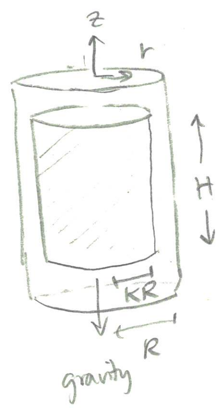
Lecture 6 - 1D Shell Momentum Balances II.

I. Another example: Falling cylinder viscometer (BSL 2C.4) p.70



* Can we calulate the viscosity of a fluid with a falling weight? what will we need to know?

* How do this? This looks like a drag force / terminal velocity problem.

* Terminal velocity (steady state) force balance on cylinder

$$0 = F_{Buoyancy} + F_{Drag} - F_{gravity} \qquad (*)$$

$$F_{Buoyancy} = \rho_L g V = \rho_L g \, \pi (KR)^2 H$$

(L ← liquid)

$$F_{gravity} = \rho_o g V = \rho_o g \, \pi (KR^2) H$$

(o ← object.)

$F_{Drag} = ?$ Recall from fluids: $F_D = \frac{1}{2} \rho u^2 C_D A_\perp$

$C_D = C_D(Re)$, $Re = \frac{\rho u D}{\mu}$

$C_D \propto$ const at $Re >> 10^2$

$C_D \propto \frac{1}{Re}$ at $Re << 10^2$

* We will very likely be at $Re << 10^2$

$$F_{Drag} = \frac{1}{2} \rho u^2 \cdot \frac{c_1}{Re} \cdot A_{\parallel}$$

← at low Re, a cylinder in this orientation is basically a flat plate! $A_{\parallel} = 2\pi KRH$

$$= \frac{1}{2} \cancel{\rho} u^{\cancel{2}} \cdot c_1 \frac{\mu}{\cancel{\rho} \cancel{u} \cancel{D}} \cdot 2\pi K \cancel{R} H$$

← $D = 2R$

$$= c_1 \frac{\pi}{2} \mu u H K$$

terminal velocity

viscosity

* Putting together our force balance

$$0 = (\rho_L - \rho_0) g \pi (kR)^2 H + \frac{c_1 \pi}{2} \mu U H K$$

solve for viscosity

$$\mu = \frac{(\rho_0 - \rho_L) g \cancel{\pi} (kR)^2 \cancel{H}}{c_1 \cancel{\pi}/2 \; U \cancel{H} K} = \frac{(\rho_0 - \rho_L) g (kR)^2}{U} \cdot \underbrace{\frac{2}{c_1 K}}$$

$c_1 = c_1(K)$ too! It depends on the geometry. Our usual drag force calculation was for external flow only

* lets do this a bit more carefully.

(1) Calculate $v_z(r)$ for the geometry

(2) Use $v_z(r)$ to calculate F_{Drag}.

we expect it will have the form: $F_D = \mu U H \cdot f(K)$

(3) Resolve Eq (*).

II. Find $v_z(r)$ for falling cylinder

* This is a shell balance problem:

(1) 1D Geometry & transport direction

(2) write Balance Equation

(3) Take limit as thickness $\to 0$

Same as last two examples → skip steps

(4) Identify Boundary conditions

(5) Math: solve ODE BVP

$$(1)-(3) \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = \frac{\mathcal{P}_L - \mathcal{P}_0}{\mu H}$$

(4) Identify Boundary conditions

$v_z(R) = 0$ no slip @ walls

$v_z(KR) = -U = -v_0$ no slip @ cylinder boundary

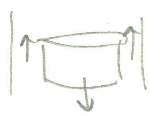
BSL notation

terminal velocity

why negative? velocity is down.

• what is v_0 ??

