Lecture 21 - ODEs and Explicit Euler

Outline

I. Intro to Differential Equations

II. The Explicit Euler Method

III. Examples in Python & Excel

I. Introduction to Differential Equations

* In an ODEs course (Math 333/334) you cover analytical methods for solving ODEs
  → Numerical Methods are quite different.

* What is a differential equation?
  → Any equation w/ derivatives in it!

ex. $m\frac{d^2x}{dt^2} = F$ (Newton's 2^{nd} law) → super common in physical sciences

$x(t)$

$c(t)$ $\leftarrow \frac{dc}{dt} = -kc^2$ (2^{nd} order chem rxn)

* How classify DEs? → Give us a language to be able to talk about.

  • **Order**: The number order of the highest derivative

$$m\frac{d^2x}{dt^2} = F \rightarrow 2^{nd} \text{ order D.E. } (2^{nd} \text{ derivative})$$

$$\frac{dc}{dt} = -kc^2 \rightarrow 1^{st} \text{ order D.E. } (1^{st} \text{ order derivative})$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - x^2 = 5 \quad ? \quad (2^{nd} \text{ order})$$

  • **linearity**: Just like algebraic Equations, DEs can be linear or nonlinear. In DE class → stick to linear.

$m\frac{d^2x}{dt^2} = F \leftarrow t,$ ~~thing~~ mg (linear) ← no powers of terms w/ x in them

$\frac{d}{dt}\left(\frac{dx}{dt}\right)$

$x(t) \rightarrow x(t)^2$

$\frac{dc}{dt} = -kc^2$ (non-linear)

$c(t)^2$

$\left(\frac{dx}{dt}\right)^2$ or $x \cdot \frac{dx}{dt}$

$\frac{dx}{dt} \cdot \frac{dx}{dt}$

(ok) → t^2

$f(cx+by) \rightarrow cf(x) + bf(y)$

$\frac{d(cf+bg)}{dt} \rightarrow$

$y\frac{dy}{dt} + 5 = 0$ (non linear)

$\frac{dy}{dt} = -t^2$ (linear)

- **Ordinary vs. Partial DEs** : an ordinary DE (ODE) has derivatives w.r.t only one variable. while a Partial D.E. has derivatives w.r.t. more than one variable

$$\frac{dc}{dt} = -kc^2 \qquad \frac{d}{dt} \rightarrow c(t) \text{ only}$$

reg. derivatives

O.D.E

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 \qquad c(x,y) \rightarrow \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}$$

partial derivatives.

P.D.E.

Goal: $c(t) = ?$ $\quad c(x,y)?$

- **Initial/boundary conditions** : All DEs need auxilliary conditions to find a unique solution

think about this:

$$\frac{dx}{dt} = v \leftarrow \text{constant}$$

speed

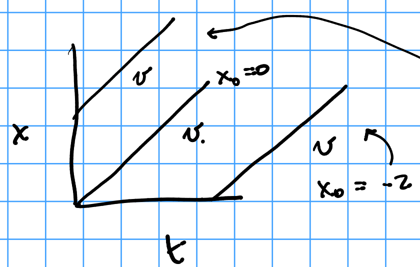
separate & integrate (indef. integral)

$$x = \int v \, dt = vt + c$$

const

$$x(t=0) = v \cdot 0 + c$$

x_0



$x_0 = c = 2$

- you need an auxiliary condition to specify a unique solution
- you need one condition for each order of derivative

$$\frac{dx}{dt} = v \rightarrow x_0$$

1

$$\frac{d^2x}{dt^2} = -g \rightarrow$$

2

x_0 - initial pos

v_0 - initial vel.

