Lecture 19 – Newton Coates Integration

* AMA / Quiz / Prayer

I. Newton Coates Integration

* In calculus, you learned how to integrate a function

$$I = \int_a^b f(x)\, dx$$

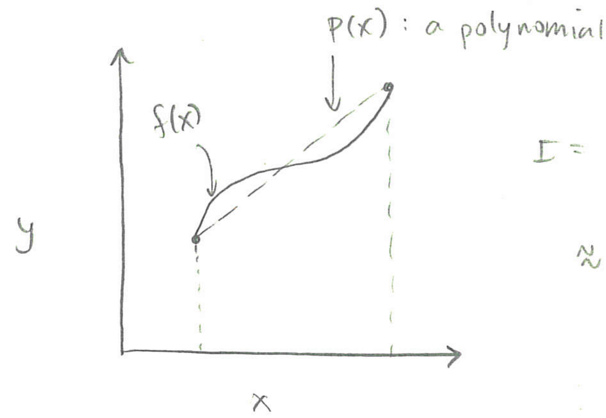
* Integrals can be difficult to evaluate analytically! Some are easy (polynomials); some can be done by special techniques (integration by parts, u-substitution); some are impossible (e.g. error function). Luckily, we can do them numerically!

* Numerical integration goes by the alternate name "quadrature" or "numerical quadrature."

* How can we do the integral

$$I = \int_a^b f(x)\, dx$$

One way (there are others) is to use the idea that <u>polynomials are easy to integrate</u>. If we replace our function $f(x)$ with a polynomial $P(x)$, then it would be easy to do.



$$I = \int_a^b f(x)\,dx \approx \int_a^b P(x)\,dx$$

* what polynomial do we use? we use one that <u>interpolates</u> the values of our function between discrete points.

<u>Example</u> : A linear polynomial

$$I = \int_a^b f(x)\,dx$$

- we need an interpolating function between $(a, f(a))$ and $(b, f(b))$. $\rightarrow$ Linear Interpolation!

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

$$P(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

- Approximate the integral using $P(x)$:

$$I \approx \int_a^b P(x)\,dx$$

$$I \approx \int_a^b \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right] dx$$

$$\approx \left[\frac{1}{2} \frac{f(b)-f(a)}{b-a}(x-a)^2 + f(a)\,x \right]_a^b$$

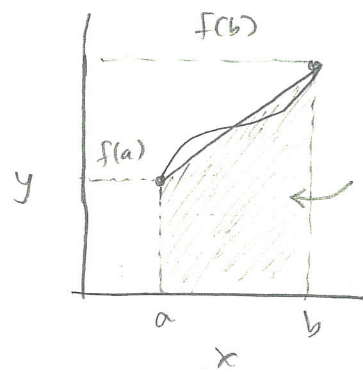
$$\approx \frac{1}{2} \frac{f(b)-f(a)}{b-a}(b-a)^2 + b f(a) - 0 - a f(a)$$

$$\approx \frac{1}{2}\left[f(b)-f(a)\right](b-a) + (b-a) f(a)$$

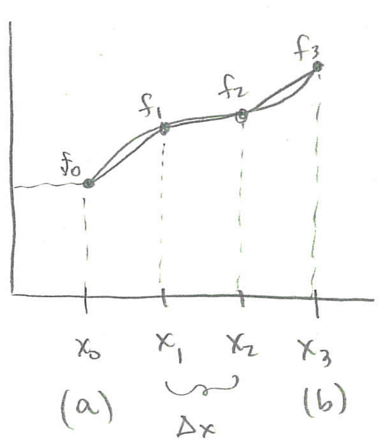
$$\approx \frac{b-a}{2}\left[f(b) - f(a) + 2f(a) \right]$$

$$\boxed{I \approx \frac{1}{2}(b-a)\left[f(a) + f(b) \right]}$$

* this is the area of a trapezoid, so this method is called the trapezoidal rule.



* Just like we saw with interpolation, it isn't very accurate to use a single polynomial to fit the entire integrand. Instead we break the interval into pieces and do piecewise interpolation.



This method is called the composite trapezoidal rule

$$I = \int_a^b f(x)\,dx \approx \int_a^b P(x)\,dx$$

$$\approx \int_{x_0}^{x_1} P(x)\,dx + \int_{x_1}^{x_2} P(x)\,dx + \int_{x_2}^{x_3} P(x)\,dx$$

assume Δx is the same.

$$\approx \frac{1}{2}\Delta x\,(f_0 + f_1) + \frac{1}{2}\Delta x\,(f_1 + f_2) + \frac{1}{2}\Delta x\,(f_2 + f_3)$$

$$\approx \frac{1}{2}\Delta x\,(f_0 + 2f_1 + 2f_2 + f_3)$$

for $n+1$ points

$$\boxed{I \approx \frac{1}{2}\Delta x\,[f_0 + 2f_1 + \ldots + 2f_{n-1} + f_n]}$$

* Using higher order polynomials leads to even better approximations! These class of methods are called Newton-Coates Integration formulas.

Quadratic Polynomials $\rightarrow$ Simpson's 1/3 rule

$$I \approx \frac{\Delta x}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 4f_{n-1} + f_n\right]$$

odd even

Cubic Polynomials $\rightarrow$ Simpson's 3/8 rule

$$I \approx \frac{3\Delta x}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \ldots + f_n\right]$$

1, 2 every 3rd one has a 2

II. Examples using NC formulas in Excel & Python

* Code in class for the students

$$I = \int_0^1 \sin(\pi x)\cos^2(\pi x)\,dx$$

(exact answer: $\frac{2}{3\pi}$)

* If time do in python & Excel.