Lecture 13 – Fixed Point Methods

Outline

I. Non-linear Equations

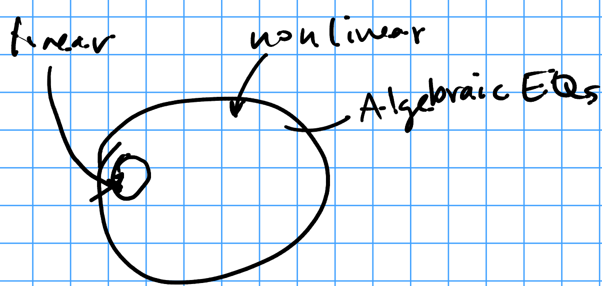
II. Fixed-Point Methods

A. Picard's method

B. Newton's method

I. Non-linear Equations

* What is a nonlinear Equation?



Anything that is <u>not</u> linear.

* typically, you encounter: polynomial terms: x^2, x^3

  transcendental : $\sin x, \ln x, e^x$

* Nonlinear EQ's are <u>common</u> in engineering.

* In linear EQ's
  - n Equations & n unknowns $\rightarrow$ unique solution

* In nonlinear EQs
  - n EQs & n unknowns $\rightarrow$ 0 solns / 1 soln / ∞ solns ?

No Guarantees! $\rightarrow$ **Hard**

* Consider 2 examples
  - an implicit equation : $\cos(2\pi x) - \frac{x}{3} = 0$ $\leftarrow$ can't isolate x

    13 solutions!

  - A polynomial : $x^2 - p = 0$ p is a const.

quad formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad a=1$, $b=0$, $c=-p$

$$x = \pm\sqrt{p}$$

0, 1, 2 solutions depending on p!

* there are special cases (e.g. quadratic polynomial) with direct solutions. But there are no general direct methods for nonlinear equations

* General problem → std format $\qquad |A \cdot x - b| \to 0$

$$\boxed{f(x) = 0}$$

eq. unknown

"standard form"

"residual form" → 0

$x = x^*$ ← root

Activity write the following in std form.

$$\log_{10} P = A - \frac{B}{T+C}$$

solve for T given P, A, B, C

✓ $-\log_{10} P + A - \dfrac{B}{T+C} = 0$

✓ $P - 10^{A - \frac{B}{T+C}} = 0$

$\cancel{A - \dfrac{B}{T+C} = 0}$

$\cancel{10^{A - \frac{B}{T+C}}} = 0$

$f(x) = 0$

$\cancel{P(T) = 0}$

II. Fixed Point Methods

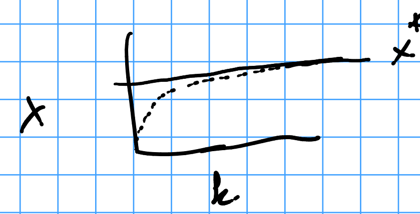
* No general direct methods (quad formula, Gauss Elim) for NLEQs

* Instead → Iterative methods
↖ "guess & check" then repeat until "converge"
(stops changing)

* Fixed point methods are a <u>class</u> of iterative methods.

* General strategy

$$f(x) = 0 \xrightarrow{\text{re-write}} x = g(x)$$



guess $x_0 \rightarrow g(x_0)$

$x_1 \rightarrow g(x_1)$

$x_2 \rightarrow g(x_2)$

$\vdots$

$$x^{(k+1)} = g(x^{(k)})$$

$$x^* = g(x^*)$$

$\downarrow$

$f(x^*) = 0$ so x^* is a root.

x is at a "fixed point" → it does not change.
$\quad \parallel$
$\quad x^*$

* How do we pick $g(x)$? → A. Picard's method.
B. Newton's method.

(Detour)

Example Pseudo code for a fixed point method.

```
x = x_guess
tol = 1e-8      # 10^-8
k = 0
res = abs(f(x))   # residual = 0 when @ root
while (res > tol):
```

f(x) : defined above
g(x) : " "

$x = g(x)$ # when $f(x) \to 0$ this will quit

k += 1 converge

res = abs(f(x))

A. Picards' Method.

* we simply add x to both sides :

$$f(x) = 0$$

$$\underbrace{x + f(x)}_{g(x)} = x \qquad g(x) = x + f(x)$$

$$\boxed{x^{(k+1)} = x^{(k)} + f(x^{(k)})}$$

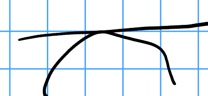
- Easy!
- Not reliable! (Doesn't always converge).

B. Newton's method.

* Derivation is in my online notes (TaylorSeries)
* $g(x) = x - \frac{f(x)}{f'(x)}$ ← Derivative $\frac{df}{dx}$

tangent line, find the zero.

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}}$$



0 ↑ (when does it break)

- harder than Picard
- way more reliable

$\boxed{\text{Example}}$ Solve for $x^2 = 4$ w/ Newton's method.

$f(x) = x^2 - 4 = 0$ (std form!)

$f'(x) = 2x$

$$x^{(k+1)} = x^{(k)} - \frac{(x^k)^2 - 4}{2x^k}$$

k	$x^{(k)}$	$g(x^{(k)})$
0	1	5/2
1	5/2	2.05
2	2.05	2.00061
3	2.00061	2.000000 (7 digits)

$\boxed{\text{Activity}}$ Solve $e^x = 2$ using Newton's method.

guess $x^{(0)} = 1$. Answer: $x = \ln 2 = 0.6931$