* It is also useful to note that mass conservation gives us another useful condition



mass flow down = mass flow up

$$\rho v_0 \pi (KR)^2 = \rho \underbrace{\iint v_z(r)\, r\,dr\,d\theta}_{\text{volumetric flow rate.}}$$

$$v_0 \pi K^2 R^2 = \int_0^{2\pi}\int_{KR}^{R} v_z\, r\,dr\,d\theta$$

$$v_0 \pi K^2 R^2 = 2\pi \int_{KR}^{R} v_z\, r\,dr$$

$$v_0 = \frac{2}{K^2R^2}\int_{KR}^{R} r v_z\,dr$$

(5) Math: solve BVP

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\Delta P}{\mu H}$$

$$v_z(R) = 0$$
$$v_z(KR) = \frac{-2}{K^2R^2}\int_{KR}^{R} r v_z\,dr$$

(integrate 1x)

$$r\frac{\partial v_z}{\partial r} = \frac{\Delta P}{\mu H}\frac{r^2}{2} + C_1 \rightarrow \frac{\partial v_z}{\partial r} = \frac{\Delta P}{\mu H}\frac{r}{2} + \frac{C_1}{r}$$

(integrate)

$$\boxed{v_z = \frac{\Delta P}{\mu H}\frac{r^2}{4} + C_1 \ln r + C_2}$$

* Finding c_1 & c_2 is a major pain

(A) Integrate $v_z(r)$ to find v_0

$$v_0 = \frac{2}{K^2R^2}\int_{KR}^{R} r v_z \, dr \qquad \int r \ln r = \frac{r^2}{2}\ln r - \frac{r^2}{4}$$

$$= \frac{2}{K^2R^2}\int_{KR}^{R}\left[\frac{\Delta P}{4\mu H} r^3 + c_1 r \ln r + c_2 r\right] dr$$

$$= \frac{2}{K^2R^2}\left[\frac{\Delta P}{4\mu H}\frac{r^4}{4} + \frac{c_1 r^2}{2}\ln r - \frac{c_1 r^2}{4} + \frac{c_2 r^2}{2}\right]_{KR}^{R}$$

$$= \frac{2}{K^2R^2}\left[\frac{\Delta P}{4\mu H}\frac{(R^4 - K^4R^4)}{4} + \frac{c_1}{2}(R^2 \ln R - K^2R^2 \ln KR)\right.$$
$$\left. + \frac{1}{2}\left(c_2 - \frac{c_1}{2}\right)(R^2 - K^2R^2)\right]$$

$$v_0 K^2R^2 = \frac{\Delta P}{4\mu H}\frac{(R^4 - K^4R^4)}{2} + c_1(R^2 \ln R - K^2R^2 \ln KR)$$
$$+ (c_2 - c_1/2)(R^2 - K^2R^2)$$

(B) simplify BC 2

$$v_z(KR) = \frac{\Delta P}{\mu H}\frac{K^2R^2}{4} + c_1 \ln(KR) + c_2 = -v_0$$

$$K^2R^2 v_z(KR) = -K^2R^2 v_0 \leftarrow \text{use above expression.}$$

$$\frac{\Delta P}{4\mu H}K^4R^4 + \cancel{c_1 K^2R^2 \ln(KR)} + \cancel{K^2R^2 c_2} = -\frac{\Delta P}{4\mu H}\frac{R^4}{2} + \frac{\Delta P K^4 R^4}{2 \cdot 4\mu H}$$

½ from canal RHS

$$- c_1 R^2 \ln R + \cancel{c_1 K^2R^2 \ln KR} - c_2 R^2 + \cancel{c_2 K^2R^2}$$
$$+ \frac{c_1 R^2}{2} - \frac{c_1 K^2R^2}{2}$$

$$0 = \frac{-\Delta P}{4\mu H}\frac{(R^4+K^4R^4)}{2} - C_1R^2\ln R - C_2R^2 + \frac{C_1R^2}{2} - \frac{C_1K^2R^2}{2} \quad \checkmark$$

(BC 2 v2)

(c) Use BC1 & BC 2 to find C_1

$$v_z(R) = 0 = \frac{\Delta P}{4\mu H}R^2 + C_1\ln R + C_2 \quad \text{(BC1)}$$

$$0 = -\frac{\Delta P}{4\mu H}\frac{(R^4+K^4R^4)}{2} - C_1R^2\ln R - C_2R^2 + \frac{C_1R^2}{2} - \frac{C_1K^2R^2}{2}$$

