(4) 1D Boundary conditions

$T(H) = T_1$, $T(0) = T_0$ } const T at top & bottom → fast heat transfer (isothermal)

• other common conditions:

- constant flux/adiabatic: $q_y = c$, ($c = 0$, adiabatic)
- symmetry/antisymmetry: $\frac{dT}{dy} = 0$ / $y = 0$
- boundedness: $T < \infty$, $\left|\frac{dT}{dy}\right| < \infty$
- convection: $q_y = h(T - T_{ext})$

(if $h \to 0$, get adiabatic
if $h \to \infty$, get const T → $T = \frac{q_y}{h} + T_{ext}$

(5) Math: Solve BVP

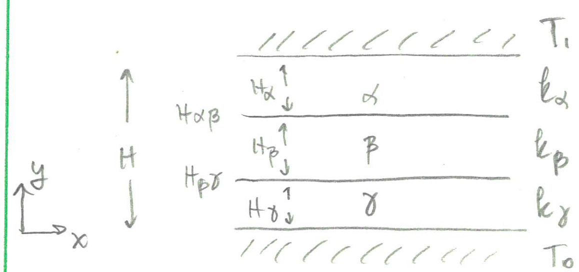
$$\frac{d^2T}{dy^2} = 0 \qquad T(H) = T_1, \quad T(0) = T_0 \qquad \text{easy!}$$

$$T = c_1 y + c_2 \qquad \left.\begin{aligned} T(H) &= c_1 H + c_2 = T_1 \\ T(0) &= c_2 = T_0 \end{aligned}\right\} \quad c_1 = \frac{T_1 - T_0}{H}$$

$$\boxed{T(y) = \frac{T_1 - T_0}{H} y + T_0}$$

III. Example: composite (similar to BSL §10.6)

* let's see if we can do a more interesting example:



← composite w/ 3 different materials

(1) Geometry & transport dir: same
(2) Balance Eq.: same
(3) Take limit & substitute eq: same.

$$\frac{d^2T}{dy^2} = 0$$ for all 3 regions. (wait where is k!?)

(4) Boundary conditions

$$T(y=H) = T_1 \quad \text{(top)}$$
$$q_\alpha \big|_{H_{\alpha\beta}} = q_\beta \big|_{H_{\alpha\beta}} \quad (\alpha\beta)$$
$$q_\beta \big|_{H_{\beta\gamma}} = q_\gamma \big|_{H_{\beta\gamma}} \quad (\beta\gamma)$$
$$T(y=0) = T_0 \quad \text{(bottom)}$$

recall $q = -k \frac{dT}{dy} \Rightarrow \frac{dq_y}{dy} = 0$ or $q_y = \text{const} = q_0$

(5) Math: solve coupled BVPs

$$T_\alpha = c_{1\alpha} y + c_{2\alpha}$$
$$T_\beta = c_{1\beta} y + c_{2\beta}$$
$$T_\gamma = c_{1\gamma} y + c_{2\gamma}$$

$$T_\gamma(y=0) = T_0 = c_{2\gamma} \longrightarrow c_{2,\gamma} = T_0$$

$$-k_\gamma \frac{dT_\gamma}{dy} = -k_\gamma c_{1,\gamma} = q_0 \longrightarrow c_{1,\gamma} = \frac{-q_0}{k_\gamma}$$

$$-k_\beta \frac{dT_\beta}{dy} = -k_\beta c_{1,\beta} = q_0 \longrightarrow c_{1\beta} = \frac{-q_0}{k_\beta}$$

$$-k_\alpha \frac{dT_\alpha}{dy} = -k_\alpha c_{1,\alpha} = q_0 \longrightarrow c_{1\alpha} = \frac{-q_0}{k_\alpha}$$

$$T_\alpha(y=H) = T_1 = \frac{-q_0}{k_\alpha} H + c_{2\alpha} \longrightarrow c_{2\alpha} = T_1 + \frac{q_0}{k_\alpha} H$$

- evaluate T_β @ $H_{\beta\gamma}$: call it $T_{\beta\gamma}$

$$T_\beta(y = H_{\beta\gamma}) = \frac{-q_0}{k_\beta} H_{\beta\gamma} + c_{2\beta} = T_{\beta\gamma} \rightarrow c_{2,\beta} = T_{\beta\gamma} + \frac{q_0}{k_\beta} H_{\beta\gamma}$$

