 0.6$

  $c(0) = c_0$ $c_0 = 1$

$$c_{n+1} = c_n + \Delta t \left(-\frac{c_n}{\tau}\right) \quad \text{(Explicit Euler)}$$

$$c = c_0 e^{-t/\tau} \quad \text{(analytical)}$$