Lecture 22 - Systems of ODEs

*Prayer / Quiz / AMA

I. Scipy's ODE solver

* Last time we learned how to solve IVPs of the form:

$$\frac{dy}{dt} = f(y,t) \quad , \quad y(0) = y_0$$

- first-order
- linear or non-linear
- ODE
- initial condition

using the Explicit Euler method in Excel & Python.

* Today we will learn Python's library function called "solve_ivp", which uses more accurate methods. (By default it uses the Runge-Kutta 45 method).

Activity

* Demonstrate the use of solve_ivp for a single, first order IVP.

* Show how to get for arbitrary times.

* let students practice.

II. Systems of ODEs.

* There are many cases where we have more than one dependent variable that we care about in a differential equation.

* For example:

$$A + B \rightarrow C$$

$$\frac{dA}{dt} = -kAB \quad , \quad A(0) = 2 \text{ mol/L}$$

$$\frac{dB}{dt} = -kAB \quad , \quad B(0) = 1 \text{ mol/L}$$

$$\frac{dC}{dt} = kAB \quad , \quad C(0) = 0 \text{ mol/L}$$

* How can we solve a system of these 1st order ODEs?

* It turns out that a system is only slightly harder than a single ODE. Let's re-write the example above using different notation:

$$A \rightarrow y_0 \quad , \quad B \rightarrow y_1 \quad , \quad C \rightarrow y_2$$

$$\frac{dy_0}{dt} = f_0(t, y_0, y_1, y_2) \quad , \quad y_0(0) = y_{0,0}$$

$$\frac{dy_1}{dt} = f_1(t, y_0, y_1, y_2) \quad , \quad y_1(0) = y_{1,0}$$

$$\frac{dy_2}{dt} = f_2(t, y_0, y_1, y_2) \quad , \quad y_2(0) = y_{2,0}$$

* Using vector notation this gets even more compact!

$$\frac{d}{dt}\underline{y} = \underline{f}(t, \underline{y}) \qquad \underline{y}(0) = \underline{y}_0$$

$$\underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{bmatrix}$$

* The vector form shows that, in fact, the problem is almost exactly the same as a single ODE! In a minute we will see how to solve this using solve_ivp.

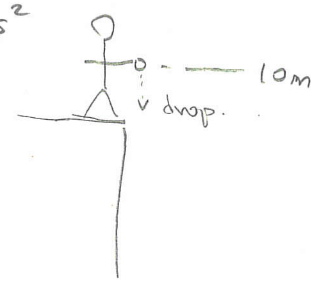
III. Higher order ODE IVPs

* There are lots of examples where we want to solve ODEs that are higher than 1^{st} order.

* For example:

$$m \frac{d^2x}{dt^2} = -g \qquad g = 9.8 \, m/s^2$$

$$x(0) = 10 \, m \, , \quad x'(0) = 0 \, m/s$$



* Lucky for us, there is a trick we can use to turn all higher order ODEs into a system of first order ODEs. To do this, we define lower order derivatives to be intermediate variables

e.g. $v \equiv \frac{dx}{dt}$ $\qquad x'(0) = v(0) = 0 \, m/s$

define

$$m \frac{d^2x}{dt^2} = m \frac{d}{dt}\left(\frac{dx}{dt}\right) = m \frac{dv}{dt}$$

notice how the initial conditions work out.

so:

$$\frac{dv}{dt} = -\frac{g}{m} \qquad v(0) = 0$$

$$\frac{dx}{dt} = v \qquad x(0) = 10$$

* So, in general, we need to:

(1) Define a new intermediate variable for each order of derivative until we have only 1st order derivatives

(2) Substitute our definition into the original equation & initial conditions.

* Another example:

$$\frac{d^3y}{dt^3} = f(t,y) \qquad y''(0) = A \ , \ y'(0) = B \ , \ y(0) = C$$

- list lower order derivatives

$$\frac{d^2y}{dt^2} \ , \ \frac{dy}{dt}$$

- define new vars:

$$a = \frac{d^2y}{dt^2} \qquad v = \frac{dy}{dt}$$

- substitute:

$$\frac{da}{dt} = f(t,y) \qquad a(0) = A$$

$$\frac{dv}{dt} = a \qquad v(0) = B$$

$$\frac{dy}{dt} = v \qquad y(0) = C$$

$\boxed{\text{Practice}}$ Express the IVP as a system of 1st order ODEs

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + ty = \cos(t)$$

$$y'(0) = -2 \qquad y(0) = 2$$

- lower order derivative: $\frac{dy}{dt} = v$.

$$\frac{dv}{dt} + 3v + ty = \cos t \quad , \quad v(0) = -2$$

$$\frac{dy}{dt} = v \quad , \quad y(0) = 2$$

- in std form!

$$\frac{dv}{dt} = -3v - ty - \cos t \quad , \quad v(0) = -2$$

$$\frac{dy}{dt} = v \quad , \quad y(0) = 2$$

Activity

* Show students how to use solve_ivp for systems of ODEs
* Show students how to use solve_ivp for higher order ODEs.
* Let students practice