

Lecture 21 - ODEs & Explicit Euler

* Prayer / Quiz / AMA

I. Introduction to Differential Equations

* This is our last type of problem we are going to solve in the class!

* Don't worry if you haven't had differential equations yet, I am going to cover what you need to know for this class. All you would need is the first day or two of Math 303 anyway :)

Math 303 / 304 - Analytical methods for ODEs

This class - Numerical methods for ODEs

* Differential Equations are equations which contain ... derivatives in them! They are very common in physics / chemistry.

e.g. $m \frac{d^2 x}{dt^2} = F$ (Newton's 2nd law)

$$\frac{dc}{dt} = -kc^2 \quad (\text{2nd order chemical reaction rate})$$

* It is important to be able to classify differential equations, so you can identify how to solve them.

- order : the order of a differential equation is the order of the highest derivative.

e.g. $m \frac{d^2x}{dt^2} = F$ - 2nd order

$$\frac{dc}{dt} = -kc^2 \text{ - 1st order}$$

what is : $\frac{d^2x}{dt^2} + \frac{dx}{dt} - x^2 = 5$ (2nd order)

- linearity : Just like algebraic equations, DEs can be linear or non-linear. In your DEs class you will only solve linear equations.

$$m \frac{d^2x}{dt^2} = F \quad \text{linear}$$

$$\frac{dc}{dt} = -kc^2 \quad \text{non-linear}$$

$$y \frac{dy}{dt} + 5 = 0 \quad \text{non-linear}$$

$$\frac{dy}{dt} = -t^2 \quad \text{linear! (in y)}$$

- Ordinary vs. Partial : An ordinary differential equation (ODE) has derivatives w.r.t only one variable (ordinary derivatives). A Partial differential equation has derivatives w.r.t. 2 or more variables.
- ⊗ we will only look at ODEs in this class (i.e. in 303/334).

$$\frac{dc}{dt} = -kc^2 \quad \text{ODE}$$

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 \quad \text{PDE}$$

- Initial / boundary conditions : All ODEs need auxiliary conditions to find a unique solution.

e.g.

$$\frac{dx}{dt} = v$$

↑ ↑
speed constant

← special case, can solve by separating & integrating (not possible for all ODEs)

$$\int dx = \int v dt$$

indefinite integral

$$x = vt + c$$

only know $x(t)$ to within a constant.

- If I know $x(0) = x_0$, then

$$x(0) = v(0) + c = x_0 \Rightarrow c = x_0$$

$$x(t) = vt + x_0$$

- The auxiliary condition $x(0) = x_0$ is needed to uniquely specify $x(t)$.
- You need one auxiliary condition for each order of derivative (e.g. 1 for 1st order, 2 for 2nd)
- If auxiliary conditions are specified at one point (e.g. $x(0) = 0$, $\frac{dx}{dt}|_{x=0}$) then the problem is an initial value problem (IVP) and the conditions are initial conditions.
- If not, the conditions are specified at different points, then the problem is a boundary value problem (BVP) & the conditions are boundary conditions (e.g. $x(0) = 0$, $x(L) = 0$).

II. The Explicit Euler method

* We will focus today only on a single, first order ODE IVP.

Next time we will do higher order IVPs & systems of IVPs.

(No PDEs, no BVPs, yes nonlinear!)

* A single, first order IVP has the form:

$$\frac{dy}{dt} = f(y, t)$$

$$y(0) = y_0$$

} You can think of this
as a "rate equation"

LHS: rate of change of y

RHS: some function

IC: initial "position"

* How solve?

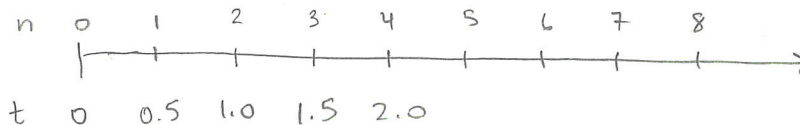
(i) make a discrete grid for time

(ii) approximate the derivative

(iii) Use grid & approximate derivative
to rewrite ODE for generic time
point.

} Explicit Euler
method
(formula below)

(i) discrete grid for time



y $y(0)$ $y(0.5)$...

(ii) approximate the derivative

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{y_{n+1} - y_n}{t_{n+1} - t_n} \quad \text{e.g.} \quad \frac{y_1 - y_0}{t_1 - t_0}$$

(iii) Putting (i) & (ii) together gives

$$\frac{dy}{dt} = f(y, t)$$

↓

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} = f(y_n, t_n)$$

use the y & t we know on the grid.

Δt is normally a constant on our grid. →

↓ Solving for y_{n+1}

$$y_{n+1} = y_n + \Delta t f(y_n, t_n)$$

Explicit Euler method

III. Excel & Python

Activity

* Demonstration of coding the Explicit Euler method in Excel & Python

* Show the effect of step size on the accuracy & stability of the solution.

↑
better
for small Δt

↑
"blows up" at large Δt

* Solve: $\frac{dc}{dt} = -\frac{c}{\tau}$ $\tau = 0.6$
 $c(0) = c_0$ $c_0 = 1$

$$c_{n+1} = c_n + \Delta t \left(-\frac{c_n}{\tau} \right) \quad (\text{Explicit Euler})$$

$$c = c_0 e^{-t/\tau} \quad (\text{analytical})$$