* multiply (BC1) by R^2 & add to (BC2 v2)

$$0 = \frac{\Delta P}{4\mu H}R^4 + \cancel{C_1R^2\ln R} + \cancel{C_2R^2}$$

$$\cancel{-\frac{\Delta P}{4\mu H}}\frac{R^4}{2} - \frac{\Delta P K^4R^4}{4\mu H\cdot 2} - \cancel{C_1R^2\ln R} - \cancel{C_2R^2} + \frac{C_1R^2}{2} - \frac{C_1K^2R^2}{2}$$

factor of ½ after cancel

$$0 = \frac{\Delta P}{4\mu H}\left(\frac{R^4}{2} - \frac{K^4R^4}{2}\right) + \frac{C_1R^2}{2} - \frac{C_1K^2R^2}{2} \qquad \text{mult by } 2/R^2$$

$$0 = \frac{\Delta P R^2}{4\mu H}(1-K^4) + C_1(1-K^2)$$

$$C_1 = \frac{-\Delta P R^2}{4\mu H}\frac{1-K^4}{1-K^2} \quad \leftarrow \quad 1-K^4 = (1-K^2)(1+K^2)$$

$$\boxed{C_1 = \frac{-\Delta P R^2}{4\mu H}(1+K^2)}$$

(D) Use BC1 to find C_2

$$v_z(R) = 0 = \frac{\Delta P}{4\mu H}R^2 + C_1\ln R + C_2$$

$$0 = \frac{\Delta P}{4\mu H}R^2 - \frac{\Delta P R^2}{4\mu H}(1+K^2)\ln R + C_2$$

$$c_2 = \frac{\Delta P R^2}{4\mu H}(1+k^2)\ln R - \frac{\Delta P R^2}{4\mu H}$$

$$\boxed{c_2 = \frac{\Delta P R^2}{4\mu H}\left[(1+k^2)\ln R - 1\right]}$$

(E) Find v_0 (use BC 2)

$$-v_0 = v_z(kR) = \frac{\Delta P}{4\mu H}k^2R^2 + c_1\ln(kR) + c_2$$

substitute in c_1 & c_2

$$v_0 = -\frac{\Delta P}{4\mu H}k^2R^2 + \frac{\Delta P R^2}{4\mu H}(1+k^2)\ln(kR) - \frac{\Delta P R^2}{4\mu H}\left[(1+k^2)\ln R - 1\right]$$

$$= \frac{\Delta P R^2}{4\mu H}\left[-k^2 + (1+k^2)\ln(kR) - (1+k^2)\ln R + 1\right]$$

$$\boxed{v_0 = \frac{\Delta P R^2}{4\mu H}\left[1 - k^2 + (1+k^2)\ln k\right]}$$

(F.) Plug this all back in to the original equation

$$v_z(r) = \frac{\Delta P}{4\mu H}r^2 + c_1\ln r + c_2$$

$$= \frac{\Delta P}{4\mu H}r^2 - \frac{\Delta P R^2}{4\mu H}(1+k^2)\ln r + \frac{\Delta P R^2}{4\mu H}\left[(1+k^2)\ln R - 1\right]$$

$$= \frac{\Delta P R^2}{4\mu H}\left[\frac{r^2}{R^2} - (1+k^2)\ln r + (1+k^2)\ln R - 1\right]$$

$$= \frac{\Delta P R^2}{4\mu H}\left[\frac{r^2}{R^2} - 1 + (1+k^2)\ln(R/r)\right]$$

get rid of this guy for v_0

$$v_z(r) = \frac{v_0}{1-\kappa^2+(1+\kappa^2)\ln\kappa}\left[\frac{r^2}{R^2}-1+(1+\kappa^2)\ln(R/r)\right]$$

$$\frac{v_z}{v_0} = \frac{r^2/R^2-1+(1+\kappa^2)\ln(R/r)}{1-\kappa^2+(1+\kappa^2)\ln\kappa}$$

cleaning up:

