Lecture 19 – Newton-Coates Integration

Outline

I. Intro to NC Integration formulas

II. Examples

I. Newton Coates Integration Formulas

* In calculus, you learned how to evaluate

$$I = \int_a^b f(x)\,dx$$

→ Analytical ~~soln~~

↳ Numerical ☺

* Notation: "Quadrature"
  ↳ Numerical

$f(x)$ ~ polynomial → easy

$f(x)$ ~ $\sin(x)$, $\exp(x)$ → tables/memorize

~ combos → u-sub, int. by parts

– impossible → $\int_0^\infty e^{-x^2}\,dx$ ← erf(x)

* How solve for I numerically?

$$I = \int_a^b f(x)\,dx$$

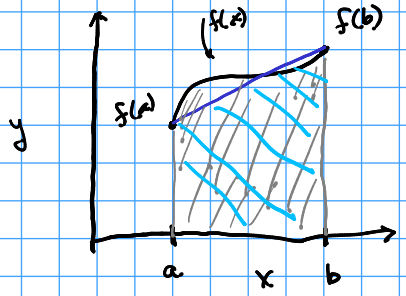
(1) $f(x)$ in general can't be done by hand

But polynomials can.

$f(x) \rightarrow P(x)$ ☺

(2) Interpolation can do this!

piecewise polynomial → approx $f(x)$



$$I = \int_a^b f(x)\,dx$$

$$I \approx \int_a^b P(x)\,dx$$

* Example: Linear Interpolation

$$I = \int_a^b f(x)\,dx$$

$$\approx \int_a^b P(x)\,dx$$

$$\approx \int_a^b \left[(x-a)\frac{f(b)-f(a)}{b-a} + f(a)\right] dx$$

$$\approx \left[\frac{(x-a)^2}{2}\frac{f(b)-f(a)}{b-a} + f(a)x\right]\Bigg|_a^b = \frac{(b-a)^2}{2}\frac{f(b)-f(a)}{b-a} + bf(a) - 0 - af(a)$$

$$\approx \frac{b-a}{2}\left[f(b)-f(a)\right] + (b-a)f(a) = \frac{(b-a)}{2}\left[f(b) - f(a) + 2f(a)\right]$$

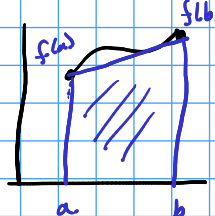
$$\boxed{I \approx \frac{b-a}{2}\left[f(a)+f(b)\right]}$$

Trapezoidal Rule

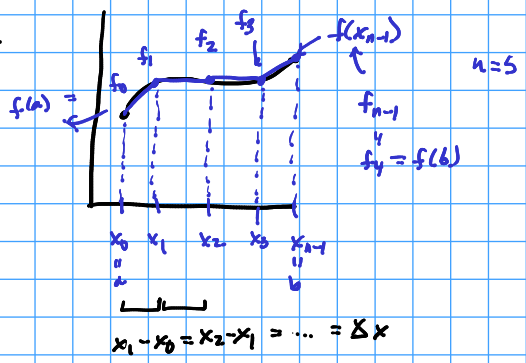
Interpolate: $(a, f(a))$ & $(b, f(b))$

linear: (equal slopes) $\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$

$$P(x) = y = (x-a)\frac{f(b)-f(a)}{b-a} + f(a)$$



* Just like interpolation, we can be more accurate with piecewise functions!



$$I = \int_a^b f(x)\,dx \approx \int_a^b P(x)\,dx$$

$$\approx \int_{x_0}^{x_1} P_0(x)\,dx + \int_{x_1}^{x_2} P_1(x)\,dx + \dots + \int_{x_{n-2}}^{x_{n-1}} P_{n-2}(x)\,dx$$

TR 0 | TR 1 | TR n-2

$$\approx \frac{x_1 - x_0}{2}[f_0 + f_1] + \frac{x_2 - x_1}{2}[f_1 + f_2] + \dots + \frac{x_{n-1} - x_n}{2}[f_{n-1} + f_{n-2}]$$

$$\approx \frac{\Delta x}{2}\left[f_0 + f_1 + f_1 + f_2 + \dots + f_{n-2} + f_{n-1}\right]$$

$$\boxed{I \approx \frac{\Delta x}{2}\left[f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-2} + f_{n-1}\right]}$$ ✓

composite trapz-rule.

C.T.R.

* What about other interpolating functions?

lines, quadratic?, cubic?

$P(x) \sim ax^2 + bx + c \rightarrow \int_a^b x^2 + x + \dots$

✓ Quadratic: $I \approx \frac{\Delta x}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 4f_{n-2} + f_{n-1}\right]$

end pts coeff 1 | odd #'s have 4 | evens have 2

Simpson's 1/3 rule

✓ Cubic: $I \approx \frac{3\Delta x}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \dots + f_{n-1}\right]$

coeff 1 | 3 | 3 | 2 | 3 | 3 | 2 | coeff 1

every 3rd coeff is a 2

Simpson's 3/8 rule (8)

↘ Newton-Coates Formulas

II. Examples: Python & Excel

$$I = \int_0^1 \sin(\pi x)\cos^2(\pi x)\,dx \xrightarrow{\text{(by hand)}} \frac{2}{3\pi}$$ (check our answer)