Lecture 11 – Gauss Elimination

Outline

I. Gauss Elimination in Python

A. Forward Elimination

B. Back Substitution

* Last time we learned a systematic method for solving a linear system of equations. This algorithm is called <u>Gauss Elimination</u>.

* A general $n \times n$ matrix:

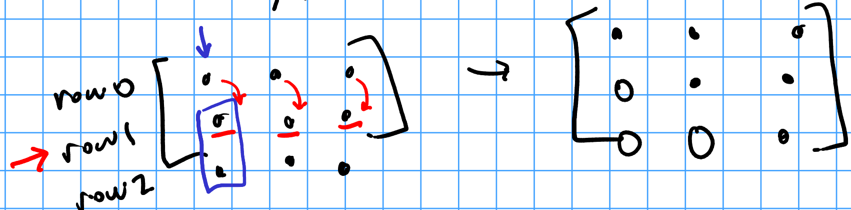
$$\underline{\underline{A}} \cdot \underline{x} = \underline{b} \quad \longrightarrow \quad \sum_{j=0}^{n-1} a_{ij}\, x_j = b_i$$

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0,n-1} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

A. Forward Elimination

* The first step is forward elimination

$$A \rightarrow U$$



Procedure:

- start in column $\overset{k}{0}$ (goal: eliminate all rows below diagonal)
- Find ratio: $\dfrac{a_{10}}{a_{00}}$ ← diagonal on bottom $(x-1)$

- multiply all elements of row 1 by ratio an then add row 0. → replace row 1. (don't forget b_1)
- Repeat for all rows to $n-1$ (i)
- Repeat for all columns to $n-2$ (k)

* we can write this more precisely w/ index notation

which "update" // which element on the diagonal

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \cdot a_{kj}^{(k)}$$

row, col

which column/diagonal I'm working on.

ratio w/ diagonal on bottom

(1 b/4 end)

$k: 0, 1, \ldots, n-2$

$i: k+1, k+2, \ldots, n-1$

$j: k, k+1, \ldots, n-1$

Rewrite:

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}$$

$$b_i^{(k+1)} = b_i^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \cdot b_k^{(k)}$$

(don't forget RHS!)

End Result:

$$\begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & \cdots & a_{0,n-1}^{(0)} \\ 0 & a_{11}^{(1)} & \cdots & a_{1,n-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & a_{n-1,n-1}^{(n-1)} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_0^{(0)} \\ b_1^{(1)} \\ \vdots \\ b_{n-1}^{(n-1)} \end{bmatrix}$$

$$\underline{\underline{u}} \cdot \underline{x} = \underline{b}_{new}$$

(updated)

B. Back Substitution

* Back sub is the 2^{nd} step in G.E.

$$\begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & \cdots & a_{0,n-1}^{(0)} \\ 0 & a_{11}^{(1)} & \cdots & a_{1,n-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & a_{n-1,n-1}^{(n-1)} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_0^{(0)} \\ b_1^{(1)} \\ \vdots \\ b_{n-1}^{(n-1)} \end{bmatrix}$$

Procedure

- start at the bottom (row $= n-1$). Solve for x_{n-1}
- Move to row $n-2$. Move known x_i & coefts to the RHS w/ b_i.
- Divide by diagonal (a_{ii}) to solve for x_i.
- Repeat moving up the matrix until top.

Write precisely w/ index notation (Back Sub.)

$$x_{n-1} = \frac{b_{n-1}}{a_{n-1,n-1}}$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=i+1}^{n-1} a_{ij} x_j \right)$$

repeat for $i = n-2, n-3, \ldots, 0$