$\xi = r/R$

$\ln(\kappa) = -\ln(1/\kappa)$

$1-\kappa^2 = -(\kappa^2-1)$

$$\frac{v_z}{v_0} = \frac{\xi^2-1+(1+\kappa^2)\ln(1/\xi)}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)}$$

$$\boxed{\frac{v_z}{v_0} = -\frac{\xi^2-1+(1+\kappa^2)\ln(1/\xi)}{\kappa^2-1+(1+\kappa^2)\ln(1/\kappa)}}$$

or

$$\boxed{\frac{v_z}{v_0} = -\frac{1-\xi^2-(1+\kappa^2)\ln(1/\xi)}{1-\kappa^2-(1+\kappa^2)\ln(1/\kappa)}}$$

multiply top & bottom by (-1)

See 2C.4 for confirmation

* A quick comment about the solution

- parabolic part: $(1-r^2/R^2)$
- logarithmic part: $\ln(R/r)$

Just like flow in an annulus.

↓

- weighted a bit differently between the two, & different prefactor.

III. Calculate F_{drag}

* The drag force is given by the z-component of the total stress on the cylinder:

$$F_z = \int_A \underline{n}\cdot\underline{\underline{\Pi}}\cdot\underline{\delta}_z \, dA$$

total stress tensor

z-unit vector

surface area of the object.

* we should be a little more careful, the stress tensor here includes the buoyancy as well, we need to take it out because we have already accounted for it:

$$\underline{F} = \int_A \underline{n} \cdot \underline{\underline{\pi}}\, dA = \underbrace{\int \underline{n} p\, dA}_{} + \int \underline{n} \cdot \underline{\underline{\tau}}\, dA$$

$\mathcal{P} = p + \rho g z$

$p = -\rho g z + \mathcal{P}$ ← definition of dyn. P.

$$= \underbrace{-\rho g \int \underline{n} z\, dA}_{\text{buoyancy terms}} + \underbrace{\int \underline{n}\, \mathcal{P}\, dA}_{\text{dynamic pressure,}} + \underbrace{\int \underline{n} \cdot \underline{\underline{\tau}}\, dA}_{\text{shear stress.}}$$

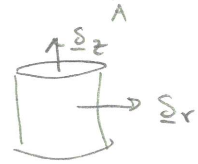
pressure from flow only (zero when $v=0$)
↑ takes out fluid statics

* Drag force is given by z-component of dynamic pressure & shear stress

$$F_z = \int_A \underline{n}\, \mathcal{P} \cdot \underline{\delta}_z\, dA + \int_A \underline{n} \cdot \underline{\underline{\tau}} \cdot \underline{\delta}_z\, dA$$

$\underline{\delta}_r \cdot \underline{\delta}_z = 0$

$\underline{\delta}_z \cdot \underline{\delta}_z = 1$



$\underline{\delta}_r \cdot \underline{\underline{\tau}} \cdot \underline{\delta}_z = \tau_{rz}$

$\tau_{\theta z} = \tau_{zz} = 0$ ← velocity gradients in θ & z are 0.

$$F_z = \int_{\text{top}} \mathcal{P}\, r\,dr\,d\theta - \int_{\text{bottom}} \mathcal{P}\, r\,dr\,d\theta + \int_{\text{sides}} \tau_{rz}\Big|_{KR} KR\, d\theta\, dz$$

$$= \mathcal{P}_0 \pi (KR)^2 - \mathcal{P}_H \pi (KR)^2 + KR \int_0^{2\pi}\int_0^H \mu \frac{\partial v_z}{\partial r}\Big|_{KR} d\theta\, dz$$

Force on cylinder is (+)

↑ independent of θ, z

$$= (\mathcal{P}_0 - \mathcal{P}_H)\pi (KR)^2 + 2\pi H K R \left(\mu \frac{\partial v_z}{\partial r}\Big|_{KR} \right)$$