- now plug in to find all 3

$$T_\alpha = \frac{-q_0}{k_\alpha} y + T_1 + \frac{q_0}{k_\alpha} H = \frac{q_0}{k_\alpha}(H-y) + T_1$$

$$T_\beta = \frac{-q_0}{k_\beta} y + T_{\beta\gamma} + \frac{q_0}{k_\beta} H_{\beta\gamma} = \frac{q_0}{k_\beta}(H_{\beta\gamma} - y) + T_{\beta\gamma}$$

$$T_\gamma = \frac{-q_0}{k_\gamma} y + T_0$$

- need to know q_0 $\rightarrow$ evaluate $T_\alpha(y = H_{\alpha\beta}) = T_{\alpha\beta}$
  $T_\beta(y = H_{\alpha\beta}) = T_{\alpha\beta}$
  $T_\gamma(y = H_{\beta\gamma}) = T_{\beta\gamma}$

$$T_{\alpha\beta} = T_\alpha(y = H_{\alpha\beta}) = \frac{q_0}{k_\alpha}(H - H_{\alpha\beta}) + T_1 = \frac{q_0}{k_\alpha} H_\alpha + T_1$$

$$T_1 - T_{\alpha\beta} = \frac{-q_0}{k_\alpha} H_\alpha \qquad (1)$$

$$T_{\alpha\beta} = T_\beta(y = H_{\alpha\beta}) = \frac{q_0}{k_\beta}(H_{\beta\gamma} - H_{\alpha\beta}) + T_{\beta\gamma} = \frac{-q_0}{k_\beta} H_\beta + T_{\beta\gamma}$$

$$T_{\alpha\beta} - T_{\beta\gamma} = \frac{-q_0}{k_\beta} H_\beta \qquad (2)$$

$$T_{\beta\gamma} = T_\gamma(y = H_{\beta\gamma}) = \frac{-q_0}{k_\gamma} H_\gamma + T_0$$

$$T_{\beta\gamma} - T_0 = \frac{-q_0}{k_\gamma} H_\gamma \qquad (3)$$

aside:

$$T_{\beta\gamma} = \frac{-q_0}{k_\gamma} H_\gamma + T_0$$

$$T_\beta = \frac{q_0}{k_\beta}(H_{\beta\gamma} - y) - \frac{q_0}{k_\gamma} H_\gamma + T_0$$

$$T_\beta = \frac{q_0}{k_\beta}\left(H_{\beta\gamma} - \frac{k_\beta}{k_\gamma} H_\gamma - y\right) + T_0$$

- sum of all 3

$$T_1 - T_0 = -q_0 \left(\frac{H_\alpha}{k_\alpha} + \frac{H_\beta}{k_\beta} + \frac{H_\gamma}{k_\gamma} \right)$$

$$\boxed{q_0 = \frac{-(T_1 - T_0)}{\frac{H_\alpha}{k_\alpha} + \frac{H_\beta}{k_\beta} + \frac{H_\gamma}{k_\gamma}}}$$

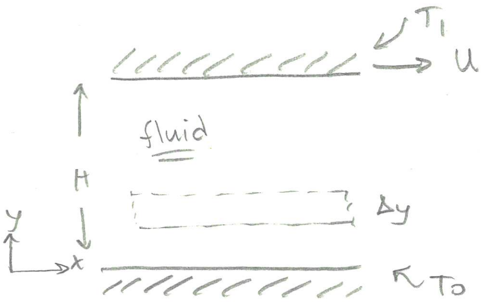
overall heat transfer coefficient

$$\boxed{q_0 = -U \Delta T} \qquad \underbrace{\boxed{\frac{1}{U} = \sum_i \frac{H_i}{k_i}}}_{\text{sum of resistances}}$$

$$\boxed{\begin{aligned}
T_\gamma &= \frac{-q_0}{k_\gamma} y + T_0 && 0 < y < H_{\beta\gamma} \\
T_\beta &= \frac{q_0}{k_\beta}\left(H_{\beta\gamma} - \frac{k_\beta}{k_\gamma} H_\gamma - y\right) + T_0 && H_{\beta\gamma} < y < H_{\alpha\beta} \\
T_\alpha &= \frac{q_0}{k_\alpha}(H - y) + T_1 && H_{\alpha\beta} < y < H
\end{aligned}}$$

* I have negative signs & BSL doesn't because my gradients go the other way.

IV. Example : Viscous Heating (BSL §10.4)



* known velocity field.

$$v_x = \frac{Uy}{H}$$

* Shell balance!

(1) Geometry & Transport Dir → same

(2) Balance EQ → same

(3) Limit & substitute $\underline{e}$ → some changes

$$\frac{d}{dy}(e_y) = 0 \qquad e_y = \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H} + \rho \hat{\Phi}\right) v_y + \tau_{yi} v_i + q_y$$

$$v_y = 0, \quad v_x \neq 0 \quad \text{so } \tau_{yx} v_x \neq 0$$

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y}$$

$$q_y = -k \frac{dT}{dy}$$