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5 = -2 \rightarrow \text{2 conditions}$$

- Initial conditions are specified at the same point.

$$\frac{dx}{dt} = v \qquad x(t=0) = 2$$

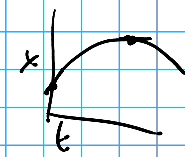
initial condition

$$\frac{d^2x}{dt^2} = -g \qquad x(t=0) = 5$$

$$\frac{dx}{dt}(t=0) = 2$$

initial conditions

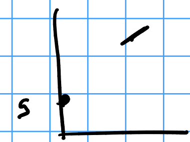
$\frac{d^2x}{dt^2} = -g$ $\quad x(t=5) = 5$, $\quad \frac{dx}{dt}(t=5) = 2$ → also Initial conditions!



A boundary condition is anything else, i.e. when conditions are at separate points

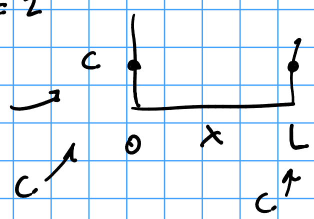
$\frac{d^2x}{dt^2} = -g$ $\quad x(t=0) = 5$, $\quad \frac{dx}{dt}(t=5) = 2$

specified at different points → Boundary condition



$\frac{d^2c}{dx^2} = 2$

$c, \frac{dc}{dx}$ → or



$C(0) = 1$, $C(L) = 7$ } specify c at 2 pts not $\frac{dc}{dx}$ at 1 pt.

B.C are harder.

- problems w/ I.C's are called Initial value problems → IVPs
- " " B.C's " " Boundary value problems → BVPs

⇒ Focus: any order, non-linear (or linear) ODEs, IVPs *

In Math dept: any order (2nd or less), linear, ODEs, IVPs
(1 week of BVPs) — 1 case of nonlinear (1st order only)

II. The Explicit Euler method

* ODEs, → today 1st order ODE, IVP.

* A single first order IVP has the form:

$$\frac{dy}{dt} = f(y,t), \quad y(0) = y_0$$

"rate equation" "speed"
LHS: rate of change of y
RHS: function
IC: initial "position"

* How solve? "Explicit Euler" method.

(i) Make a discrete time grid.

(ii) approximate the derivative on this grid

(iii) use grid & derivative to rewrite the ODE.

$$\frac{dy}{dt} = f(y,t)$$
$$\boxed{y(t=0) = y_0}$$

(i) Make a discrete grid

n	0	1	2	3	4	5	
t	0	0.5	1.0	1.5	2.0	2.5	
y	$\boxed{y(0)}$	y(0.5)	y(1.0)	......			
	y_0	y_1	y_2				

(ii) approx the derivative on my grid

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} \approx \frac{y_{n+1} - y_n}{t_{n+1} - t_n} \qquad \text{eg. } \frac{y_1 - y_0}{t_1 - t_0}$$

(iii) Put (i) & (ii) together into $\boxed{}$

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} = f(y_n, t_n) \qquad \text{ex } \frac{y_1 - y_0}{t_1 - t_0} = f(y_0, t_0)$$

$$t_{n+1} - t_n = \Delta t$$

$$y_{n+1} = y_n + \underbrace{(t_{n+1} - t_n)}_{\Delta t} \cdot f(y_n, t_n)$$

$$\boxed{y_{n+1} = y_n + \Delta t \cdot f(y_n, t_n)}$$ Explicit Euler method

III. Example to code in Python & Excel.

1st order ODE ✓
IVP

$$\begin{cases} \dfrac{dc}{dt} = -\dfrac{c}{\tau} \\ c(0) = c_0 \end{cases} \qquad \tau = 0.6 \quad c_0 = 1 \quad \longrightarrow \quad \frac{dc}{dt} = f(t,c) = -\frac{c}{\tau}$$

can solve by hand.

$$c = c_0 e^{-t/\tau}$$

$$\frac{c_{n+1} - c_n}{\Delta t} = -\frac{c_n}{\tau} \longrightarrow c_{n+1} = c_n + \Delta t \cdot \left(\frac{-c_n}{\tau}\right)$$

$$c_{n+1} = c_n - \frac{\Delta t\, c_n}{\tau} = c_n\left(1 - \frac{\Delta t}{\tau}\right)$$