* Evaluate shear stress @ the wall

(*) f(k) see below

$$\frac{\partial v_z}{\partial r} = \frac{\partial}{\partial r}\left[\frac{v_0}{f(k)}\left(\frac{r^2}{R^2}-1+(1+k^2)\ln(R/r)\right)\right]\Bigg|_{r=kR}$$

$$= \frac{v_0}{f(k)}\left[\frac{2r}{R^2}-(1+k^2)\frac{1}{r}\right]\Bigg|_{r=kR}$$

$$= \frac{v_0}{f(k)}\left[\frac{2k\cancel{R}}{R^{\cancel{2}}}-(1+k^2)\frac{1}{kR}\right]$$

$$= \frac{v_0}{Rf(k)}\left[2k-\frac{1+k^2}{k}\right] = \frac{v_0}{Rf(k)}\left[\cancel{2}k-\frac{1}{k}-\cancel{k}\right]$$

$k - 1/k$

$$= \frac{v_0}{kRf(k)}(k^2-1)$$

(*) $f(k) = 1-k^2+(1+k^2)\ln k$

* Plug into F_z

$$F_z = (\mathcal{P}_0-\mathcal{P}_H)\pi(kR)^2 + 2\pi\cancel{kR}H\cdot\left(\mu\frac{v_0}{\cancel{kR}}\frac{k^2-1}{1-k^2+(1+k^2)\ln k}\right)$$

$$F_z = \underbrace{(\mathcal{P}_0-\mathcal{P}_H)\pi(kR)^2}_{} + 2\pi\mu v_0 H\frac{k^2-1}{1-k^2+(1+k^2)\ln k}$$

use our relation in II (E) on p.6

$$\frac{\Delta\mathcal{P}R^2}{4\mu H} = \frac{v_0}{1-k^2+(1+k^2)\ln k}, \quad \Delta\mathcal{P} = \mathcal{P}_H-\mathcal{P}_0$$

$$(\mathcal{P}_0-\mathcal{P}_H) = -\frac{4\mu v_0 H/R^2}{1-k^2+(1+k^2)\ln k}$$

$$(\mathcal{P}_0-\mathcal{P}_H)\pi k^2R^2 = -\frac{4\pi\mu v_0 Hk^2}{1-k^2+(1+k^2)\ln k}$$

$$F_z = -\frac{4\pi\mu v_0 Hk^2}{1-k^2+(1+k^2)\ln k} + \frac{2\pi\mu v_0 H\,(k^2-1)}{1-k^2+(1+k^2)\ln k}$$

$$F_z = \frac{\cancel{2\pi\mu v_o H k^2} - \cancel{4}^{2}\pi\mu v_o H k^2 - 2\pi\mu v_o H}{1 - k^2 + (1+k^2)\ln k}$$

$$F_z = -\frac{2\pi\mu v_o H (k^2+1)}{1-k^2+(1+k^2)\ln k} = \frac{2\pi\mu v_o H (k^2+1)}{k^2-1+(1+k^2)\ln(1/k)}$$

IV. Expression for viscosity

$$0 = F_{Buoyancy} - F_{gravity} + F_{Drag}$$

$$0 = (\rho_L - \rho_o) g \pi (kR)^2 H + 2\pi\mu v_o H \frac{k^2+1}{k^2-1+(1+k^2)\ln 1/k}$$

↑ solve for viscosity

$$\mu = \frac{(\rho_o - \rho_L) g \cancel{\pi} (kR)^2 \cancel{H}}{2\cancel{\pi} v_o \cancel{H}} \frac{k^2-1+(1+k^2)\ln(1/k)}{k^2+1}$$

$$\mu = \frac{(\rho_o - \rho_L) g (kR)^2}{2v_o}\left[\frac{k^2-1}{k^2+1} + \frac{\cancel{1+k^2}}{\cancel{k^2+1}}\ln(1/k)\right]$$

Book swaps around : $k^2-1 = -(1-k^2)$

$$\boxed{\mu = \frac{(\rho_o - \rho_L) g (kR)^2}{2v_o}\left[\ln(1/k) - \left(\frac{1-k^2}{1+k^2}\right)\right]}$$

Eq. 2.C4-2 p. 70

- Same as our Eq on pg. 2 except it has a different numerical prefactor & fn. of geometry (k)
- We need to know ρ_o, ρ_L, k, R, v_o ← terminal velocity (measure)

ρ_o: object density; ρ_L: liquid density; k, R: visiometer geometry (cylinder radius & gap)